On the Doubly Transitive Permutation Representations of $\mathrm{Sp}(2n, \mathbb{F}_2)$

N.S. Narasimha Sastry[1]
Math.-Stat. Division, Indian Statistical Institute
8th Mile, Mysore Road, R.V. College Post
Bangalore - 560 059, India

Peter Sin[2]
Department of Mathematics, University of Florida,
358, Little Hall, PO Box 118105
Gainesville, FL 32611–8105, USA

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Abstract

Each symplectic group over the field of two elements has two exceptional doubly transitive actions on sets of quadratic forms on the defining symplectic vector space. This paper studies the associated 2-modular permutation modules. Filtrations of these modules are constructed which have subquotients which are modules for the symplectic group over an algebraically closed field of characteristic 2 and which, as such, have filtrations by Weyl modules and dual Weyl modules having fundamental highest weights. These Weyl modules have known submodule structures. It is further shown that the submodule structures of the Weyl modules are unchanged when restricted to the finite subgroups $\mathrm{Sp}(2n, 2)$ and $\mathrm{O}^{\pm}(2n, 2)$.

1 Introduction

The problem of determining the submodule structure of the doubly transitive permutation modules was first made explicit in [18]. In that paper, a list of the known doubly transitive groups was given and the cases where the permutation module has three or fewer composition factors were determined. Some of the other cases had already been considered in a different context (see for example [16]) or could be treated by existing methods, while others ([11], [4]) have been solved since, so that the submodule structures of many of the doubly transitive permutation modules in all characteristics are now known. Important classes where the submodule structure has not yet been investigated in detail include the two famous series of doubly transitive actions of the symplectic groups over the field of two elements. One class of these permutation representations is studied in Jordan's famous treatise [13] and the other first appeared in work of Steiner. We shall therefore follow Mortimer (see [18, 3(D), p.8]) in calling them the *Jordan-Steiner* actions. These are the actions on the quadratic forms of non-maximal and maximal index respectively which polarize to the given symplectic form. Papers in which these actions have been studied include [14] and [19].

We know from [18] that a Jordan-Steiner permutation module will have a complicated structure only in characteristic 2. As we shall see, the structure is then extremely complicated and there does not seem to us to be any reasonable way to describe the complete submodule lattice. This is not very surprising, since for modular representations in general, it is the exception rather than the rule when such a complete answer is possible. In view of this, we may look instead for suitable filtrations of the modules keeping in mind the following two conflicting aims. The subquotients of the filtration must be simple enough that we can give a detailed description of their submodule structure. At the same time the filtration must be as coarse as possible in order for much of the structure of the whole module to be captured in the subquotients. For example we could consider the radical and socle filtrations. Another idea, from the representation theory of reductive algebraic groups is the notion of a filtration by *Weyl* modules ([12, p.251]) or one by their duals (called a *good* filtration; see [12, p.238]).

In this paper, we will construct and study filtrations of the Jordan-Steiner permutation modules which are closely related to Weyl filtrations and good filtrations and have a mixture of Weyl modules and dual Weyl modules as subquotients (Theorem 7.2 and Corollary 6.3). The group $G = \mathrm{Sp}(2n, 2)$ is embedded in the obvious way in the algebraic group $G(k) = \mathrm{Sp}(2n, k)$, where k is an algebraically closed extension of $\mathbb{F}_2$. Therefore each Weyl module for $G(k)$ is a kG-module by restriction. Now the permutation modules (with coefficients in k) are certainly not modules for $G(k)$ but we shall construct kG-filtrations on them such that the subquotients are restrictions of $G(k)$-modules which have either Weyl or good filtrations. The Weyl modules involved turn out to have fundamental highest weights and consequently, we can show that their submodule structures with respect to G and $G(k)$ are identical. Moreover, the submodule structure of these Weyl modules has been determined in [2]. In this way, we obtain an incomplete but still useful picture of the permutation modules, which includes the characters and multiplicities of the composition factors and some information about how they fit together.

It has long been known (See (5) below) that restriction of a Jordan-Steiner action to the orthogonal group $\mathrm{O}(f) \cong \mathrm{O}^{\pm}(2n, 2)$ fixing one of the forms f being permuted is isomorphic to the natural action of the orthogonal group on the set of zeroes of f. Our filtrations of the Jordan-Steiner permutation modules therefore yield similar $k\mathrm{O}(f)$-filtrations on the module of functions on a quadric. In §8 we show that the submodule structures of the Weyl modules remain unchanged upon this further restriction from G to $\mathrm{O}(f)$.

2 Notation and background

2.1

Let V be a vector space of dimension $2n$ over a perfect field K of characteristic 2. To avoid trivial exceptions, we shall assume $n \geq 2$, except in §5. The spaces $S^2(V^*)$ and $\wedge^2(V)^* \cong \wedge^2(V^*)$ are, respectively, the vector space of all quadratic forms and the space of all symplectic bilinear forms on V. By definition, a quadratic form has an associated bilinear form

$$\theta(q)(v, u) = q(v + u) - q(v) - q(u) \tag{1}$$

and this formula defines the *polarization* homomorphism from $S^2(V^*)$ to $\wedge^2(V^*)$. We recall that the "Frobenius twist" $V^{(2)}$ of V is the vector space with the same additive group as V but scalar multiplication of $v \in V^{(2)}$ by $\lambda \in K$ gives the same element as scalar multiplication by $\sqrt{\lambda}$ when v is considered as an element of V. The quadratic forms with zero polarization can be identified with $V^{(2)*}$ and we have a short exact sequence of $\mathrm{GL}(V)$-modules

$$0 \longrightarrow V^{(2)*} \longrightarrow S^2(V^*) \xrightarrow{\theta} \wedge^2(V^*) \longrightarrow 0. \qquad (2)$$

2.2

Fix a nondegenerate symplectic bilinear form b on V and let $\mathrm{Sp}(V)$ be its symplectic group. Every element of $V^{(2)*} = \ker\theta$ can be written as $b(-,x)^2$ for some $x \in V$. So if q is any quadratic form polarizing to b, then

$$\theta^{-1}(b) = \{q + b(-,x)^2 \mid x \in V\}. \qquad (3)$$

Now the group $\mathrm{GL}(V)$ acts on $S^2(V^*)$ by the formula $(gq)(v) = q(g^{-1}v)$ for all $g \in G$, $q \in S^2(V^*)$ and $v \in V$. Two quadratic forms in the same orbit are called *equivalent*. The equivalence class of a quadratic form f is determined by its *Arf invariant* $\Delta(f) \in K/J$ [15, Theorem 27, p.33], where $J = \{\lambda^2 + \lambda \mid \lambda \in K\}$. If two forms q and $q' \in \theta^{-1}(b)$ are equivalent, the conjugating element of $\mathrm{GL}(V)$ must belong to $\mathrm{Sp}(V)$, so the $\mathrm{Sp}(V)$-orbits in $\theta^{-1}(b)$ are also determined by the Arf invariant. By (3) we know that there is a unique $x \in V$ such that $q' = q + b(-,x)^2$, and it follows from the formula for the Arf invariant [15, p.30] that $\Delta(q) - \Delta(q') = q(x)$ in K/J. (See also [6, p. 65], [9, Lemma 1].)

The map $\lambda \mapsto \lambda^2 + \lambda$ is an additive homomorphism of K with kernel $\{0,1\}$. So if K is finite there are two $\mathrm{Sp}(V)$-orbits.

Let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be a symplectic basis for V with respect to b and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be the dual basis for V^*. Thus,

$$b(e_i, f_j) = x_i(e_j) = y_i(f_j) = \delta_{i,j}, \quad b(e_i, e_j) = b(f_i, f_j) = 0. \qquad (4)$$

Then when K is finite the quadratic forms $q^+ = \sum_{i=1}^n x_i y_i$ and $q^- = x_1^2 + \alpha y_1^2 + \sum_{i=1}^n x_i y_i$, where $\alpha \notin J$, are representatives of the two $\mathrm{Sp}(V)$-orbits. These orbits consist of the forms of Witt index n and $n-1$ respectively.

