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The Cohomology in Degree 1 of the Group F_4 in Characteristic 2 with Coefficients in a Simple Module

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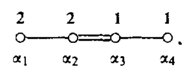
We compute the 1-cohomology with coefficients in a simple module for the simple algebraic group of type F_4 in characteristic 2.

INTRODUCTION

In this paper we compute the groups $H^1(G, V)$, where G is the algebraic group of type F_4 over an algebraic closure F of the field $\mathbf{F}_2$ of two elements and V is a simple rational G-module over F.

It is well known [Steinberg 2, p. 157] that G has an exceptional τ whose square is the Frobenius endomorphism for some $\mathbf{F}_2$-structure. G possesses a τ-stable Borel subgroup B containing a τ-stable maximal torus T. Thus τ operates on the group $X = X(T)$ of (rational) characters of T. Since G is both simply connected and of adjoint type we may identify X with the weight lattice of a root system Φ of type F_4 and this coincides with the lattice spanned by the roots. The subgroup B determines a positive system in Φ (such that the root subgroups of B correspond to negative roots) and hence also a base $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, the set X_+ of dominant weights and a set $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of fundamental dominant weights, defined by $\langle \lambda_i, 2\alpha_j/\langle \alpha_j, \alpha_j \rangle \rangle = \delta_{ij}$, where "$\langle\ ,\ \rangle$" is the inner product for Φ. We choose our numbering according to the following diagram:



Thus α_1 and α_2 are long roots and α_3 and α_4 are short.

As is well known, the simple rational G-modules are parametrized by X_+ according to their highest weights. Let $L(\lambda)$ be the simple module

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with highest weight λ. The first cohomology groups are described by the following statement.

THEOREM. *Let $\lambda \in X_+$. Then the F-vector space $H^1(G, L(\lambda))$ is one dimensional if λ belongs to the orbit under $\tau: X \to X$ of λ_3, $\lambda_1 + \lambda_4$ or $\lambda_3 + 2\lambda_4$ and is zero otherwise.*

One of the main ideas is to exploit properties proved in [CPSK] of the restriction map from G-cohomology to the cohomology of the finite groups

$$G(n) = G^{\tau^n} = \begin{cases} F_4(2^{n/2}) & \text{if } n \text{ is even} \\ {}^2F_4(2^n) & \text{if } n \text{ is odd} \end{cases}.$$

The groups $G(n)$ for odd values of n are the Ree groups of type F_4. For given dominant weights λ and μ, there is a certain threshold for n, beyond which the map

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{G(n)}(L(\lambda), L(\mu))$$

is injective and a second bound such that the map becomes an isomorphism when n exceeds it. These facts allows us to make use of the injectivity of the Steinberg module for the finite group to prove the triviality of most of the cohomology groups in the theorem.

We recall [Steinberg 1, Sect. 11] that the simple $FG(n)$-modules are the restrictions of the simple G-modules $L(\lambda)$ for λ in a certain subset of X_+. It follows from our discussion of the restriction map above that if one could compute all of the groups $H^1(G(n), L(\lambda))$ for all of the simple modules for all of the finite groups, then one would also know all of the groups $H^1(G, L(\lambda))$. The goal of the present paper is a much more modest one, however, because in computing $H^1(G, L(\lambda))$ for a given λ by this method, we are free to choose n as large as we wish, and the "difficult" weights for a particular group $G(n)$ may be "easy" if we replace n by a larger value. In this way we manage to avoid most of the combinatorial complications of the kind which appear in the related papers [Sin 1–4], at the cost of obtaining less sharp results for the finite groups.

Remarkably, a large part of our calculations relies only on knowledge of the multiplication table of the Grothendieck ring of finite-dimensional G-modules. This can be computed in principle from the characters of the simple G-modules, which have been known for some time. We devote the first section of this paper to collecting together information about the simple modules and their tensor products and some Weyl modules. Using this, we treat the main case in Section 2, showing that nearly all of the cohomology groups in question vanish. The remaining ones are analyzed piecemeal in the final section.

1. Tensor Products of Simple Modules

It is straightforward to check that under $\tau: X(T) \to X(T)$ we have $\lambda_4 \mapsto \lambda_1 \mapsto 2\lambda_4$ and $\lambda_3 \mapsto \lambda_2 \mapsto 2\lambda_3$. We call the set $X_\tau = \{0, \lambda_3, \lambda_4, \lambda_3 + \lambda_4\}$ the set of *τ-restricted* weights. Every dominant weight has a "τ-adic" expression

$$\lambda = \sum_{i=0}^{r} \tau^i v_i, \qquad v_i \in X_\tau, \qquad \text{with} \quad v_r \neq 0 \quad \text{if} \quad r \geqslant 1. \tag{1.1}$$

