THE PERMUTATION REPRESENTATION OF $\mathbf{Sp}(2m, \mathbb{F}_p)$ ACTING ON THE VECTORS OF ITS STANDARD MODULE.

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Abstract. This paper studies the permutation representation of a finite symplectic group over a prime field of odd characteristic on the vectors of its standard module. The submodule lattice of this permutation module is determined. The results yield additive formulae for the p-ranks of various incidence matrices arising from the finite symplectic spaces.

Introduction

In this paper, we study the action of the symplectic group $\mathrm{Sp}(2m, p)$ on the set of vectors in its standard module. The composition factors of this permutation module have been known for some time ([9], [12], [13]) and so the problem we address here is that of describing the submodule lattice. This turns out to be quite similar to the known structure of this module under the action of the general linear group (See [3] and references cited there). This structural information yields additive formulae for the p-ranks of the incidence matrices between points and isotropic subspaces of fixed dimension in $(2m-1)$-dimensional projective space over $\mathbb{F}_p$. This generalizes recent work [5] of de Caen and Moorhouse, who worked out the p-rank of the point-line incidence when $m = 2$.

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§1. Functions on a finite vector space

1.1. Let p be an odd prime and let V be a $2m$-dimensional $\mathbb{F}_p$-vector space with a nonsingular alternating bilinear form $\langle -, - \rangle$. We shall assume $m \geq 2$ to avoid trivial exceptions. We fix a symplectic basis $e_1, \ldots e_m, f_m, \ldots f_1$ and corresponding coordinates $X_1, \ldots, X_m, Y_m, \ldots Y_1$ so that $\langle e_i, f_j \rangle = \delta_{ij}$.

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Let k be an algebraic closure of $\mathbb{F}_p$ and let

$$A = k[X_1, \dots , X_m, Y_m, \dots , Y_1]/(X_i{}^p - X_i, Y_i{}^p - Y_i)_{i=1}^m \tag{1}$$

be the ring of functions on V. This is the principal object of our study.

1.2. Structure of A as a $k\,\mathbf{GL}(V)$-module. The the action of $\mathrm{GL}(V)$ on A is induced from its action on the polynomial ring $k[X_1, \dots , X_m, Y_m, \dots , Y_1]$ through linear substitutions of the variables. The $k\,\mathrm{GL}(V)$-module structure of A is well known; we will give a brief description. When we factor out by the inhomogeneous ideal $(X_i{}^p - X_i, Y_i{}^p - Y_i)_{i=1}^m$ the grading on $k[X_1, \dots X_m, Y_m, \dots , Y_1]$ is destroyed, leaving only a filtration $\{F_e\}_{e=0}^{2m(p-1)}$, where

$$F_e = \text{ Image in } A \text{ of polynomials of degree} \leq e, \tag{2}$$

and a $\mathbb{Z}/(p-1)\mathbb{Z}$-grading (from the action of the scalar matrices)

$$A = \oplus_{d=0}^{p-2} A[d], \tag{3}$$

where $A[d]$ is the image of all homogeneous polynomials of degree congruent to d modulo $p-1$.

Denote by $S(e)$ the component of degree e in the graded ring

$$S = k[X_1, \dots , X_m, Y_1, \dots , Y_m]/(X_i{}^p, Y_i{}^p)_{i=1}^m. \tag{4}$$

Here e ranges from 0 to $2m(p-1)$. The dimension of $S(e)$ is

$$s(e) = \sum_{i=0}^{\lfloor \frac{e}{p} \rfloor} (-1)^i \binom{2m}{i} \binom{2m-1+e-ip}{2m-1}. \tag{5}$$

The modules $S(e)$ are simple modules for $\mathrm{GL}(V)$ and since the graded module of A associated with the filtration $\{F_e\}$ is isomorphic to S, this filtration is in fact a composition series of A. The filtration $\{F_e\}$ also induces composition series on each direct summand $A[d]$. We have $F_e \cap A[d] = F_{e-1} \cap A[d]$ unless $e \equiv d \bmod p-1$, so we set $A[d]_t = F_{d+t(p-1)} \cap A[d]$ (and $A[d]_{-1} = \{0\}$). Then for $[d] \neq [0]$

$$\{0\} \subset A[d]_0 \subset A[d]_1 \subset \cdots \subset A[d]_{2m-1} \tag{6}$$

is a composition series of $A[d]$, with

$$A[d]_t/A[d]_{t-1} \cong S(d + t(p-1)); \tag{7}$$

the same holds for $A[0]$ except that the series has one extra term $A[0]_{2m}$.

The following has been known for a long time in one form or other, e.g. [7], [10]. A generalization to all finite fields and further references can be found in [3].

Lemma 1.