Our main interest is in the case $K = \mathbb{F}_2$. In this case we denote $\mathrm{Sp}(V)$ by G and we choose q^- with $\alpha = 1$. We denote the G-orbits containing q^+ and q^- by $\mathcal{Q}^+$ and $\mathcal{Q}^-$ respectively, and we denote by $\mathcal{Q}_f$ the G-orbit of $f \in \theta^{-1}(b)$. The foregoing discussion shows that

$$\mathcal{Q}_f = \{f + b(-,v)^2 \mid v \in V, f(v) = 0\}. \qquad (5)$$

The stabilizer in G of f is the orthogonal group $\mathrm{O}(V, f)$. By Witt's Lemma this group has two orbits on $V \setminus \{0\}$ consisting of isotropic and anisotropic vectors. Further, the map sending v to $f + b(-, v)^2$ is an isomorphism of $\mathrm{O}(V, f)$-sets from V to $\theta^{-1}(b)$ taking the set of isotropic vectors to $\mathcal{Q}_f$. It follows that G acts doubly transitively on $\mathcal{Q}_f$, which we shall call a *Jordan-Steiner set*.

The first proof of the double transitivity of the G-action on the forms of maximal index appears in [13, p. 236]. The argument we have given is standard.

Our approach to the study of the permutation modules for the Jordan-Steiner sets will be based on the characterization (5), which will allow us to view these sets as affine subsets of a suitable module for G.

Remark 2.1 We note in passing that when $K = \mathbb{F}_2$ Witt's Lemma and the $\mathrm{O}(V, f)$-isomorphism above also imply that each of the groups $\mathrm{O}(V, q^+)$ and $\mathrm{O}(V, q^-)$ acts transitively on the Jordan-Steiner set of the opposite type, namely $\mathcal{Q}^-$ and $\mathcal{Q}^+$ respectively.

Our study of the Jordan Steiner permutation modules will be motivated in part by the following heuristical remarks. The quotient of the module $\theta^{-1}(Kb)$ by the submodule $V^{(2)*}$ is a one-dimensional trivial module. Then if K is finite, the $K\mathrm{Sp}(V)$-permutation modules given by the action of the group on each of the cosets of $V^{(2)*}$ in $\theta^{-1}(Kb)$, will have the same composition factors. This is because the composition factors are determined by the elements of odd order and the actions of an element of odd order on the $|K|$ cosets are all isomorphic. When $K = \mathbb{F}_2$, we have $V^{(2)*} \cong V^*$ and the nontrivial coset is $\theta^{-1}(b) = \mathcal{Q}^+ \cup \mathcal{Q}^-$. Therefore, in the Grothendieck group of $K\mathrm{Sp}(V)$-modules, the sum of the two Jordan-Steiner permutation modules is equal to the permutation module on the set V^*. The latter is well known to have a filtration (by polynomial degree) such that the graded module is isomorphic to the exterior algebra $\wedge(V^*)$ of V^*. (This is in fact the classical construction of the Reed-Muller codes [17].) The exterior powers $\wedge^r(V^*)$ are fundamental and have been studied in detail by several authors (e.g. [2], [3], [7] and [10]).

The remainder of this paper is an attempt to make the relation between the Jordan-Steiner permutation modules and $\wedge(V^*)$ precise.

3 Orthogonal spaces of dimension $2n+1$

In this section, we study the well known correspondence which exists over perfect fields of characteristic 2 between the geometry of an odd-dimensional orthogonal space and that of a symplectic space of one dimesion lower. Our purpose in doing this is to produce a kG-module where the action of G on the Jordan-Steiner set can be described in coordinates and directly compared with the action of G on $\wedge(V^*)$. In the process, we also give proofs of some classical facts since this can be done with no extra effort.

3.1

Let $\widehat{V}$ denote a vector space of dimension $2n+1$ over K. Fix a nonzero vector $\widehat{d} \in \widehat{V}$ and take V above to be the space $\widehat{V}/K\widehat{d}$, with $\pi : \widehat{V} \to V$ the natural map. Let $\widehat{b}$ be the symplectic bilinear form on $\widehat{V}$ defined by

$$\widehat{b}(u,v) = b(\pi(u), \pi(v)) \qquad \text{for } u,\, v \in \widehat{V}. \tag{6}$$

Let $\widehat{\mathcal{Q}}$ denote the set of all nondegenerate quadratic forms $\widehat{q}$ on $\widehat{V}$ such that $\widehat{q}(\widehat{d}) = 1$ and $\widehat{b}(u,v) = \widehat{q}(u+v) - \widehat{q}(u) - \widehat{q}(v)$ for $u,\, v \in \widehat{V}$. Let $\widehat{q} \in \widehat{\mathcal{Q}}$ and $v \in V$. Then the map

$$\widehat{v} \mapsto \tilde{v} = \widehat{v} + \sqrt{\widehat{q}(\widehat{v})}\widehat{d} \tag{7}$$

is constant on $\pi^{-1}(v)$. The element $\tilde{v}$ satisfies $\pi(\tilde{v}) = v$ and $\widehat{q}(\tilde{v}) = 0$ and is the unique element of $\widehat{V}$ with these properties.

Let $\Gamma(\widehat{V})$ denote the group of linear automorphisms of $\widehat{V}$ which preserve $\widehat{d}$ and $\widehat{b}$. Let $\mathcal{H}$ be the set of all hyperplanes of $\widehat{V}$ which do not contain $\widehat{d}$.

Lemma 3.1 *(i) For any $\widehat{q} \in \widehat{\mathcal{Q}}$, we have $\widehat{\mathcal{Q}} = \{\widehat{q} + \lambda^2 \mid \lambda \in \widehat{V}^*, \lambda(\widehat{d}) = 0\}$.*

(ii) $\Gamma(\widehat{V})$ acts transitively on $\widehat{\mathcal{Q}}$.

(iii) $\Gamma(\widehat{V})$ acts transitively on $\mathcal{H}$.

Proof: Part (i) follows from the fact that the difference of two quadratic forms with the same polarization is the square of a linear functional. For $\lambda \in \widehat{V}^*$ with $\lambda(\widehat{d}) = 0$, the transformation $t_\lambda : v \mapsto v + \lambda(v)\widehat{d}$ of $\widehat{V}$ belongs to $\Gamma(\widehat{V})$ and its induced action on quadratic forms sends $\widehat{q} + \lambda^2$ to $\widehat{q}$. This proves (ii). To prove (iii), we identify $\mathcal{H}$ with $\{\nu \in \widehat{V}^* \mid \nu(\widehat{d}) = 1\}$. For μ, ν in this set, let $\lambda = \mu - \nu$. Then in its action on $\widehat{V}^*$ the element t_λ sends μ to

ν. ☐

Lemma 3.2 *Let $\widehat{q} \in \widehat{\mathcal{Q}}$ and $H \in \mathcal{H}$.*

(i) The set of $\mathrm{O}(\widehat{q})$*-orbits on $\mathcal{H}$ is in bijection with the set of* $\mathrm{Sp}(H, \widehat{b}|_H)$*-orbits on $\widehat{\mathcal{Q}}$.*

(ii) The permutation action of $\mathrm{Sp}(H, \widehat{b}|_H)$ *on $\widehat{\mathcal{Q}}$ is isomorphic to the action of* $\mathrm{Sp}(V)$ *on $\theta^{-1}(b)$.*

(iii) The set of $\mathrm{O}(\widehat{q})$*-orbits on $\mathcal{H}$ is in bijection with the set of* $\mathrm{Sp}(V)$*-orbits on $\theta^{-1}(b)$.*