The τ^n-restricted weights X_{τ^n} are defined to be those for which the upper limit r in (1.1) is less than n. For $\lambda \in X_{\tau^n}$, the simple G-module $L(\lambda)$ remains simple upon restriction to $G(n)$ and the 4^n modules thus obtained form a complete set of nonisomorphic simple $FG(n)$-modules (see [Steinberg 1, Sect. 11]). Steinberg's Tensor Tensor Product Theorem [Steinberg 1, Sect. 11] states that for λ as in (1.1) there is a G-module isomorphism

$$L(\lambda) \cong L(v_0) \otimes L(v_1)^\tau \otimes \cdots \otimes L(v_r)^{\tau^r}, \tag{1.2}$$

where the superscripts "τ^{i}" indicate twisting the G-module structure be the endomorphism τ^i of G. It follows that the characters of all the simple rational G-modules and simple $FG(n)$-modules are completely determined by those of $L(\lambda_3)$, $L(\lambda_4)$ and $L(\lambda_3 + \lambda_4)$. Likewise, the multiplication tables for the Grothendieck rings of the categories of finite-dimensional modules for G and $FG(n)$ are determined by the products of the classes of these three modules, because forming tensor products commutes with twisting the factors by τ. The characters of the simple modules were first given in [Veldkamp], and are reproduced in Table I. The Weyl group of Φ is denoted by W.

Since $-1 \in W$, all of the simple modules are self-dual.

Let $V(\lambda)$ denote the Weyl module with highest weight $\lambda \in X_+$. It has a unique maximal submodule with quotient isomorphic to $L(\lambda)$. The character of $V(\lambda)$ is given by Weyl's character formula. Tables of these characters, including all of the ones we shall need, appear in [Bremner–Moody–

TABLE I

	dim	0	λ_4	λ_1	λ_3	$2\lambda_4$	$\lambda_1 + \lambda_4$	λ_2	$\lambda_3 + \lambda_4$
W-orbit size		1	24	24	96	24	144	96	192
$L(0)$	1	1	0	0	0	0	0	0	0
$L(\lambda_4)$	26	2	1	0	0	0	0	0	0
$L(\lambda_3)$	246	6	4	2	1	0	0	0	0
$L(\lambda_3 + \lambda_4)$	4096	64	40	24	14	8	4	2	1

Patera]. The entries in Table I can be calculated using Jantzen's Sum Formula [Jantzen, p. 314].

The module $V(\lambda_4) \cong L(\lambda_4)$ has an interesting interpretation as the reduction mod 2 of the submodule of elements of trace zero in an exceptional Jordan Algebra over $\mathbf{Z}$. The module $V(\lambda_3+\lambda_4) \cong L(\lambda_3+\lambda_4)$ is the Steinberg module for $G(1) \cong {}^2F_4(2)$. Let $\rho = \lambda_1+\lambda_2+\lambda_3+\lambda_4$. The module $L(\rho)$ is the first Steinberg module for G. Since by (1.2) we have $L(\rho) \cong L(\lambda_3+\lambda_4) \otimes L(\lambda_3+\lambda_4)^\tau$, we may regard $L(\lambda_3+\lambda_4)$ as the $\frac{1}{2}$th Steinberg module for G.

It will be convenient to introduce the following notation. Let $E = L(\lambda_4)$, $M = L(\lambda_3)$ and $S = L(\lambda_3+\lambda_4)$. For any G-module V, let V_i be the module obtained through twisting by τ^i and for any finite set I of natural numbers, let $V_I = \bigotimes_{i \in I} V_i$. With this notation the simple G-modules are the modules $E_I \otimes M_J \otimes S_K$ for disjoint finite subsets I, J and K of natural numbers. For the finite group $G(n)$, we have $V_i \cong V_{i+n}$ and the simple modules with highest weights belonging to X_{τ^n} are those for which the subsets I, J and K of the new notation are contained in the set $N = \{0, 1, ..., n-1\}$. The simple $FG(n)$-module S_N is injective; it is the Steinberg module for $G(n)$.

We shall write $[V:L(\lambda)]$ for the multiplicity of $L(\lambda)$ as a composition factor of the G-module V if V has finite dimension. The numbers $[L(\lambda) \otimes L(\mu):L(\nu)]$ for $\lambda, \mu \in X_\tau$ are given in Table II. It is clear that these multiplicities can be computed from Table I in principle, so we discuss briefly some practical aspects. Let $\{e_i \mid 1 \leqslant i \leqslant 4\}$ be an orthonormal basis for the euclidean space $(X \otimes_{\mathbf{Z}} \mathbf{R}, \langle\ ,\ \rangle)$. Then we may take Φ to be the set $\{\pm e_i;\ \pm e_i \pm e_j;\ \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \mid 1 \leqslant i < j \leqslant 4\}$ with $\alpha_1 = (1, -1, 0, 0)$, $\alpha_2 = (0, 1, -1, 0)$, $\alpha_3 = (0, 0, 1, 0)$, $\alpha_4 = \frac{1}{2}(-1, -1, -1, 1)$, so that the fundamental dominant weights are $\lambda_1 = (1, 0, 0, 1)$, $\lambda_2 = (1, 1, 0, 2)$, $\lambda_3 = \frac{1}{2}(1, 1, 1, 3)$ and $\lambda_4 = (0, 0, 0, 1)$. If σ_i denotes the reflection relative to α_i then $W = \langle \sigma_i \mid 1 \leqslant i \leqslant 4 \rangle$. In computing with W both for the present calculation and for calculations with the Jantzen Sum Formula it is often helpful to consider the subgroup $\Sigma = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4\sigma_3\sigma_2\sigma_3\sigma_4 \rangle$ of index 3 which consists of all signed permutations of the basis elements. The advantage of using Σ is that it is possible to recognize instantly whether or not two weights lie in the same Σ-orbit. In all of the calculations we made it was very easy to determine the Σ-orbits contained in the same W-orbit, by considering lengths of vectors, for example.