(a) *For $d \neq 0$, $A[d]$ is a uniserial module of dimension $\frac{p^{2n}-1}{p-1}$.*

(b) *In $A[0]$ the top and bottom factors $A[0]_0 \cong k$ and $A[0]_{2m}/A[0]_{2m-1} \cong k$ split off and*

$$A[0] \cong k \oplus k \oplus M, \tag{8}$$

where $M \cong A[0]_{2m-1}/A[0]_0$ is uniserial of dimension $\frac{p^{2n}-1}{p-1} - 1$.

Since we will be looking closely at the module M, we let M_r be the image in M of $A[0]_r$ for $1 \leq r \leq 2m-1$. (This is the subspace of M generated by images of polynomials of degree $\leq r(p-1)$.) Then the series

$$\{0\} \subset M_1 \subset \cdots \subset M_{2m-1} = M \tag{9}$$

is the unique composition series of M as a $k\,\mathrm{GL}(V)$-module with

$$M_t/M_{t-1} \cong S(t(p-1)), \qquad 1 \leq t \leq 2m-1. \tag{10}$$

(By convention we set $M_0 = \{0\}$.)

The submodule lattice for $\mathrm{SL}(V)$ is identical, since this subgroup has index prime to p and all composition factors remain simple when restricted to $\mathrm{SL}(V)$.

1.3. Structure of A as a $k\,\mathrm{Sp}(V)$-module. The $k\,\mathrm{Sp}(V)$-composition factors of A, or, what amounts to the same thing, of the modules $S(e)$, were given independently in [9] and [12].

Lemma 2. *The modules $S(e)$ all remain simple for $Sp(V)$ with the single exception of the middle degree $e = m(p-1)$, where $S(m(p-1))$ is the direct sum of two simple modules $S^+(m(p-1))$ and $S^-(m(p-1))$ of dimensions $\frac{1}{2}(s(m(p-1)+p^m))$ and $\frac{1}{2}(s(m(p-1)-p^m))$ respectively.*

More information can be found in [9] an [12]. Since $S(e)$ and $S(2m(p-1)-e)$ are dual as $k\,\mathrm{GL}(V)$-modules they are isomorphic for $\mathrm{Sp}(V)$.

We are now ready to state our main result concerning the $k\,\mathrm{Sp}(V)$-submodule lattice of A. Taking into account the decompositions (3) and (8), it suffices to describe the submodule lattices of $A[d]$, $[d] \neq [0]$ and of the nontrivial summand M of $A[0]$. Since the simple modules in the layers $A[d]_r/A[d]_{r-1}$ and M_r/M_{r-1} have already been described above, we will not repeat that information here.

Theorem 1.

(a) *For $[d] \neq [0]$ the module $A[d]$ is uniserial.*

(b) *The modules M_t form the socle (and radical) filtration of M. The quotients $M_t/M_{t-1} \cong$ are simple except that M_m/M_{m-1} is the direct sum of two simple modules.*

Remarks.(1) In other words, the theorem says that the socle and radical series of A are the same for $\mathrm{GL}(V)$ and $\mathrm{Sp}(V)$, with all composition factors remaining irreducible, except for one which splits into two.
(2) Pictorially, M has the following structure.

$$\begin{gathered}
S((2m-1)(p-1)) \\
S((2m-2)(p-1)) \\
\vdots \\
S((m-1)(p-1)) \\
S^{+}(m(p-1)) \oplus S^{-}(m(p-1)) \\
S((m-1)(p-1)) \\
\vdots \\
S(2(p-1)) \\
S(p-1).
\end{gathered}$$

(3) As has already been mentioned, we have $S((2m-r)(p-1)) \cong S(r(p-1))$, $(1 \leq r \leq m)$.

1.4. The permutation module on projective space and p-rank problems.

Now $A[0]$ is the subspace of functions on V which are unchanged by scalar multiplication of the coordinates, it may be considered as the space of functions on the disjoint union of the projective space $\mathbb{P}(V)$ and the zero subspace of V. Thus, the permutation module $k[\mathbb{P}(V)]$ on $\mathbb{P}(V)$ is isomorphic to $k \oplus M$

The structure of the permutation module $k[\mathbb{P}(V)]$ follows immediately from part (b) of the theorem.

In finite projective geometry, the incidence relations between objects of two types is encoded in an incidence matrix, with rows labeled by the objects of one type and columns by those of the second and entries 1 or 0 according to whether or not the corresponding row and column labels are incident. It is natural to ask about the rank of this matrix, over any field. When the geometry arises from a field of characteristic p, then one may be interested in the rank over a field of characteristic p, or p-rank. An important geometry of this kind is the geometry of a symplectic vector space V in characteristic p and a typical problem is to determine the p-rank of the incidence between points and isotropic r-dimensional linear subspaces (isotropic r-flat for short) of the projective space $\mathbb{P}(V)$. When $2m = 4$ and $r = 1$, the geometry of points and isotropic lines is an example of a *generalized quadrangle* In this case, the p-rank has been found recently by de Caen and Moorhouse [5].

