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The Green Ring and Modular Representations of Finite Groups of Lie Type[1]

PETER SIN

Department of Mathematics, University of Chicago, Chicago Illinois 60637

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In the setting of the Green ring, the well-known "inclusion–exclusion" principle of elementary combinatorial theory is applied to study the set of indecomposable summands of the permutation module of a Sylow p-subgroup, yielding results about these modules and their endomorphism rings.

INTRODUCTION

Let G be a finite group of Lie type of characteristic p, and k a sufficiently large field of the same characteristic. The permutation module $Y = \mathrm{ind}_{U,G}(k)$ on the right cosets of a Sylow p-subgroup U has been studied by several authors; the fundamental results are due to Curtis and Richen (see [5], whose used the module Y to describe the irreducible kG-modules. Later their results were reinterpreted by Sawada [8] and Green [7] by consideration of the endomorphism ring $E = \mathrm{End}_{kG}(Y)$. This was also the approach of Carter and Lusztig [4]. The indecomposable direct summands of Y were studied by Tinberg in [11], in which she determined their dimensions and vertices. Recently, Cabanes [1] has computed the Green correspondents of the summands of Y by calculating the Brauer morphisms between certain endomorphism rings, thereby verifying a general conjecture of Alperin in the case of groups with split BN-pairs over fields of the defining characteristic.

In this paper, starting from the basic theorem of Curtis, we shall give new calculations of the dimensions and vertices of the summands of Y (Section 4), and their Green correspondents (Section 6). In Section 8, we compute the dimensions of the modules of kG-homomorphisms among the summands of the permutation module $\mathrm{ind}_{B,G}(k)$ on the right cosets of the

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Borel subgroup, from which we observe a duality operation on these modules induced by set-theoretic complementation in the set of fundamental roots (Section 9). Our methods are quite different from those of [1] and [11], the main novelty being the use of the Green group. All of our results are obtained as consequences of a simple formula in this group, which we derive in Section 3 from the Curtis–Richen theory by means of the well-known inclusion–exclusion principle (Möbius inversion). Formally, our arguments resemble those of L. Solomon [10], which supports the adage that Coxeter groups are degenerate forms of Chevalley groups.

1. The Green Group of a Finite Group G

This is the free abelian group $a(G)$ on the set of indecomposable finite dimensional kG-modules, where, by the Krull–Schmidt property, we may identify modules with "positive" elements of this group. By extending linearly the definition

$$(M, N) = \dim_k \operatorname{Hom}_{kG}(M, N)$$

we obtain a bilinear form on $a(G)$. We shall be concerned with the subgroup spanned by direct summands of permutation modules. It is well known that such summands may be lifted to characteristic zero, and that for any two, we have

$$(M, N) = (\mu, \nu),$$

where the right-hand side is the usual inner product of the characters afforded by lifts of M and N.

2. Groups with a Split BN-Pair

We adopt the usual notation; thus we have subgroups B, N, H, U, with $U \in \mathrm{Syl}_p(G)$, $B = U \rtimes H$, H being abelian of order prime to p, $H = B \cap N \lhd N$, and a Weyl group $W = N/H = \langle w_i \mid i \in R \rangle$, where R is the set of fundamental roots corresponding to B. For each subset J of R, we have a standard parabolic subgroup G_J, such that $G_\varnothing = B$ and $G_R = G$, and such that the map $J \mapsto G_J$ is an isomorphism of the posets $\{G_J \mid J \subseteq R\}$ and $\mathscr{P}(R)$, ordered by inclusion. For each $J \subseteq R$, we set $W_J = \langle w_i \mid i \in J \rangle$. Each W_J has a unique element of maximal length (see [2]), denoted w_J, which has square equal to the identity.