Proof: The stabilizer in $\Gamma(\widehat{V})$ of $\widehat{q}$ is the orthogonal group $\mathrm{O}(\widehat{q})$ and the stabilizer of H is the symplectic group $\mathrm{Sp}(H, \widehat{b}|_H)$. Therefore, (i) follows from (ii) and (iii) of Lemma 3.1. Each quadratic form on H polarizing to $\widehat{b}|_H$ has a unique extension to an element of $\widehat{\mathcal{Q}}$ and since π induces an isometry of H with V, we have (ii). Then (iii) follows from (i) and (ii). ☐

Example 3.3 Let $\widehat{d}$, $\widehat{e}_i$, $\widehat{f}_i$, $i = 1, \ldots, n$, be a basis of $\widehat{V}$ such that $\pi(\widehat{e}_i) = e_i$ and $\pi(\widehat{f}_i) = f_i$ and let $\widehat{z}$, $\widehat{x}_i$, $\widehat{y}_i$, $i = 1, \ldots, n$, be the dual basis for $\widehat{V}^*$. If K is finite and H is the hyperplane $\widehat{z} = 0$ then the restrictions to H of $\widehat{q}^+ = \sum_{i=1}^n \widehat{x}_i \widehat{y}_i$ and $\widehat{q}^+ + \widehat{x}_1^2 + \alpha \widehat{y}_1^2$ ($\alpha \notin J$) correspond under π to the representatives q^+ and q^- of the $\mathrm{Sp}(V)$-orbits in $\theta^{-1}(b)$.

3.2

Fix $\widehat{q} \in \widehat{\mathcal{Q}}$. Then π defines a group homomorphism $\sigma_{\widehat{q}}$ from $\mathrm{O}(\widehat{q})$ to $\mathrm{Sp}(V)$. (The well known fact that this is an isomorphism will follow from our discussion.) Let $U = \theta^{-1}(Kb)$. Via $\sigma_{\widehat{q}}$, we may think of the $\mathrm{Sp}(V)$-module U as a module for $\mathrm{O}(\widehat{q})$. Also we have natural injections $i : V^{(2)*} \to U$ and $\pi^{(2)*} : V^{(2)*} \to \widehat{V}^{(2)*}$.

Theorem 3.4 *There exists a $K\mathrm{O}(\widehat{q})$-module isomorphism $\zeta_{\widehat{q}}$ from $\widehat{V}^{(2)*}$ to U such that the following diagram commutes.*

$$\begin{tikzcd} \widehat{V}^{(2)*} \arrow[r, "\zeta_{\widehat{q}}"] & U \\ V^{(2)*} \arrow[u, hook, "\pi^{(2)*}"] \arrow[r, equal] & V^{(2)*} \arrow[u, hook', "i"'] \end{tikzcd} \tag{8}$$

For $\lambda \in \widehat{V}^{(2)*}$ the quadratic form $\zeta_{\widehat{q}}(\lambda)$ is given by the formula

$$\zeta_{\widehat{q}}(\lambda)(v) = \lambda(\widehat{v}) + \widehat{q}(\widehat{v})\lambda(\widehat{d}), \tag{9}$$

where $\widehat{v} \in \widehat{V}$ is any preimage of $v \in V$ under π.

Proof: For $v \in V$ let $\tilde{v}$ be its preimage in $\widehat{V}$ given in (7). For $\lambda \in \widehat{V}^{(2)*}$, we define $T_{\widehat{q}}(\lambda) : V \to K$ given by $T_{\widehat{q}}(\lambda)(v) = \lambda(\tilde{v})$. Then $T_{\widehat{q}}(\lambda)$ is a quadratic form which polarizes to $\lambda(\widehat{d})^2 b$, so $T_{\widehat{q}}(\lambda) \in U$. Then the map $\zeta_{\widehat{q}} : \widehat{V}^{(2)*} \to U$ given by $\zeta_{\widehat{q}}(\lambda) = T_{\widehat{q}}(\lambda)$ is an injective homomorphism of $K\mathrm{O}(\widehat{q})$-modules, hence an isomorphism since both $V^{(2)*}$ and U have dimension $2n + 1$. The commutativity of the diagram is easily checked. $\square$

Remark 3.5 Theorem 3.4 gives us the following commutative diagram

$$\begin{tikzcd} \mathrm{O}(\widehat{q}) \arrow[r, "\rho"] \arrow[d, "\sigma_{\widehat{q}}"] & \mathrm{GL}(\widehat{V}^{(2)*}) \\ \mathrm{Sp}(V) \arrow[r] & \mathrm{GL}(U) \arrow[u, "\approx"] \end{tikzcd} \tag{10}$$

in which ρ is the representation of $\mathrm{O}(\widehat{q})$ on $V^{(2)*}$ and the right vertical isomorphism is induced by $\zeta_{\widehat{q}}$. The commutativity of the square shows that ρ may be factored through $\sigma_{\widehat{q}}$. Thus $V^{(2)*}$ has the structure of a $\mathrm{Sp}(V)$-module. As a homomorphism of abstract groups, the injectivity of ρ implies that $\sigma_{\widehat{q}}$ is injective. Since $\mathrm{Sp}(V)$ acts faithfully on U, it follows that $\sigma_{\widehat{q}}$ is also surjective. Thus, we have proved the well known isomorphism (of abstract groups) of $\mathrm{O}(\widehat{q})$ with $\mathrm{Sp}(V)$. In the case of algebraic groups, the above diagram also shows that the Frobenius map (obtained by composing ρ with inverse-transpose) of $\mathrm{O}(\widehat{q})$ factors through the noncentral infinitesimal isogeny $\sigma_{\widehat{q}}$.

4 Coordinates and filtrations

In this section we shall apply the results of the previous sections when the field K is either $\mathbb{F}_2$ or an algebraically closed extension k. We wish to view the $\mathbb{F}_2$ theory as embedded inside the theory for k, so we need first to clarify our notation. We will keep the notation V, U, $\widehat{V}$, $\widehat{\mathcal{Q}}$, $\zeta_{\widehat{q}}$, $\theta^{-1}(b)$, etc. to mean the objects with $K = \mathbb{F}_2$ and use parallel notation such as V_k, U_k, $\widehat{V}_k$, $\widehat{\mathcal{Q}}_k$, $\zeta_{\widehat{q}_k}$ for the corresponding k-objects. Further, we shall consider the $\mathbb{F}_2$-objects such as V, $\widehat{\mathcal{Q}}$ as subsets of V_k, $\widehat{\mathcal{Q}}_k$, etc.

4.1

We now fix some additional notation. We choose bases for $\widehat{V}$ and V and their duals as in Example 3.3 and use the same bases for the extensions to k. Then under our convention that $V_k^{(2)}$ is the same abelian group as V_k, the elements e_i, f_i also form a basis for $V_k^{(2)}$, but because of the different scalar multiplication, we will denote these elements by capitals E_i and F_i when thinking of them this way. The dual basis will consist of the elements $X_i = x_i^2$ and $Y_i = y_i^2$. We apply the same convention to $\widehat{V}_k$. Thus, our chosen basis for $\widehat{V}_k^{(2)*}$ is $\widehat{Z}$, $\widehat{X}_i$, $\widehat{Y}_i$, $i = 1, \ldots, n$.