In general, the rank is equal to the dimension of the $k\,\mathrm{Sp}(V)$-submodule of $k[\mathbb{P}(V)]$ generated by the characteristic function of a fixed r-flat. This submodule can be found without much trouble thanks to part (b) of the theorem, yielding the following numerical result.

Theorem 2. *The p-rank of the incidence matrix between points and isotropic r-flats ($r = 0, \ldots, 2m-1$) of $\mathbb{P}(V)$ is equal to*

$$\begin{cases} 1 + \sum_{i=1}^{2m-1-r} s(i(p-1)) & \text{for } r \neq m-1, \\ 1 + \sum_{i=1}^{m} s(i(p-1)) + \frac{1}{2}(s(m(p-1)) + p^m) & \text{for } r = m-1. \end{cases}$$

(The numbers $s(e)$ were defined in (5) above.)

The formula for $r \neq m-1$ agrees with Hamada's formula [7] for the p-rank of the incidence between points and all r-flats This shows that the code generated by isotropic r-flats is equal to the code generated by all r-flats, except when $r = m-1$.

§2. Technical preliminaries

2.1. Characters of the diagonal subgroup. The diagonal subgroup H of the symplectic group $\mathrm{Sp}(V)$ consists of all matrices of the form $\mathrm{diag}(\mu_1, \ldots, \mu_m, \mu_m^{-1} \ldots, \mu_1^{-1})$ where the μ_i are nonzero elements of $\mathbb{F}_p$. Let δ_i, $1 \leq i \leq m$, be the map which sends such a matrix to its diagonal entry μ_i^{-1}. The maps δ_i generate the group of characters of H and each character is uniquely expressible as a product $\prod_{i=1}^m \delta_i^{c_i}$, with $c_i \in \mathbb{Z}/(p-1)\mathbb{Z}$.

Each monomial $\prod_{i=1}^m X_i^{a_i} Y_i^{b_i}$ (or its image in A) is a simultaneous eigenvector for H, affording the character $\prod_{i=1}^m \delta_i^{[a_i - b_i]}$, where $[a]$ denotes the congruence class modulo $p-1$ of the integer a.

2.2. Induced modules. Let $G = \mathrm{Sp}(V)$. For a subgroup X of G and a kX-module L, Let $ind_X^G L$ denote the induced kG-module $kG \otimes_{kX} L$. If $L = k$ then the induced module is just the permutation module of G acting on the left cosets of X. In this subsection and the next, we collect together some facts about modules induced from one-dimensional modules of stablizers of flags of isotropic subspaces. These subgroups are parabolic subgroups and so our statements are really special cases of the general theory of such modules developed in [4].

Let $G_r \leq G$ denote the stabilizer of the r-dimensional isotropic subspace $W_r = \langle e_1, \ldots, e_r \rangle$ of V.

Lemma 3. *There exists a unique nontrivial simple kG-module $L(r, k)$ which contains a trivial kG_r-submodule. Moreover different choices of r give nonisomorphic modules.*

Proof. This is a standard fact in the theory of representations of Chevalley groups [11], so we will give a brief summary of the relevant facts in lieu of a proof. Let ω_r be the r-th fundamental weight in the usual order where ω_1 is the highest weight of V. We view ω_r as a character of the diagonal subgroup of the algebraic group $\mathrm{Sp}(V \otimes_{\mathbb{F}_p} k)$. The simple modules for $\mathrm{Sp}(V \otimes_{\mathbb{F}_p} k)$ are parametrized by their highest weights. Each nonnegative integral combination $\sum n_j \omega_j$ is the highest weight of a unique simple module. The modules whose highest weights are one of the p^m combinations with $0 \leq n_j \leq p-1$, remain simple on restriction to $\mathrm{Sp}(V)$ and form a full set of simple $k\,\mathrm{Sp}(V)$-modules. The simple modules on which G_r leaves a line invariant are precisely those with highest weight a multiple of ω_r and the stable line is the highest weight space. Thus, the ω_r weight space in the module with highest weight ω_r is a one-dimensional kG_r-module. This representation generates the group (under tensor product) of one-dimensional representations of G_r. Thus, the high weight spaces in the modules of highest weight $n\omega_r$, $0 \leq n \leq p-1$ give all $p-1$ one dimensional representations of G_r, with the trivial representation occuring for the two values $n=0$ and $n=p-1$. So the module $L(r,k)$ is the simple module with highest weight $(p-1)\omega_r$.

Lemma 4.

$$\mathrm{ind}_{G_r}^{G} k \cong k \oplus M(r,k) \tag{11}$$

where $M(r,k)$ has a unique simple submodule and a unique simple quotient. Both submodule and quotient are isomorphic to $L(r,k)$.