3. The Modules $Y(\chi, J)$

Let $\chi \in \mathrm{Hom}(H, k^*)$ and let $M(\chi) = \{ i \in R \mid \chi|_{H_i} = 1 \}$; we shall consider *pairs* (χ, J), with $J \subseteq M(\chi)$. Let L_χ be the one-dimensional module which affords χ (considered as a character of B). Then $\mathrm{ind}_{U,B}(k) \cong \bigoplus_\chi L_\chi$, so that in order to study $Y = \mathrm{ind}_{U,G}(k)$ we may as well study the modules $\mathrm{ind}_{B,G}(L_\chi)$. The following statement is reformulation of Curtis' theorem ([5], Sections 6.6, 6.8):

Theorem 1. *Let $\chi \in \mathrm{Hom}(H, k^*)$. Then*

(i) *L_χ extends uniquely to each subgroups of $G_{M(\chi)}$.*

(ii) *We have*

$$\mathrm{ind}_{B,G}(L_\chi) \cong \bigoplus_{J \subseteq M(\chi)} Y(\chi, J),$$

where the modules $Y(\chi, J)$ are indecomposable and mutually nonisomorphic.

(iii) *For $K \subseteq M(\chi)$ we have*

$$\mathrm{ind}_{G_k,G}(L_\chi) \cong \bigoplus_{K \subseteq J \subseteq M(\chi)} Y(\chi, J).$$

Moreover, the socles of the $Y(\chi, J)$ are precisely the simple kG-modules, without repeats, as are the heads of the $Y(\chi, J)$.

We remark that for $S \subseteq R$, Theorem 1 holds with G_S in place of G if we replace every $J \subseteq R$ by $J \cap S$. In this case we denote the summands of $\mathrm{ind}_{B,G}(L_\chi)$ by $Y_S(\chi, J \cap S)$.

We now reinterpret Theorem 1 in terms of the Green group: we regard the modules $Y(\chi, J)$ and $\mathrm{ind}_{G_K,G}(L_\chi)$ as funtions on $\mathscr{P}(M(\chi))$. Theorem 1 then says that $\mathrm{ind}_{G_{(-)},G}(L_\chi)$ is the partial sum function for the function $Y(\chi, -)$. Applying Möbius inversion, we obtain

Theorem 2. *In $a(G)$ the following equation holds:*

$$Y(\chi, J) = \bigoplus_{J \subseteq K \subseteq M(\chi)} (-1)^{|K \setminus J|} \mathrm{ind}_{G_K,G}(L_\chi). \qquad (*)$$

The following is now obtained by applying Theorem 2 to the groups G and $G_{M(\chi)}$:

Corollary 3.

$$Y(\chi, J) = \mathrm{ind}_{G_{M(\chi)},G}(Y_{M(\chi)}(\chi, J)).$$

4. DIMENSIONS AND VERTICES OF THE $Y(\chi, J)$

As our first application of $(*)$, we shall compute the dimension and vertices of $Y(\chi, J)$. This will be done by looping at the restriction to B, which turns out to have a particularly simple form. First, we introduce some notation: for $J \subseteq L \subseteq R$ set

$$X_{L,J} = \{w \in W_L \mid w^{-1}(J) > 0\},$$
$$V_{L,J} = \{w \in W_L \mid w^{-1}(J) > 0 \text{ and } w^{-1}(L \backslash J) < 0\}.$$

We shall write X_K and V_K for $V_{R,K}$, respectively. It is clear that $K \subseteq J$ if, and only if, $X_{L,J} \subseteq X_{L,K}$, and that W_L is the disjoint union of the sets $V_{L,K}$, $K \subseteq L$. It is well known that the set $\{n_w \mid w \in X_{L,J}\}$ is a set of (G_J, B) double coset representatives in G_L (see [3]). Since the group $G_J^{n_w}$ is independent of our choice of n_w, we shall follow the convention of writing G_J^w for this group. Set $q^w = |U: U \cap U^{n_w}|$, and $U_J = U \cap U^{n_{w_J}}$. Then $U_J = \mathbf{O}_p(G_J)$ is the unipotent radical of G_J.

THEOREM 4 (Tinberg). *We have*

(i) $\dim_k Y(\chi, J) = |G : G_{M(\chi)}| \sum_{w \in V_{M(\chi),J}} q^w$.

(ii) $Y(\chi, J)$ *has vertex* $U_{M(\chi)\backslash J}$.