We now proceed to describe the G-orbits on $\theta^{-1}(b) \subseteq U_k$ in coordinates by means of Theorem 3.4. That theorem gives different isomorphisms $\zeta_{\widehat{q}_k}$ for different choices of $\widehat{q}_k \in \widehat{\mathcal{Q}}_k$. Since $\pi_k^{(2)*}(X_i) = \widehat{X}_i$ and $\pi_k^{(2)*}(Y_i) = \widehat{Y}_i$, the commutativity of the diagram in Theorem 3.4 yields $\zeta_{\widehat{q}_k}(\widehat{X}_i) = X_i$ and $\zeta_{\widehat{q}_k}(\widehat{Y}_i) = Y_i$ for all $\widehat{q}_k \in \widehat{\mathcal{Q}}_k$. The image of $\widehat{Z}$ under $\zeta_{\widehat{q}_k}$ will depend on $\widehat{q}_k$. Let $f \in \theta^{-1}(b)$ and $\mathcal{Q}_f$ be its G-orbit. Theorem 3.4 shows that there is a unique choice of $\widehat{q}_f \in \widehat{\mathcal{Q}}_k$ such that $\zeta_{\widehat{q}_f}(\widehat{Z}) = f$ and of course we have $\widehat{q}_f \in \widehat{\mathcal{Q}}$.

Let $\mathcal{Z}_f$ be the G-orbit of $\widehat{Z}$ in $\widehat{V}^{(2)*} \subseteq \widehat{V}_k^{(2)*}$ under the action defined by $\zeta_{\widehat{q}_f}$. For $v = \sum_{i=1}^n (\alpha_i e_i + \beta_i f_i) \in V$, α_i, $\beta_i \in \mathbb{F}_2$, we have

$$b(-, v)^2 = \sum_{i=1}^n (\beta_i X_i + \alpha_i Y_i) \in V^{(2)*} \subseteq V_k^{(2)*} \tag{11}$$

so by (3)

$$\zeta_{\widehat{q}_f}^{-1}(\theta^{-1}(b)) = \{\widehat{Z} + \sum_{i=1}^n \beta_i \widehat{X}_i + \alpha_i \widehat{Y}_i \mid \alpha_i, \beta_i \in \mathbb{F}_2\}. \tag{12}$$

Since, by (5), $f + b(-,v)^2 \in \mathcal{Q}_f$ if and only if $v \in V$ and $f(v) = 0$, we have

$$\mathcal{Z}_f = \{\widehat{Z} + \sum_{i=1}^{n} \beta_i \widehat{X}_i + \alpha_i \widehat{Y}_i \mid \alpha_i, \beta_i \in \mathbb{F}_2,\ f(\sum_{i=1}^{n} \alpha_i e_i + \beta_i f_i) = 0\}. \tag{13}$$

Thus, $\mathcal{Z}_f$ is the set of $\mathbb{F}_2$-rational points of the affine subset of $\widehat{V}_k^{(2)*}$ given by the equations $\widehat{D} = 1$, $f(\widehat{F}_i, \widehat{E}_i) = 0$. Here we have identified $\widehat{V}_k^{(2)}$ with its double dual and $f(\widehat{F}_i, \widehat{E}_i)$ means the quadratic form on $\widehat{V}_k^{(2)*}$ obtained by substituting $\widehat{F}_i$ for x_i and $\widehat{E}_i$ for y_i in f.

4.2

The action of $\mathrm{Sp}(V_k)$ on $\widehat{V}_k^{(2)}$ extends to a homogeneous action of $\mathrm{Sp}(V_k)$ on the polynomial ring

$$\widehat{R} = k[\widehat{E}_1, \ldots, \widehat{E}_n, \widehat{F}_1, \ldots, \widehat{F}_n, \widehat{D}]. \tag{14}$$

Since $\mathrm{Sp}(V_k)$ preserves $\widehat{D}$, the subspaces

$$\mathcal{F}_t = \{h \in \widehat{R} \mid \text{degree of } h \text{ in } \widehat{E}_i,\ \widehat{F}_i \text{ is at most } t\} \tag{15}$$

define a filtration of $\widehat{R}$ by $\mathrm{Sp}(V_k)$-modules, as t ranges over the natural numbers. The filtrations for its quotient rings S considered below are the images of this filtration under natural surjections. We denote the image of $\mathcal{F}_t$ in S by $\mathcal{F}_t(S)$ and the quotient space $\mathcal{F}_t(S)/\mathcal{F}_{t-1}(S)$ by $\mathrm{gr}_t(S)$.

Let I denote the ideal of $\widehat{R}$ generated by $\widehat{D}$, $\widehat{E}_1\widehat{F}_1 + \cdots + \widehat{E}_n\widehat{F}_n$, $\widehat{E}_i^2$,$\widehat{F}_i^2$, $i = 1,\ldots,n$. This ideal is stable under $\mathrm{Sp}(V_k)$. For $f \in \theta^{-1}(b)$, let I_f denote the ideal of $\widehat{R}$ generated by $\widehat{D} - 1$, $f(\widehat{F}_i, \widehat{E}_i)$, $\widehat{E}_i^2 - \widehat{E}_i$, $\widehat{F}_i^2 - \widehat{F}_i$, $i = 1,\ldots, n$. This ideal is stable under the action of G. Let $A = \widehat{R}/I$ and $A_f = \widehat{R}/I_f$.

Lemma 4.1 *There exists a surjective kG-module homomorphism φ_f from* $\mathrm{gr}(A)$ *to* $\mathrm{gr}(A_f)$.

Proof: Let t be a nonnegative integer and M_t denote the k-subspace $k[\widehat{E}_i, \widehat{F}_i]_{i=1}^n \cap \mathcal{F}_t$ of $\widehat{R}$. Consider the diagram

$$\begin{tikzcd} M_t \arrow[r, hook, "i"] \arrow[dr, "\eta_t \circ i"'] & \mathcal{F}_t \arrow[r, two heads, "\psi_t"] \arrow[d, "\eta_t"] & \mathcal{F}_t(A_f)/\mathcal{F}_{t-1}(A_f) \\ & \mathcal{F}_t(A) \arrow[ur, dashed, two heads, "\overline{\psi}_t"'] & \end{tikzcd} \tag{16}$$

where i is the inclusion map and η_t, ψ_t are the natural surjections. Note that $\eta_t \circ i$ is onto with kernel $M_t \cap I$. The map $\psi_t \circ i$ takes this kernel to zero because its elements can be written as $\sum_{i=1}^{n}(\widehat{E}_i^2 H_i + \widehat{F}_i^2 K_i) + (\sum_{i=1}^{n} \widehat{E}_i \widehat{F}_i)L$ with $H_i, K_i, L \in M_{t-2}$ and $\psi_t \circ i$ takes $\widehat{E}_i^2, \widehat{F}_i^2$ and $\sum_{i=1}^{n} \widehat{E}_i \widehat{F}_i$ and all multiples of them to zero. Therefore a k-linear surjective homomorphism $\overline{\psi}_t$ exists and the diagram commutes. Since ψ_t and η_t are kG-module homomorphisms and η_t is surjective, it follows that $\overline{\psi}_t$ also is a kG-module homomorphism. Since ψ_t is surjective, so is $\overline{\psi}_t$. Since $\mathcal{F}_{t-1} \subseteq \mathrm{Ker}\psi_t$, it follows that $\overline{\psi}_t$ maps $\mathcal{F}_{t-1}(A)$ to zero. Thus there exists a kG-module surjective homomorphism from $\mathrm{gr}_t(A)$ to $\mathrm{gr}_t(A_f)$. $\square$

5 The exterior algebra and the spin module

The results of this section are needed in the proof of Theorem 7.2 to determine the kernel of φ_t but may also be of general interest. They hold for any field K of characteristic two. Though Proposition 5.1 below can be extracted from [10], we present a proof of this because our approach is different and makes the presentation of Proposition 5.2 easier.

We consider b as an element of degree two in the exterior algebra $\wedge(V^*)$. In characteristic two we have $b^2 = 0$. So the map δ given by multiplication by b makes $\wedge(V^*)$ into a complex. It is best for us to keep the standard grading on $\wedge(V^*)$ so δ has degree two. Obviously, the complex decomposes into a direct sum $\wedge(V^*) = \wedge(V^*)^{even} \bigoplus \wedge(V^*)^{odd}$. The symplectic group $\mathrm{Sp}(V)$ which preserves b acts on this complex, hence also on its homology groups.