Proof. These statements follow from Lemma 3 by applying Frobenius reciprocity and self-duality of $\mathrm{ind}_{G_r}^{G} k$, using the fact that all simple kG-modules are isomorphic to their duals.

To relate this to our earlier notation, we observe that $\mathrm{ind}_{G_r}^{G} k$ is the permutation module on the set of r-dimensional subspaces of V, so in particular $M(1,k) = M$, the nontrivial summand of $k[\mathbb{P}(V)]$. The simple modules $L(r,k)$ can also be recognized; comparison of highest weights shows that $L(r,k) \cong S(r(p-1))$, for $r \neq m$ and $L(m,k) \cong S^{+}(m(p-1))$.

2.3. Incidence maps. For any subset X of V we denote by $\chi_X \in A$ its characteristic function. If X happens to be a homogeneous subset, such as a linear subspace, we will use the same notation for the corresponding characteristic function on $\mathbb{P}(V)$. Thus, we may regard characteristic funtions of subspaces as elements of the permutation module $\mathrm{ind}_{G_1}^{G}(k) = k[\mathbb{P}(V)]$.

For each $r = 1, \ldots, m$, we have incidence maps from the permutation module on isotropic r-subspaces to the permutation module on projective space given by

$$\alpha_r : \mathrm{ind}_{G_r}^{G}(k) \to k[\mathbb{P}(V)], \qquad W \mapsto \chi_W \tag{12}$$

and

$$\beta_r : \mathrm{ind}_{G_r}^{G}(k) \to k[\mathbb{P}(V)], \qquad W \mapsto \chi_{W^\perp}. \tag{13}$$

Of course, $\alpha_m = \beta_m$. These are maps of kG-modules.

All the modules $\mathrm{ind}_{G_r}^{G}(k)$ have a trivial summand (the constant functions) and a more interesting summand $M(r,k)$. Moreover it is easily checked that the incidence maps map the constant functions onto the constant functions. Therefore, it is technically more convenient to work modulo the trivial summands and consider the maps

$$M(r,k) \to \mathrm{ind}_{G_r}^G(k) \xrightarrow{\alpha_r,\beta_r} \mathrm{ind}_{G_1}^G(k) \to M(1,k) = M, \tag{14}$$

induced by restriction and projection. We shall call these maps $\overline{\alpha}_r$ and $\overline{\beta}_r$ and refer to them also as incidence maps.

Let $\mathcal{C}_{r-1}$ the image of α_r and let $\mathcal{C}_{2m-r-1}$ be the image of β_r. (The subscripts are the projective dimensions of the supports of the characteristic functions.) The images of $\overline{\alpha}_r$ and $\overline{\beta}_r$ will be denoted by $\overline{\mathcal{C}}_{r-1}$ and $\overline{\mathcal{C}}_{2m-r-1}$ respectively. Thus

$$\mathcal{C}_t \cong k \oplus \overline{\mathcal{C}}_t \tag{15}$$

for all $t = 0, \ldots, 2m-2$. An important property of the module $\overline{\mathcal{C}}_{r-1}$ is that, being a homomorphic image of $M(r,k)$, it has a unique maximal submodule.

Lemma 5. *For $0 \le t < t' \le 2m-1$, we have $\mathcal{C}_t' \subsetneq \mathcal{C}_t$ and $\overline{\mathcal{C}}_t' \subsetneq \overline{\mathcal{C}}_t$.*

Proof. We may assume $t' = t+1$. If $0 \le r \le m-1$, $\mathcal{C}_r$ is spanned by (characteristic functions of) isotropic r-flats, while if $m \le r \le 2m-2$, it is generated by the orthogonal complements of isotropic $2m-2-r$-flats. These will be the only two types of flats we will consider in this proof. Let W be a fixed $t+1$-flat and P a point of W. Then it is a simple count to check that the number of t-flats in W which contain P is congruent to 1 mod p. Thus the sum of the characteristic functions of those t-flats is equal to the characteristic function of W. This proves that $\mathcal{C}_{t+1} \subseteq \mathcal{C}_t$ and strict containment follows from the fact that the unique simple quotients of $\mathcal{C}_t$ and $\mathcal{C}_{t+1}$ are not isomorphic, by the last assertion of Lemma 3. The lemma is proved.

In the proof of Theorem 1(a) will also want to consider maps from induced modules into the modules $A[d]$. The relevant parabolic subgroups are the stabilizers $G_{r-1,r} = G_r \cap G_{r-1}$ of the flags $W_{r-1} \subset W_r$.