Proof. By Corollary 3, we are reduced to the case $M(\chi) = R$. In this case, $(*)$ yields

$$Y(\chi, J) = \sum_{J \subseteq K \subseteq R} (-1)^{|K \backslash J|} \operatorname{ind}_{G_K, G}(L_\chi).$$

Restricting to B and applying the Mackey formula, we have

$$\operatorname{res}_{G,B}(Y(\chi, J)) = \sum_{J \subseteq K \subseteq R} (-1)^{|K \backslash J|} \sum_{w \in X_K} \operatorname{ind}_{G_K^w \cap B, B}(L_\chi \otimes_{kG_K} w).$$

Now $G_K^w \cap B = B^w \cap B$ (see [3]), and $L_\chi \otimes_{kG_K} w = L_\chi$ since $M(\chi) = R$. Thus,

$$\operatorname{res}_{G,B}(Y(\chi, J)) = \sum_{J \subseteq K \subseteq R} (-1)^{|K \backslash J|} \sum_{w \in X_K} \operatorname{ind}_{B^w \cap B, B}(L_\chi).$$

Since $K \subseteq I$ exactly when $X_I \subseteq X_K$, the coefficient of $\operatorname{ind}_{B^w \cap B, B}(L_\chi)$ is zero unless $w \in X_J$ but $w \notin X_K$ for any $K \supsetneq J$, and is 1 in this case. Thus

$$\operatorname{res}_{G,B}(Y(\chi, J)) = \sum_{w \in V_J} \operatorname{ind}_{B^w \cap B}(L_\chi). \qquad (**)$$

Now $|B: B^w \cap B| = q^w$ is a power of p and L_χ is indecomposable. Thus a vertex of $Y(\chi, J)$ is given by an element of smallest length in V_J. The element $w_{R\setminus J}$ clearly has this property. Therefore in the case $M(\chi) = R$, the subgroup $U_{R\setminus J}$ is a vertex of $Y(\chi, J)$, and the dimension is given by (**). The general case follows from this special case by Corollary 3.

5. The Steinberg Module

Setting $J = \emptyset$ in Theorem 2, we have

$$Y(1, \emptyset) = \sum_{J \subseteq R} (-1)^{|J|} \operatorname{ind}_{G_J, G}(k). \tag{***}$$

The module $Y(1, \emptyset)$ is the Steinberg module St_G. Thus we see that the definition ([3], Chap. 6) of the Steinberg character may be extended to give an alternative definition of the Steinberg module. This is not clear *a priori*, but is probably known to specialists. We shall make use of (***) to compute the Green correspondents of the $Y(\chi, J)$. The results of the last section tell us that St_G is projective of dimension $|U|$, which is of course well known. Finally, we remark that (***) and similar formulae arising from the action of a finite group on a simplicial complex (in this case the Tits building) may be used to prove induction theorems for finite groups [9].

6. The Green Correspondents of the $Y(\chi, J)$

We have seen that $Y(\chi, J)$ has vertex $U_{M(\chi)\setminus J}$. We now compute its Green correspondent in $G_{M(\chi)\setminus J} = N_G(U_{M(\chi)\setminus J})$. For $S \subseteq R$, let $\bar{G}_S = G_S/U_s$. We regard $\mathrm{St}_{\bar{G}_S}$ as a module for kG_S; as such it has vertex U_S.

THEOREM 5. *The Green correspondent of $Y(\chi, J)$ in $G_{M(\chi)\setminus J}$ is*

$$L_\chi \otimes_k \mathrm{St}_{\bar{G}_{M(\chi)\setminus J}}.$$

Proof. Clearly $L_\chi \otimes_k \mathrm{St}_{\bar{G}_{M(\chi)\setminus J}}$ has vertex $U_{M(\chi)\setminus J}$. By applying (***) to $\bar{G}_{M(\chi)\setminus J}$, we have

$$L_\chi \otimes_k \mathrm{St}_{\bar{G}_{M(\chi)\setminus J}} = \sum_{S \subseteq M(\chi)\setminus J} (-1)^{|S|} \operatorname{ind}_{G_S, G_{M(\chi)\setminus J}}(L_\chi).$$

Therefore,

$$\begin{aligned}\operatorname{ind}_{G_{M(\chi)\setminus J}, G}(L_\chi \otimes_k \mathrm{St}_{\bar{G}_{M(\chi)\setminus J}}) &= \sum_{S \subseteq M(\chi)\setminus J} (-1)^{|S|} \operatorname{ind}_{G_s, G}(L_\chi) \\ &= \sum_{S \subseteq M(\chi)\setminus J} (-1)^{|S|} \sum_{S \subseteq T \subseteq M(\chi)} Y(\chi, T),\end{aligned}$$

by Theorem 1(iii), and this simplifies to

$$\sum_{T \subseteq J} Y(\chi, T).$$

Since $Y(\chi, J)$ is in this sum and has vertex $U_{M(\chi)\setminus J}$, we are done.