Proposition 5.1 *$H^i(\wedge(V^*), \delta) = 0$ unless $i = n$, in which case it affords an irreducible representation of $\mathrm{Sp}(V)$ of dimension 2^n.*

Proof: First consider the case when $n = 1$. Here the even part is just V^* concentrated in degree 1, while the odd part is $\delta : K \to K$. So the proposition is true in this case. For the general case, we consider the decomposition

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_n, \qquad b = b_1 + b_2 + \cdots + b_n \tag{17}$$

into hyperbolic planes. We have the tensor factorization

$$\wedge(V^*) = \otimes_{j=1}^{n} \wedge (W_j^*) \tag{18}$$

under which multiplication by b on the left becomes the map $b_1 \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes b_2 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes b_n$. Therefore, (18) is an isomorphism of complexes. (There is no sign to worry about!)

In light of the case $n = 1$, we now see that the complex $\wedge(V^*)$ is a direct sum of subcomplexes of the form $M = M_1 \otimes M_2 \otimes \cdots \otimes M_n$, where each M_j is either W_j^* concentrated in degree 1, or $\delta_j : K \to K$. Now the tensor product of exact complexes is exact, as is the tensor product of an exact complex with one concentrated in a single degree, while the tensor product of two complexes each concentrated in a single degree results in one of the same kind. Thus, all subcomplexes of M above are exact, with the sole exception of the one in which all the M_j are taken to be W_j^*, which is the complex $\otimes_{j=1}^n W_j^*$ concentrated in degree n. Thus, $H^i(\wedge(V^*), \delta) = 0$ if $i \neq n$, while

$$H^n(\wedge(V^*), \delta) = \otimes_{j=1}^n W_j^*. \tag{19}$$

This proves all statements about dimensions. The decomposition into subcomplexes M above is equivariant with respect to the subgroup $\mathrm{SL}(W_1) \times \cdots \times \mathrm{SL}(W_n)$ of $\mathrm{SL}(V)$, and so (19) is an isomorphism of modules for this group, in which the right hand side is obviously simple. This completes the proof of the proposition. $\square$

If K is algebraically closed, then $\mathrm{Sp}(V)$ is a simple algebraic group (of type C_n) and the module L we have just constructed is a rational one. Its highest weight must be equal to the highest weight in the module $\wedge^n(V^*)$, since this weight does not occur in any other exterior power. It is easy to see that this weight is the fundamental weight corresponding to the long fundamental root in the C_n root system. The module L is called the *spin module* for $\mathrm{Sp}(V)$.

With e_i, f_i , x_i, y_i as in 2.2, we have $b = x_1 \wedge y_1 + \cdots + x_n \wedge y_n$ and the space $\wedge^n(V^*)$ has a basis consisting of elements $y_I x_J$, where I and J are subsets of $N = \{1, \cdots, n\}$ such that $|I| + |J| = n$, $x_I = \prod_{i \in I} x_i$ and $y_I = \prod_{i \in I} y_i$. Let $\overline{I}$ denote the complement of I in N and $[y_I x_J]$ denote the image of $y_I x_J$ in L.

Proposition 5.2 *The 2^n elements $[y_I x_{\overline{I}}]$ for $I \subseteq N$ form a basis for the spin module L.*

Proof: The elements $[y_I x_{\overline{I}}]$ are clearly annihilated by δ. It is not difficult to see they do not lie in $\delta(\wedge^{n-2}(V^*))$ so they have nonzero images in L. To see that these images are linearly independent, there is no loss in assuming that K is algebraically closed. Then we see that they are simultaneous eigenvectors affording distinct characters of the maximal torus of $\mathrm{Sp}(V)$ which is the product of the diagonal subgroups of the $\mathrm{SL}(W_j)$ in the hyperbolic decomposition described in the proof of Proposition 5.1. This proves their linear independence and the proposition. $\square$

6 Good filtrations and Weyl filtrations

In this section we assume k to be an algebraically closed extension of the field K of the previous section. Later we will apply the results in the case $K = \mathbb{F}_2$. Let $V_k = k \otimes_K V$. It is known [7, Appendix A] that as modules for the algebraic group $\mathrm{Sp}(V_k)$, the exterior powers have filtrations by Weyl modules ([12, p.251] and, by self-duality, also good filtrations (by duals of Weyl modules); see [12, p.238]).

Proposition 6.1 *(i)* $\wedge^r(V_k)/\mathrm{Im}\delta \cap \wedge^r(V_k)$ *has a good filtration for* $0 \leq r \leq n$ *and a Weyl filtration for* $n+1 \leq r \leq 2n$.

(ii) $\wedge^r(V_k)/\mathrm{Ker}\delta \cap \wedge^r(V_k)$ *has a good filtration for* $0 \leq r \leq n-1$ *and a Weyl filtration for* $n \leq r \leq 2n$.

Proof: We shall use the general fact [12, II.4.17] that in a short exact sequence

$$0 \to W' \to W \to W'' \to 0 \tag{20}$$

of rational modules for a reductive algebraic group, if W' and W have good filtrations then so does W'', together with the dual statement that if W'' and W have Weyl filtrations then so does W'. This will allow us to argue by induction. It is clear that (i) holds for $r = 0$ and $r = 1$. By Lemma 5.1, for $2 \leq r \leq n$, we have the exact sequence

$$0 \to \wedge^{r-2}(V_k)/\mathrm{Im}\delta \cap \wedge^{r-2}(V_k) \to \wedge^r(V_k) \to \wedge^r(V_k)/\mathrm{Im}\delta \cap \wedge^r(V_k) \to 0. \tag{21}$$

Then $\wedge^r(V_k)$ has a good filtration, as mentioned above, and $\wedge^{r-2}(V_k)/\mathrm{Im}\delta \cap \wedge^{r-2}(V_k)$ has one by the inductive hypothesis. So by the general fact mentioned, we have proved (i) by induction. The remainder of the proof of (i) is the same argument using duality and reverse induction and (ii) is proved in the same way as (i). □

6.1 Characters

Let ω_i, $1 \leq i \leq n$ be the fundamental dominant weights and let $V(i)$ be the Weyl module with highest weight ω_i. We label the Dynkin diagram in the usual way so that $V(1)$ is the natural module. We denote by $H(i)$ and $L(i)$ the corresponding dual Weyl module and simple module respectively. In addition, $V(0)$, $H(0)$ and $L(0)$ all mean the trivial module.

By the independence of characters of a maximal torus, the multiplicities of the factors which occur in the good (or Weyl) filtration of $\wedge^r(V_k^*)$ are the same as in characteristic zero; here it is a classical fact that the degree 2 map given by multiplication by the form is injective for degrees $< n$ and that the dual Weyl module is its cokernel. Therefore, we have $H(r) = \wedge^r(V_k^*) - \wedge^{r-2}(V_k^*)$ for $0 \leq r \leq n$ in the Grothendieck group of $Sp(V_k)$-modules, where, as usual, we take modules to be zero if they are indexed by numbers outside the range $[0, 2n]$. Inverting this equation yields

$$\wedge^r(V_k^*) = \sum_{t=0}^{\lfloor r/2 \rfloor} H(r-2t). \tag{22}$$

By the same token, the multiplicities of the good (and Weyl) filtration factors of Cokerδ and Imδ depend only on the (formal) characters. They can therefore be computed using (22) and Proposition 5.1.

Proposition 6.2 *The Weyl modules and dual Weyl modules which occur in the filtrations of Proposition 6.1 each occur with multiplicity one and are as follows.*

(i) For $0 \leq r \leq n$, $\wedge^r(V_k)/\mathrm{Im}\delta \cap \wedge^r(V_k)$ has factors $H(r-4t)$, with $0 \leq t \leq \lfloor r/4 \rfloor$.