Let λ be a one-dimensional representation of $G_{r-1,r}$. By Frobenius reciprocity,

$$\mathrm{Hom}_G(\mathrm{ind}_{G_{r-1,r}}^G \lambda, A[d]) \cong \mathrm{Hom}_{G_{r,r-1}}(\lambda, A[d]). \tag{16}$$

Thus, such homomorphisms exist precisely when $A[d]$ has a one-dimensional $kG_{r-1,r}$-submodule isomorphic to λ and in this case, the image of the kG-homomorphism is generated by this one-dimensional space.

Now $G_{r-1,r}$ acts on W_r leaving the hyperplane W_{r-1} invariant. Hence in its action on the dual of W_r, $G_{r-1,r}$ stablizes the one-dimensional subspace spanned by the image of X_r. We denote this one-dimensional representation by λ_r

Consider the function $\chi_{W_r} X_r^d$ for $0 \leq d \leq p-2$. Its image in A lies in $A[d]$ (since $\chi_{W_r} \in A[0]$). Moreover, the span of this function is $G_{r-1,r}$-stable and affords the representation λ_r^d.

In this way, we have for each $r = 1, \ldots, m$, and $0 \leq d \leq p-2$ we have homomorphisms

$$\alpha_r[d] : \mathrm{ind}_{G_{r-1,r}}^G \lambda_r^d \to A[d], \tag{17}$$

with image generated by $\chi_{W_r} X_r^d$.

Now the one-dimensional $kG_{r-1,r}$-module $W_{r-1}^\perp / W_r^\perp$ is spanned by the image of Y_r, and isomorphic to the dual λ_r^* of λ_r. It follows that for $0 \leq d \leq p-2$ the function $\chi_{W_{r-1}^\perp} Y_r^d$ spans a $kG_{r-1,r}$-submodule of $A[d]$ isomorphic to λ_r^{*d} and generates the image of a kG-module homomorphism

$$\beta_r[d] : \mathrm{ind}_{G_{r-1,r}}^G \lambda_r^{*d} \to A[d], \tag{18}$$

Note that for $d = 0$, the image is $\mathcal{C}_r$. In fact, one can show that $\alpha_r[0]$ (resp. $\beta_r[0]$) is the composite of α_r (resp. β_r) with the natural projection $\mathrm{ind}_{G_{r-1,r}}^G k \to \mathrm{ind}_{G_r}^G k$ sending a flag to its r-dimensional (resp. $(n-r)$-dimensional) member. In this sense, we have generalized the incidence maps above.

The main property we shall need is the following [4, Thm 6.13].

Lemma 6. *For $d \neq 0$ the modules $\mathrm{ind}_{G_{r-1,r}}^G \lambda_r^d$ and $\mathrm{ind}_{G_{r-1,r}}^G \lambda_r^{*d}$ have unique maximal submodules.*

Thus, the images of the maps $\alpha_r[d]$ and $\beta_r[d]$ provide $2m$ submodules of $A[d]$, each with a unique maximal submodule. These submodules will be important in our proof of Theorem 1(a) because, as we already know, $A[d]$ has exactly $2m$ composition factors.

§3. Proofs of Theorems

The following lemmas will be combined to prove Theorem 1(b) and Theorem 2.

The characteristic function of the linear subspace W defined by the vanishing of coordinates X_i, $i \in I$ and Y_j, $j \in J$ is given by

$$\chi_W = \prod_{j \in J} (1 - Y_j^{p-1}) \prod_{i \in I} (1 - X_i^{p-1}) \tag{19}$$

Therefore, the characteristic function of an $(r-1)$-flat of $\mathbb{P}(V)$ is the image of a polynomial of degree $2m - r$, making it clear that $\overline{\mathcal{C}}_t \subseteq M_{2m-1-t}$ for $1 \leq t \leq 2m-1$.

Lemma 7. *For $t \leq m-1$ $\overline{\mathcal{C}}_{2m-1-t} = M_t$*

Proof. It is immediate from Lemma 5 and (10) that the submodule $\overline{\mathcal{C}}_{2m-1-t}$ of M_t has at least as many composition factors as M_t itself, so the two must be equal.

Lemma 8. *M_{m-1} and M/M_m are uniserial modules.*

Proof. Since the modules $\overline{\mathcal{C}}_r$ are images of the modules $M(r,k)$, they have unique maximal submodules. Therefore, by Lemma 7, each of the modules in the series $M_1 \subset M_2 \subset \cdots M_{m-1}$ has a unique maximal submodule. The uniseriality of M/M_m follows by duality.

In view of Lemma 8, we can focus attention on the subquotient M_{m+1}/M_{m-1}. In particular, we wish to determine the submodules $\overline{\mathcal{C}}_{m-1}/M_{m-1} \leq M_m/M_{m-1}$ and $\overline{\mathcal{C}}_{m-2}/M_{m-1} \leq M_{m+1}/M_{m-1}$

Lemma 9. $\overline{\mathcal{C}}_{m-1}/M_{m-1} \cong S^+(m(p-1))$.