We remark that since $G_{M(\chi)\setminus J} = N_G(U_{M(\chi)\setminus J})$ is parabolic, so is every subgroup which contains it, so that the theorem also gives the Green correspondence with respect to any intermediate subgroup (see the remarks following Theorem 1).

7. Alperin's Conjecture

J. Alperin has defined a *projective weight* of a finite group to be an irreducible projective module for a section $N_G(P)/P$, where P is a p-subgroup, and has conjectured that for any finite group and any prime p, the number of conjugacy classes of projective weights is equal to the number of isomorphism classes of irreducible p-modular representations. It is general fact that when regarded as a module for $kN_G(P)$, a projective weight has a Green correspondent which is a direct summand of the permutation module on the cosets of a Sylow p-subgroup (see [1]). Thus Theorems 5 and 1 together with this fact give a natural bijection

$$\begin{aligned} \text{projective weights} &\leftrightarrow \text{irreducible modules} \\ M &\leftrightarrow \text{soc(Green correspondent of } M), \end{aligned}$$

which checks the conjecture in the case of split BN-pairs in the defining characteristic. This observation is due to Cabanes [1].

8. Homomorphisms between the $Y(\chi, J)$

Proposition 6. $\mathrm{Hom}_{kG}(\mathrm{ind}_{B,G}(L_\chi), \mathrm{ind}_{B,G}(L_\psi)) = 0$ *unless there exists* $w \in W$ *such that for every* $h \in H$, *we have* $\chi(h^w) = \psi(h)$.

Proof. This is a simple consequence of Frobenius reciprocity, Mackey decomposition, and the fact that the set $\{n_w \mid w \in W\}$ is a set of (B, B) double coset representatives in G.

Theorem 7.

$$\dim_k \mathrm{Hom}_{kG}(Y(1, J), Y(1, K)) = |V_J \cap V_K^{-1}|.$$

Proof. The $Y(\chi, J)$ are all summands of the permutation module Y, so by the remarks of Section 1 the number we wish to compute is

$$\left(\sum_{J \subseteq S \subseteq R} (-1)^{|S \setminus J|} 1_{G_S}^G, \sum_{K \subseteq T \subseteq R} (-1)^{|T \setminus K|} 1_{G_T}^G \right).$$

Using the fact that $\{n_w \mid w \in X_S \cap X_T^{-1}\}$ is a set of (G_S, G_T) double coset representatives in G, and applying the Mackey formula, this expression becomes

$$\sum_{J \subseteq S \subseteq R} \sum_{K \subseteq T \subseteq R} (-1)^{|S \setminus J|} (-1)^{|T \setminus K|} (1_{G_S}^G, 1_{G_T}^G)$$

$$= \sum_{J \subseteq S \subseteq R} \sum_{K \subseteq T \subseteq R} (-1)^{|S \setminus J|} (-1)^{|T \setminus K|} \sum_{w \in X_S \cap X_T^{-1}} (1_{G_S^w \cap G_T}, 1_{G_S^w \cap G_T}).$$

In order to simplify this expression, we note that if $I \subseteq S$, $Q \subseteq T$, then we have

$$\begin{array}{ccc} X_S \cap X_T^{-1} & \subseteq & X_I \cap X_T^{-1} \\ \cap\!| & & \cap\!| \\ X_S \cap X_Q^{-1} & \subseteq & X_I \cap X_Q^{-1}. \end{array}$$

Thus the contribution due to w is zero unless $w \in X_J \cap K_K^{-1}$, but $w \notin X_S \cap X_T^{-1}$ for any $S \supsetneq J$ or $T \supsetneq K$, in which case the contribution is one. But the above condition is precisely the condition for w to lie in $V_J \cap V_K^{-1}$, so the theorem is proved.