(ii) For $n < r \leq 2n$, $\wedge^r(V_k)/\mathrm{Im}\delta \cap \wedge^r(V_k)$ has factors $V((2n-r-2)-4t)$, with $0 \leq t \leq \lfloor (2n-2-r)/4 \rfloor$.

(iii) For $0 \leq r < n$, $\wedge^r(V_k)/\mathrm{Ker}\delta \cap \wedge^r(V_k)$ has factors $H(r-4t)$, with $0 \leq t \leq \lfloor r/4 \rfloor$.

(iv) *For* $n \leq r \leq 2n$, $\wedge^r(V_k)/\mathrm{Ker}\delta \cap \wedge^r(V_k)$ *has factors* $V((2n-r-2)-4t)$, *with* $0 \leq t \leq \lfloor (2n-2-r)/4 \rfloor$.

(Here, $\lfloor x \rfloor$ *denotes the 'integral part' of* x*.)*

□

Corollary 6.3 *In the Grothendieck ring of* $\mathrm{Sp}(V_k)$*-modules,*

(i)

$$\begin{aligned}\sum_{r=0}^{n} \mathrm{Coker}\delta_r &= \sum_{t=0}^{\lfloor n/4 \rfloor} (t+1)[H(n-4t) + H(n-4t-1) \\ &\quad + H(n-4t-2) + H(n-4t-3)];\end{aligned}$$

(ii)

$$\begin{aligned}\sum_{r=n+1}^{2n} \mathrm{Coker}\delta_r &= \sum_{t=0}^{\lfloor (n-3)/4 \rfloor} (t+1)[V(n-4t-3) + V(n-4t-4) \\ &\quad + V(n-4t-5) + V(n-4t-6)].\end{aligned}$$

(iii) $\mathrm{Coker}\delta = L + \mathrm{Im}\delta$.

□

As we have seen, it is an easy matter to determine which Weyl modules and dual Weyl modules appear in these filtrations. A harder question is to give the composition factors of the Weyl modules and harder still is the problem of describing the submodule lattices of the Weyl modules. Fortunately, these questions have been answered in the papers [2] and [3] and we will briefly describe the results here. The composition factors of the Weyl modules $V(i)$ also have fundamental highest weights and their multiplicities $|V(i) : L(j)|$ are either 0 or 1. The multiplicity is 1 if and only if $i \geq j$, $i-j$ is even and the binary digits of $\frac{1}{2}(i-j)$ are among those of $n-j+1$. The fact that the Weyl modules involved are multiplicity-free is of great help in describing their submodule structure. In general, this means that the submodule lattice is the latttice of ideals with respect to a natural ordering on the set of composition factors. The problem then is to give a combinatorial description of the ordering. For the Weyl modules with fundamental highest weights, this has been accomplished by Adamovich [3].

Finally we note that since all of the composition factors of the Weyl modules have fundamental, hence 2-restricted, highest weights, it follows from [1, Proposition 2.7] that when $K = \mathbb{F}_2$ the submodule structures of the Weyl modules are the same for the finite group $\mathrm{Sp}(V)$ as they are for the algebraic group $\mathrm{Sp}(V_k)$.

7 Functions on the Jordan-Steiner sets

We now resume the hypotheses and notations of §4 and work with $\mathbb{F}_2$ and its algebraically closed extension k.

7.1

Let f be one of the G-orbit representatives q^+, q^- in $\theta^{-1}(b)$ given in 2.2. We now consider the rings A and A_f from 4.2.

Then $\mathrm{Sp}(V_k)$ stabilizes the nondegenerate symplectic bilinear form $\sum_{i=1}^n F_i \wedge E_i$ on $V_k^{(2)*}$ and we have

$$A \cong \frac{k[E_1, \cdots, E_n, F_1, \cdots, F_n]}{< E_i^2, F_i^2, \sum_{i=1}^n E_i F_i >} \cong \frac{\wedge(V_k^{(2)})}{< \sum_{i=1}^n F_i \wedge E_i >} \tag{23}$$

as $\mathrm{Sp}(V_k)$-modules.

Let δ denote the $\mathrm{Sp}(V_k)$-module endomorphism of $\wedge(V_k^{(2)*})$ given by the left multiplication by the above symplectic bilinear form. Then (23) and Proposition 5.1 now yield the following lemma.

Lemma 7.1 *(i)* $A \cong \mathrm{Coker}(\delta)$ *as* $\mathrm{Sp}(V_k)$*-modules;*

(ii) A *contains a simple* $\mathrm{Sp}(V_k)$*-submodule isomorphic to the spin module* $L = \mathrm{Ker}\delta/\mathrm{Im}\delta$*;*

(iii) $\dim_k \mathrm{Im}(\delta) = 2^{n-1}(2^n - 1)$ *and* $\dim_k(\mathrm{Ker}\delta) = 2^{n-1}(2^n + 1)$*; and*

(iv) $\dim_k(A) = 2^{n-1}(2^n + 1)$.

We view A_f as the space of all k-valued functions on V_f.

Theorem 7.2 *Let φ_t be the kG-module homomorphism from $\mathrm{gr}_t(A)$ onto $\mathrm{gr}_t(A_f)$ defined in Lemma 4.1.*

(i) If $f = q^+$, then φ_t is an isomorphism for all t.

(ii) If $f = q^-$, then φ_t is an isomorphism for degrees t with $t \neq n$. The kernel of φ_n is the image in $\mathrm{gr}_n(A)$ of the simple kG-submodule of Lemma 7.1(ii).

Thus, we have $\mathrm{gr}(A_{q^+}) \cong \mathrm{Coker}\delta$ and $\mathrm{gr}(A_{q^-}) \cong \mathrm{Im}\delta$ as kG-modules.

Proof: Since $\dim_k \mathrm{gr}(A_f) = \dim_k A_f = |V_f| = |\{a \in V \mid f(a) = 0\}| = 2^{n-1}(2^n + 1)$ or $2^{n-1}(2^n - 1)$ according as $f = q^+$ or $f = q^-$, part (i) follows from the surjectivity of φ_t and (iv) of Lemma 7.1.

By Proposition 5.2 the simple $\mathrm{Sp}(V_k)$-submodule M of A given in Lemma 7.1(ii) which corresponds to $\mathrm{Ker}\delta/\mathrm{Im}\delta$ in $\mathrm{Coker}\delta$ is generated as a kG-module by the image in A of the product $\widehat{E}_1 \cdots \widehat{E}_n \in \mathcal{F}_n$. We must show that the image of $\widehat{E}_1 \cdots \widehat{E}_n$ in $\mathrm{gr}_n(A_{q^-})$ is zero. Consider the polynomial

$$h(\widehat{E}_1, \cdots, \widehat{E}_n) = \widehat{E}_1 \prod_{i=2}^{n} (1 + \widehat{E}_i) \tag{24}$$

in $\widehat{R}$. We will prove that this function vanishes on V_{q^-}. Then, on expanding h, it will be immediate that the image of $\widehat{E}_1 \cdots \widehat{E}_n$ in A_{q^-} belongs to $\mathcal{F}_{n-1}(A_{q^-})$. Let $m \in V_{q^-}$. Then $m = \widehat{Z} + \sum_{i=1}^{n} (\beta_i \widehat{X}_i + \alpha_i \widehat{Y}_i)$ with α_i, $\beta_i \in \mathbf{F}_2$ and $q^-(\sum_{i=1}^{n} (\alpha_i e_i + \beta_i f_i)) = 0$. Then, $\prod_{i=2}^{n} (1 + \widehat{E}_i)$ takes a nonzero value at m if and only if $\beta_i = 0$ for $i \geq 2$. This happens if and only if $\beta_1^2 + \alpha_1^2 + \beta_1 \alpha_1$ is zero, equivalently, if and only if $\alpha_1 = \beta_1 = 0$. So h vanishes on V_{q^-}.