Proof. By definition, $\overline{\mathcal{C}}_{m-1}/M_{m-1}$ is the submodule of $M_m/M_{m-1} \cong S^+(m(p-1)) \oplus S^-(m(p-1))$ generated by the image of $\chi_{1,\dots,m}$. But modulo M_{m-1} we have $\chi_{1,\dots,m} = (-1)^m X_1^{p-1} \cdots X_m^{p-1}$, by (19). This monomial is in the $S^+(m(p-1))$ summand of M_m/M_{m-1} (where it is in fact the highest weight vector).

We now come to the key lemma.

Lemma 10. $\overline{\mathcal{C}}_{m-2} = M_{m+1}$

Proof. We begin by examining the case $m = 2$. In this case $\mathcal{C}_{m-2} = k^P$ so the result is trivial. For $m > 2$ we shall argue by induction. Let v^+ denote the monomial $X_1^{p-1} \cdots X_m^{p-1}$ and let v^- be the binomial $X_1^{p-1} \cdots X_{m-2}^{p-1} X_{m-1}^{p-2} X_m^{p-1} Y_m - X_1^{p-1} \cdots X_{m-2}^{p-1} X_{m-1}^{p-1} X_m^{p-2} Y_{m-1}$.

The results of [9] and [12] show that the images in $M_m/M_{m-1} \cong S^+((m-1)(p-1)) \oplus S^-((m-1)(p-1))$ of v^+ and v^- lie in the $S^+((m-1)(p-1))$ and $S^+((m-1)(p-1))$ components respectively.

$\overline{\mathcal{C}}_{m-2}$ is the $k\,\mathrm{Sp}(V)$-submodule of M generated by the image of the characteristic function of the m-plane where $X_1 = \cdots = X_m = Y_m = 0$. This function is given by

$$\chi_{1,\dots,m,m} = (1 - Y_m^{p-1}) \prod_{i=1}^{m} (1 - X_i^{p-1}) \tag{20}$$

The image of this function in M/M_{m-1} is

$$\overline{\chi}_{1,\dots,m,m} = (-1)^{m+1} (X_1^{p-1} \dots X_m^{p-1} Y_m^{p-1} - X_1^{p-1} \dots X_m^{p-1} - \sum_{|I|=m-1} X_I^{p-1} Y_m^{p-1}), \tag{21}$$

where X_I denotes the product of the variables X_i with $i \in I$. Let $\mathbf{m} = \{1, \dots, m\}$

Applying the symplectic transvection $X_1 \mapsto X_1 + \eta Y_1$ to $(-1)^{m+1}\overline{\chi}_{1,\dots,m,m}$ yields

$$\left(\sum_{r=0}^{p-1} \eta^r \binom{p-1}{r} X_1^{p-1-r} Y_1^r\right) \times \left(X_{\mathbf{m}\setminus\{1\}}^{p-1} Y_m^{p-1} - X_2^{p-1} \cdots X_m^{p-1} - \sum_{\substack{1 \notin J \subseteq \mathbf{m} \\ |J| = m-2}} X_J^{p-1} Y_m^{p-1} \right) - X_{\mathbf{m}\setminus\{1\}}^{p-1} Y_m^{p-1}. \tag{22}$$

The $\delta_1^{[-2]}$, component is

$$\left(\eta(p-1)X_1^{p-2}Y_1 + \eta^{\frac{p+1}{2}} \binom{p-1}{\frac{p+1}{2}} X_1^{\frac{p-3}{2}} Y_1^{\frac{p+1}{2}}\right) \times \left(X_{\mathbf{m}\setminus\{1\}}^{p-1} Y_m^{p-1} - X_2^{p-1} \cdots X_m^{p-1} - \sum_{\substack{1 \notin J \subseteq \mathbf{m} \\ |J| = m-2}} X_J^{p-1} Y_m^{p-1} \right) \tag{23}$$

By subtracting this expression for $\eta = 1$ from the corresponding expression when η is a primitive p-th root of unity (so that $\eta^{\frac{p-1}{2}} = -1$) after first dividing the latter by $-\eta$, we see that the $k\,\mathrm{Sp}(V)$-module generated by $\overline{\chi}_{1,\dots,m,m}$ contains the element

$$X_1^{p-2}Y_1 \left(X_{\mathbf{m}\setminus\{1\}}^{p-1} Y_m^{p-1} - X_2^{p-1} \cdots X_m^{p-1} - \sum_{\substack{1 \notin J \subseteq \mathbf{m} \\ |J| = m-2}} X_J^{p-1} Y_m^{p-1} \right) \tag{24}$$