Remark. If $w^2 = 1$, then $w \in V_K \cap V_K^{-1}$ for some unique subset $K \subseteq R$, because W is the disjoint union of the subsets $V_J \cap V_K^{-1}$, $J, K \subseteq R$. In the extreme cases we have $V_R \cap V_R^{-1} = \{1\}$, and $V_\emptyset \cap V_\emptyset = \{w_R\}$. It seems of interest to ask when the set $V_K \cap V_K^{-1}$ consist only of elements whose square is 1. For example, this is true when $|R| = 2$, as can be seen by plane geometry.

The computation of $\dim_k \operatorname{Hom}_{kG}(Y(\chi, J), Y(\psi, K))$ appears to be more complicated in general, and will appear elsewhere.

9. "Duality"

THEOREM 8. *The map $c\colon Y(1, J) \mapsto Y(1, R \setminus J)$ is an isometry of order 2 of the subgroup of $a(G)$ spanned by the summands of* $\operatorname{ind}_{B,G}(k)$.

Proof. By Theorem 7 we have

$$(Y(1, S), Y(1, T)) = |V_S \cap V_T^{-1}|,$$

where

$$V_J = \{w \in W \mid w^{-1}(J) > 0, w^{-1}(R \setminus J) < 0\}.$$

It is obvious from this that right multiplication by w_R sends $V_S \cap V_T^{-1}$ bijectively onto $V_{R\setminus S} \cap V^{-1}_{-w_R(R\setminus T)}$. Now $|X_{R\setminus S} \cap X^{-1}_{-w_R(R\setminus T)}| = |X_{R\setminus S} \cap X^{-1}_{R\setminus T}|$, since they are double coset transversals for $(W_{R\setminus S}, W^{w_R}_{R\setminus T})$ and $(W_{R\setminus S}, W_{R\setminus T})$ respectively. It follows that $|V_{R\setminus S} \cap V^{-1}_{-w_R(R\setminus T)}| = |V_{R\setminus S} \cap V^{-1}_{R\setminus T}|$. Thus,

$$(cY(1, S), cY(1, T)) = (Y(1, S), Y(1, T)).$$

This duality is related to Alvis–Curtis duality [6] in the ring of complex characters; for instance, the operation c interchanges the trivial module k and the Steinberg module. The operation c commutes with induction in the sense that if $S \subseteq J$, then

$$c(\mathrm{ind}_{G_J,G}(Y_J(1, S))) = \mathrm{ind}_{G_J,G}(Y_J(1, J\setminus S)),$$

which can be proved using Theorems 1 and 2. We have

$$Y(1, K) = \sum_{K \subseteq J \subseteq R} (-1)^{|J\setminus K|} \mathrm{ind}_{G_J,G}(k),$$

by $(*)$, and by the previous paragraph, we have

$$Y(1, R\setminus K) = \sum_{K \subseteq J \subseteq R} (-1)^{|J\setminus K|} \mathrm{ind}_{G_J,G}(\mathrm{St}_{\bar{G}_J}).$$

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References

1. M. Cabanes, Brauer morphism between modular Hecke algebras, preprint, Paris.
2. R. W. Carter, "Simple Groups of Lie Type," Wiley, Chichester, 1971.
3. R. W. Carter, "Finite Groups of Lie Type," Wiley, Chichester, 1985.
4. R. W. Carterand G. Lusztig, Modular representations of finite groups of Lie type, *Proc. London Math. Soc.* **32** (1976), 347–384.
5. C. W. Curtis, "Modular Representations of Finite Groups with Split *BN*-Pairs," Lecture Notes in Mathematics, Vol. 131, pp. 57–95, Springer-Verlag, Heidelberg, 1969.
6. C. W. Curtis, Truncation and duality in the character ring of a finite group of Lie type, *J. Algebra* **62** (1980), 320–332.
7. J. A. Green, On a theorem of H. Sawada, *J. London Math. Soc.* **18** (1978), 247–252.
8. H. Sawada, A characterisation of the modular representations of finite groups with split *BN*-pairs, *Math. Z.* **155** (1977), 29–41.
9. P. Sin, The Steinberg module and applications to induction theorems, preprint, Oxford (1986).
10. L. Solomon, A decomposition of the group algebra of a finite Coxeter group, *J. Algebra* **9** (1968), 220–239.
11. N. Tinberg, Some indecomposable modules for finite groups with split *BN*-pairs, *J. Algebra* **61** (1979), 508–526.