Thus, $\mathrm{Ker}\varphi_n$ has the submodule M, of dimension 2^n. Since φ_n is surjective and its image $\mathrm{gr}(A_{q^-})$ has dimension $2^{n-1}(2^n - 1)$, it follows that $\mathrm{Ker}\varphi_n = M$. This proves (ii). The last statement follows from (i) for q^+ and from (ii) for q^-, since the submodule M corresponds to $\mathrm{Ker}\delta/\mathrm{Im}\delta$. $\square$

Theorem 7.2 and Proposition 6.1 yield filtrations of $\mathrm{gr}(A_{q^+})$ and $\mathrm{gr}(A_{q^-})$ by kG-modules which are the restrictions to G of Weyl modules of $\mathrm{Sp}(V_k)$ and their duals. The multiplicites of the factors are given by Corollary 6.3.

Of course, since the permutation modules are self-dual, we can obtain additional filtrations simply by dualizing the above filtrations.

8 Module structure for $\mathrm{O}^+(2n, 2)$ and $\mathrm{O}^-(2n, 2)$

The equation (5) shows that the restriction of the kG-module A_f to the orthogonal group $\mathrm{O}(V, f) \cong \mathrm{O}^{\pm}(2n, 2)$ is isomorphic to the permutation module on the set of zeroes of f in V. Our results therefore give us $k\mathrm{O}(V, f)$-module filtrations of this permutation module by Weyl modules and dual Weyl modules of the algebraic group $\mathrm{Sp}(V_k)$. In order for these filtrations to be useful in studying the $k\mathrm{O}(V, f)$-submodule structure of this permutation module, it is desirable to know the $k\mathrm{O}(V, f)$-submodule structure of the Weyl modules involved. We have already seen at the end of § 6 that the submodule lattices of these Weyl modules remain unchanged when the group action is restricted from $\mathrm{Sp}(V_k)$ to $\mathrm{Sp}(V)$. The aim of this section is to show that the same holds when we restrict to $\mathrm{O}(V, f)$.

In this section we will need to consider the noncentral infinitesimal isogenies $\tau : \mathrm{Sp}(V_k) \to \mathrm{Spin}(\widehat{V}_k)$ and $\sigma : \mathrm{Spin}(\widehat{V}_k) \to \mathrm{Sp}(V_k)$ such that the compositions $\tau \circ \sigma$ and $\sigma \circ \tau$ are the absolute Frobenius maps of the respective groups. These maps and their ramifications for the representation theory of the above and related groups have been studied in detail in [8].

Lemma 8.1 $H^1(\mathrm{Sp}(V_k), L(n)) = 0$.

Proof: The module $L(n)$ is the pullback along τ of the spin module for $\mathrm{Spin}(2n + 1, k)$, which we shall denote here by S. (In the notation of [8], the letter G is used for $\mathrm{Spin}(2\ell + 1, k)$ and the spin module is $L(\omega_\ell)$.) Then the two general reduction steps in [8, I.6.3] show that $H^1(\mathrm{Sp}(V_k), L(n)) \cong H^1(\mathrm{Spin}(\widehat{V}_k), S)$. Let G_σ be the (scheme-theoretic) kernel of σ. Then S is a simple injective module for G_σ and it follows immediately from the "inflation-restriction" sequence [8, I.6.2(3)] that $H^1(\mathrm{Spin}(\widehat{V}_k), S) = 0$. □

Proposition 8.2 *Each Weyl module $V(i)$ ($i = 1,\dots,n$) of the algebraic group $\mathrm{Sp}(V_k)$ satisfies the following properties. The restrictions of its composition factors to the subgroup $\mathrm{O}(V, f)$ remain simple and distinct as modules for this subgroup. Furthermore, the lattice of submodules remains the same.*

That is, the groups $\mathrm{Sp}(V_k)$*,* $\mathrm{O}(V,f)$ *leave invariant the same subspaces of each* $V(i)$*.*

Proof: The simple modules which appear as composition factors of the Weyl modules $V(i)$ are the trivial module and the modules $L(j)$. The highest weights of these modules are *restricted* weights (see [1, §1.]) for $\mathrm{Sp}(V_k)$ and so they are also restricted for the subgroup $\Omega(V_k,f)$, of type D_n, since they have the same maximal torus and the roots for $\Omega(V_k,f)$ are a subset of those for $\mathrm{Sp}(V_k,f)$. (Note however that ω_n is not 2-restricted for the covering group $\mathrm{Spin}(V_k,f)$.)

Moreover, for $r \neq n$ the weights ω_r are τ*-restricted*. (See [8, I.4.1].) This implies that the modules $L(r)$ ($r \neq n$) are simple for $\Omega(V_k,f)$. Then, by Steinberg's theorem, they remain simple upon restriction to the finite subgroup $\Omega(V,f)$.

The restriction of the module $L(n)$ to $\Omega(V_k)$ is isomorphic to the direct sum of two simple modules which as modules for the covering group $\mathrm{Spin}(V_k)$ are isomorphic to the first Frobenius twists of the half-spin modules. Thus, their highest weights are not 2-restricted with respect to $\mathrm{Spin}(V_k)$. However, these weights *are* 2-restricted with respect to $\Omega(V_k,f)$. Again by Steinberg's theorem, the two simple modules remain simple for the finite groups $\Omega(V,f)$, for which they are isomorphic to the sum of the two half-spin modules, Therefore, their direct sum is simple for the slightly larger groups $\mathrm{O}(V,f)$, since these groups have elements which interchange the two types of maximal isotropic subspaces in the orthogonal space V_k. Thus, we have proved the first property.

In order to prove that the submodule structure is unchanged when we restrict to the finite groups $\mathrm{O}(V,f)$, we will show that any nonsplit extension

$$0 \to L(i) \to E \to L(j) \to 0 \tag{25}$$

of simple $\mathrm{Sp}(V_k)$-modules with fundamental or zero highest weight remains nonsplit when restricted to $\mathrm{O}(V,f)$. By [12, II.2.12(1)], we can assume that $i \neq j$, and by Lemma 8.1 and duality that $\{i,j\} \neq \{0,n\}$. The latter assumption implies that the (scheme-theoretic) kernel $\tilde{G}_\tau$ of τ acts irreducibly and non-trivially on at least one of $L(i)$ and $L(j)$. This is because the modules $L(r)$ for $r \neq n$ are τ*-restricted*. Since all the simple modules are self-dual and since (25) splits if and only the dual sequence does, we may assume without loss that $L(j)$ is a nontrivial, simple module for $\tilde{G}_\tau$, and that

$\mathrm{Hom}_{\tilde{G}_\tau}(L(j), L(i)) = 0$. Then $\mathrm{Hom}_{\tilde{G}_\tau}(L(j), E)$ has dimension ≤ 1 and so is trivial for any perfect group which acts on it. This shows first that

$$\mathrm{Hom}_{\tilde{G}_\tau}(L(j), E) = \mathrm{Hom}_{\mathrm{Sp}(V_k)}(L(j), E) = 0. \tag{26}$$

Secondly, $\tilde{G}_\tau$ is also a normal subgroup of $\Omega(V_k, f)$. (See [8, I.3.1]; in the notation there $\Omega(V_k, f)$ is denoted by $\tilde{D}$.) So we have

$$\mathrm{Hom}_{\Omega(V_k,f)}(L(j), E) = \mathrm{Hom}_{\tilde{G}_\tau}(L(j), E) = 0, \tag{27}$$

which shows that the extension (25) does not split for $\Omega(V_k, f)$. Now, as explained above, the composition factors of E are 2-restricted modules for $\Omega(V_k, f)$, so by [1, Proposition 2.7] E remains nonsplit on restriction to the finite groups $\Omega(V, f)$. It follows that for the slightly large groups $\mathrm{O}(V, f)$, the module E is a nonsplit extension of two simple modules (even in the case $i = n$ when there are three composition factors for $\Omega(V, f)$). $\square$