On applying the symplectic transvection $Y_1 \mapsto Y_1 + X_1$ and taking fixed points of H, we see that the module in question contains

$$X_1^{p-1} \left(X_{\mathbf{m}\setminus\{1\}}^{p-1} Y_m^{p-1} - X_2^{p-1} \cdots X_m^{p-1} - \sum_{\substack{1 \notin J \subseteq \mathbf{m} \\ |J| = m-2}} X_J^{p-1} Y_m^{p-1} \right). \tag{25}$$

Write $V = V_1 \oplus V_{m-1}$ where V_1 is the 2-dimensional hyperbolic space where $X_i = Y_i = 0$ for $i > 1$ and V_{m-1} is its orthogonal complement. Inside $\mathrm{Sp}(V)$ is the subgroup $\mathrm{Sp}(V_1) \times$

$\mathrm{Sp}(V_{m-1})$, preserving this decomposition. With respect to this group, we have a tensor product decomposition

$$A = A_1 \otimes A_{m-1}, \tag{26}$$

with $A_{m-1} = k[X_2, \dots X_m, Y_m, \dots, Y_2]/(X_i^p - X_i, Y_i^p - Y_i)_{i=2}^m$. Then (25) is equal to $X_1^{p-1}\overline{\chi}_{2,\dots,m,m}$, where $\chi_{2,\dots,m,m} \in A_{m-1}$ is the characteristic function of an $m-2$ dimensional subspace of V_{m-1}. By induction, we know that the images in A_{m-1} of the monomial $v_{m-1}^+ = X_{\mathbf{m}\setminus\{1\}}^{p-1}$ and the binomial

$$v_{m-1}^- = X_{\mathbf{m}\setminus\{1,m-1,m\}}^{p-1} X_{m-1}^{p-2} X_m^{p-1} Y_m - X_{\mathbf{m}\setminus\{1,m-1,m\}}^{p-1} X_{m-1}^{p-1} X_m^{p-2} Y_{m-1} \tag{27}$$

lie in the $k\,\mathrm{Sp}(V_{m-1})$-module generated by $\overline{\chi}_{2,\dots,m,m}$, modulo terms of degree less than $(m-2)(p-1)$.

Therefore, in A, the images of $v^+ = X_1^{p-1} v_{m-1}^+$ and $v^- = X_1^{p-1} v_{m-1}^-$, lie in the $k\,\mathrm{Sp}(V_{m-1})$-submodule generated by $X_1^{p-1}\overline{\chi}_{2,\dots,m,m}$, modulo terms of degree less than $(m-1)(p-1)$. Therefore, we have proved that the images of both v^+ and v^- lie in the $k\,\mathrm{Sp}(V(p))$-submodule of M_{m+1} generated by $\overline{\chi}_{1,\dots,m,m}$, which shows that both composition factors of $S(m(p-1))$ are composition factors of $\overline{\mathcal{C}}_{m-2}$ and hence proves that $\overline{\mathcal{C}}_{m-2} \supseteq M_m$. Since the characteristic function of an isotropic $(m-2)$-flat lies in $\overline{\mathcal{C}}_{m-2}$ and has nonzero image in the simple module M_{m+1}/M_m, it follows that $\overline{\mathcal{C}}_{m-2} = M_{m+1}$.

Proof of Theorem 1(b).

It follows from Lemma 10 that M_m is the unique maximal submodule of M_{m+1} and by duality that M_{m-1}/M_{m-2} is the unique simple submodule of M/M_{m-2}. The submodule lattice of M is determined.

Proof of Theorem 2. The p-rank for points versus r-flats is the dimension of $\mathcal{C}_r$. From (15) and Lemmas 8, 9 and 10 we obtain the composition factors and from (5) we have the dimension of each composition factor.

Proof of Theorem 1(a). Finally, we turn to the proof of Theorem 1 (a).

First, we note that the image of $\alpha_r[d]$ lies in $A[d]_{2m-r}$ and has nonzero image mod $A[d]_{2m-r-1}$. Thus, the unique simple quotient of $\mathrm{ind}_{G_{r-1,r}}^G \lambda_r^d$ is isomorphic to $S(d+(2m-r)(p-1))$. Similarly, $\beta_r[d]$ maps $\mathrm{ind}_{G_{r-1,r}}^G \lambda_R^{*d}$ into $A[d]_{r-1}$ and the image maps onto the simple module $A[d]_{r-1}/A[d]_{r-2} \cong S(d+(r-1)(p-1))$. In order to prove the uniseriality of $A[d]$, it suffices to prove that of $A[d]_r/A[d]_{r-2}$ for $1 \le r \le 2m-1$. We start by assuming that $1 \le r \le m-1$. Then we know that the image of $\beta_{r+1}[d]$ has a unique maximal submodule, so it it is enough to show that $A[d]_r/A[d]_{r-2}$ is generated as a $k\,\mathrm{Sp}(V)$-submodule by the image of $Y_{r+1}^d \chi_{W_r^\perp}$, which in turn will follow if we can show that this submodule contains a nonzero image of a polynomial of degree $d + (r-1)(p-1)$. We observe that $Y^d_{r+1}\chi_{W_r^\perp}$ is a polynomial in the Y_i's only. We consider the decomposition