Remark 8.3 The assertion in the proof of Proposition 8.2 that the nonsplit extension (25) of $\mathrm{Sp}(V_k)$ modules remains nonsplit for $\mathrm{O}(V, f)$ would be false if we were to allow the composition factors of E to be Frobenius twists of $L(i)$ and $L(j)$, even though their restrictions to $\mathrm{O}(V, f)$ are isomorphic to $L(i)$ and $L(j)$. For example, the $(2n+1)$-dimensional module $\widehat{V}_k$ (with $\mathrm{Sp}(V_k)$ acting via τ) is a nonsplit extension of the first Frobenius twist of $L(1)$ by $L(0)$, which clearly splits when restricted to $\mathrm{O}(V, f)$. The example just given may appear at first sight to contradict the assertion. After all, the trivial composition factor can also be viewed as a Frobenius twist of itself, suggesting that $\widehat{V}_k$ might be the Frobenius twist of an $\mathrm{Sp}(V_k)$-module E of the form in (25), contrary to the assertion. However, no such module really exists. The above observations are explained by the fact that while the first cohomology group of $\mathrm{Sp}(V_k)$ with values in V_k is trivial, the cohomology group with values in a Frobenius twist of V_k is nontrivial. This fact is a consequence of the exceptional non-vanishing [12, p. 371, Remark] of the first cohomology group of the Frobenius kernel of $\mathrm{Sp}(V_k)$ with values in k.

We end with some results about the socles of the Jordan-Steiner permutation modules and of the modules of functions on $\mathbb{F}_2$-rational points of quadrics. We recall that the socle, $\mathrm{soc}M$, of a module M is the maximal semisimple submodule and that the radical, rad$M$, is the intersection of all maximal submodules (whence the maximal semisimple quotient is $M/\mathrm{rad}M$). We also use the notation $(\mathrm{soc}^2/\mathrm{soc})M$ and $(\mathrm{rad}/\mathrm{rad}^2)M$ for the socle of $M/\mathrm{soc}M$ and $\mathrm{rad}M/\mathrm{rad}(\mathrm{rad}M)$ respectively.

Proposition 8.4 *Let $k^{\mathcal{Q}_f}$ denote the kG-permutation module on a Jordan-Steiner set and let $k^{\mathcal{Z}_f}$ denote the $k\mathrm{O}(V,f)$-module of functions on the set of zeroes of f in V.*

(i) $\mathrm{soc}(k^{\mathcal{Q}_f}) \cong k \cong k^{\mathcal{Q}_f}/\mathrm{rad}(k^{\mathcal{Q}_f})$. *We have* $\mathrm{soc}(k^{\mathcal{Q}_f}) \subseteq \mathrm{rad}(k^{\mathcal{Q}_f})$.

(ii) $(\mathrm{soc}^2/\mathrm{soc})(k^{\mathcal{Q}_f}) \cong V \cong (\mathrm{rad}/\mathrm{rad}^2)(k^{\mathcal{Q}_f})$.

(iii) $\mathrm{soc}(k^{\mathcal{Z}_f}) \cong k \oplus k \oplus V \cong k^{\mathcal{Z}_f}/\mathrm{rad}(k^{\mathcal{Z}_f})$.

Proof: By Frobenius reciprocity, the simple modules which appear as quotients of $k^{\mathcal{Q}_f}$ are those which have fixed points for $k\mathrm{O}(V,q)$. By Proposition 8.2 only the trivial module has this property. Since $k^{\mathcal{Q}_f}$ is a transitive permutation module, there is a unique trivial submodule. By the self-duality of permutation modules, the isomorphisms in (i) are proved. The last statement of (i) is merely the fact that the cardinality of the Jordan-Steiner set is even.

Next we consider (iii). Now $k^{\mathcal{Z}_f}$ is isomorphic to the direct sum of the functions supported at the origin and the permutation module on the one-dimensional isotropic subspaces. By Frobenius reciprocity and self-duality, it is enough to show that the stabilizer of a one-dimensional isotropic subspace of V fixes no other vectors in V or in any other nontrivial simple module. This is well known and can be seen by explicit computation, or by [5, Theorem 6.13]. The latter implies that a transitive permutation module for a finite group of Lie type with maximal parabolic subgroups as stabilizers has a socle which is the direct sum of the trivial module and one nontrivial simple module. It remains to prove (ii). We have already seen (5) that the restriction of $k^{\mathcal{Q}_f}$ to $\mathrm{O}(V,f)$ is isomorphic to $k^{\mathcal{Z}_f}$. Consider $\mathrm{rad}(k^{\mathcal{Q}_f})/\mathrm{soc}(k^{\mathcal{Q}_f})$. By (i) and (iii), its head and socle are isomorphic to V as $k\mathrm{O}(V,f)$-modules, hence also as kG-modules, since by Proposition 8.2 we know that distinct simple kG-modules restrict to distinct simple $k\mathrm{O}(V,f)$-modules. $\square$

9 Examples

To illustrate our results, we work out the case $n = 3$. In this case the dimensions of the simple modules $L(r)$ for $r = 0$, 1, 2 and 3 are respectively 1, 6, 14 and 8.

Figure 1 shows the graded modules and in this small case, Propsition 8.4 is sufficient to recover the submodule structures of the Jordan-Steiner permutation modules. Figure 2 shows the submodule structures.

$\wedge(V)$:

$$1 \oplus 6 \oplus (1 \oplus 14) \oplus \begin{array}{c} 6 \\ | \\ 8 \\ | \\ 6 \end{array} \oplus (14 \oplus 1) \oplus 6 \oplus 1$$

Cokerδ :

$$1 \oplus 6 \oplus 14 \oplus \begin{array}{c} 6 \\ \Big| \\ 8 \end{array} \oplus 1$$

Figure 1:

$k^{\mathcal{Q}^+}$:

$$\begin{array}{ccc} & 1 & \\ & | & \\ & 6 & \\ / & & \backslash \\ 8 & & 14 \\ \backslash & & / \\ & 6 & \\ & | & \\ & 1 & \end{array}$$

$k^{\mathcal{Q}^-}$:

$$\begin{array}{c} 1 \\ | \\ 6 \\ | \\ 14 \\ | \\ 6 \\ | \\ 1 \end{array}$$

Figure 2: Submodule structures for $\mathrm{Sp}(6, 2)$.

For $n > 3$, our results are not strong enough to recover the submodule structure from the structure of the graded modules. Figure 3 shows the structure of Coker$\delta \cong \mathrm{gr}(k^{\mathcal{Q}^+})$ for $n = 4$ and $n = 5$. Again, the composition factors are indicated by their dimensions.

References

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$n = 4:$

$$1 \oplus 8 \oplus \begin{array}{c} 1 \\ | \\ 26 \end{array} \oplus 48 \oplus \begin{array}{ccc} & 26 & \\ \diagup & & \diagdown \\ 1 & & 16 \end{array} \oplus 8 \oplus 1$$

$n = 5:$

$$1 \oplus 10 \oplus 44 \oplus \begin{array}{c} 10 \\ | \\ 100 \end{array} \oplus \begin{array}{c} 1 \\ | \\ 164 \\ | \\ 1 \end{array} \oplus \begin{array}{ccc} & 100 & \\ \diagup & & \diagdown \\ 10 & & 32 \end{array} \oplus 44 \oplus 10 \oplus 1$$

Figure 3: Submodule structure of $\mathrm{gr}(k^{\mathcal{Q}^+})$ for $n = 4, 5$.

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