$$V = V_e \oplus V_f = \langle e_1, \dots, e_m\rangle \oplus \langle f_1, \dots, f_m\rangle, \tag{28}$$

which gives the factorization $A = A(e) \otimes A(f)$, where $A(e)$ and $A(f)$ are the images of A of all polynomials in the X_i's and Y_i's respectively. The stabilizer of this decomposition induces the full general linear group on V_e, hence any linear substitution among the Y_i's. Since $A(f)$ is isomorphic to the ring of functions on V_f we can make use of the known structure of this module for $\mathrm{GL}(V_e)$. Namely, the component $A(f)[d]$, which is the image of polynomials with degrees congruent to d mod $p-1$ is uniserial. This means the $k\,\mathrm{GL}(V_f)$-submodule generated by any polynomial in the Y_i's of degree $d+r(p-1)$ which is nonzero mod $A[d]_{r-1}$ contains the images of all monomials in the Y_is of degree $d + (r-1)(p-1)$ Thus, the same is true for the $k\,\mathrm{Sp}(V)$ submodule of $A[d]$, generated by such an element. Therefore, we have established that $A[d]_r/A[d]_{r-2}$ is uniserial for $1 \le r \le m-1$. Since $A[d]$ is dual to $A[p-1-d]$ for all d it follows that $A[d]_r/A[d]_{r-2}$ is uniserial for $m+1 \le r \le 2m-1$.

It remains to show that $A[d]_m/A[d]_{m-2}$ is uniserial. Since the submodule genearted by the image of $X^d_m\chi_{W_m}$ has a unique maximal submodule, it is enough to show that this submodule contains the nonzero image of a polynomial of degree $d+(m-1)(p-1)$. Without loss, we may replace $X^d_m\chi_{W_m}$ by $X^d_1\chi_{W_m}$. Modulo $A[d]_{m-2}$, we have

$$(-1)^m X_1^d\chi_{W_m} = X_1^d\left(\prod_{i=1}^m Y_i^{p-1} - \prod_{i=2}^m Y_i^{p-1} - \sum_J Y_1^{p-1}Y_J^{p-1}\right) \tag{29}$$

where J runs over all subsets of size $m-2$ of $\{2,\dots m\}$. Since terms of degree $d+(m-2)(p-1)$ are zero, the right hand side of (29) can be written as

$$X_1^d(Y_1^{p-1}-1)q(Y_2,\dots,Y_m) \tag{30}$$

for some polynomial $q(Y_2,\dots,Y_m)$, of degree $(m-1)(p-1)$.

We shall consider again the decompositions $V = V_1 \oplus V_{m-1}$, $A = A_1 \otimes A_{m-1}$ and $\mathrm{Sp}(V) = \mathrm{Sp}(V_1) \times \mathrm{Sp}(V_{m-1})$ from (26) above. The subring A_1 is isomorphic to the ring of functions on a 2-dimensional $\mathbb{F}_p$-vector space and its structure under the action of $\mathrm{Sp}(V_1) = \mathrm{SL}(V_1)$ is given by 1.2. In particular, the component $A_1[d]$ (consisting of images of polynomials in X_1 and Y_1 of degree congruent to d mod $p-1$) is a nonsplit extension of two simple modules. The simple submodule has a basis of images of monomials of degree d and the quotient has a basis of images of monomials of degree $d+(p-1)$. Moreover, $A_1[d]$ is generated by $X_1^d(Y_1^{p-1}-1)$. (Note that $-(Y_1^{p-1}-1)$ is the characteristic function of the subspace spaaned by e_1 in V_1.) Returning to (30), we now see that since the $k\,\mathrm{Sp}(V_1)$-submodule of $A_1[d]$ generated by $X_1^d(Y_1^{p-1}-1)$ contains X_1^d, the $k\,\mathrm{Sp}(V_1)$-submodule of $A[d]_m/A[d]_{m-2}$ generated by (30) contains $X_1^d q(X_Y,\dots,Y_m)$, which is a nonzero element of $A[d]_{m-1}/A[d]_{m-2}$. Our proof is complete.

Concluding remarks. The cases of the same problem for $p = 2$ or with p replaced by a prime power are, to my knowledge, unsolved. For $p = 2$, the modules $S(e)$ are the exterior powers of the standard module, so an answer would include the submodule structure of these modules. The composition factors can be found using [8] and are given explcitly in [1]. The paper [2] also gives the submodule structure of the Weyl modules with fundamental highest weights. The exterior powers are known to be filtered by these modules [6, Appendix A].

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