In the group ring $\mathbf{Z}[X]$, the Grothendieck ring of finite-dimensional rational T-modules, the character of a finite-dimensional rational G-module can be decomposed into a positive integral linear combination of W-orbit sums of weights, as in Table I and each of the W-orbit sums can be written as a sum of Σ-orbit sums. To obtain Table II we first decomposed the characters in Table I into Σ-orbit sums. Then, by lengthy but routine calculations, we obtained the products of these Σ-orbit sums as positive

TABLE 2

Products of τ-Restricted Modules

	$E\otimes E$	$E\otimes M$	$E\otimes S$	$M\otimes M$	$M\otimes S$	$S\otimes S$
F	2	2	28	18	92	848
E	4	2	18	12	54	344
E_1	2	4	48	32	176	1352
M	2	2	10	4	18	108
E_2	1	2	26	16	112	1176
$E_{\{0,1\}}$	0	2	24	16	88	640
M_1	0	1	18	12	74	648
E_3	0	0	8	6	48	536
S	0	1	2	0	2	12
$E_1\otimes M$	0	0	6	4	24	176
$E_{\{0,2\}}$	0	0	6	4	32	308
$E_{\{1,2\}}$	0	0	6	4	32	432
$E\otimes M_1$	0	0	3	2	24	236
M_2	0	0	2	1	12	220
$E_{\{0,3\}}$	0	0	0	0	10	128
S_1	0	0	0	0	6	72
$E_{\{1,3\}}$	0	0	0	0	0	40
$E_2\otimes M$	0	0	1	0	6	64
$E_1\otimes S$	0	0	0	0	2	16
$M_{\{0,1\}}$	0	0	0	0	3	48
$E_3\otimes M$	0	0	0	0	0	20
E_4	0	0	0	0	4	72
$E_2\otimes M_1$	0	0	0	0	2	60
$E\otimes M_2$	0	0	0	0	1	20
$E_{\{2,3\}}$	0	0	0	0	0	32
$E_{\{0,1,2\}}$	0	0	0	0	4	64
$E\otimes S_1$	0	0	0	0	0	12
$E_1\otimes M_2$	0	0	0	0	0	16
M_3	0	0	0	0	0	4
$E_2\otimes S$	0	0	0	0	0	4
$E_{\{1,2\}}\otimes M$	0	0	0	0	0	8
$M_1\otimes S$	0	0	0	0	0	2
$M_{\{0,2\}}$	0	0	0	0	0	2
$E_{\{0,4\}}$	0	0	0	0	0	4
$E_{\{1,4\}}$	0	0	0	0	0	4
$E_{\{0,2\}}\otimes M_1$	0	0	0	0	0	2
S_2	0	0	0	0	0	1

integral combinations of other Σ-orbit sums. Next the W-conjugacy of all the Σ-orbits involved was determined, which yielded the product of the W-orbit sums in terms of W-orbit sums and also enabled us to extend Table I to all of the simple modules, 37 in all, having their highest weights in one of these orbits. This 37×37 unitriangular integer matrix (or rather its inverse) and the information on products of W-orbit sums then easily led to Table II.

We define the *mass* of a simple G-module by

$$m(E_1 \otimes M_J \otimes S_K) = 2\,|I| + 3\,|J| + 5\,|K|$$

and that of an arbitrary finite-dimensional module to be the greatest of the masses of its composition factors. The mass of a module is clearly invariant under twisting by τ and under taking duals. It also behaves well with respect to tensor products, as the following result shows.

LEMMA 1.3. *For simple G-modules* $E_I \otimes M_J \otimes S_K$ *and* $E_A \otimes M_B \otimes S_C$, *we have*

$$m((E_1 \otimes M_J \otimes S_K) \otimes (E_A \otimes M_B \otimes S_C)) \leqslant 2(|I| + |A|) + 3(|J| + |B|) + 5(|K| + |C|)$$

with equality if and only if $(I \cup K) \cap (A \cup C) = \varnothing = (J \cup K) \cap (B \cup C)$.

In terms of weights $\lambda = \sum \tau^i v_i$ *and* $\mu = \sum \tau^i v_i'$ *(given as τ-adic expressions) we have*

$$m(L(\lambda) \otimes L(\mu)) \leqslant m(L(\lambda)) + m(L(\mu))$$

with equality if and only if $v_i + v_i' \in X_\tau$ *for all i.*

Proof. Inspection of Table II shows the lemma to be true when the simple modules have τ-restricted highest weights. The general result then follows from the Tensor Product Theorem (1.2) and an obvious induction on the masses of the tensor factors.

One can similarly define the mass of an $FG(n)$-module V. Since $V \cong_{FG(n)} V_n$, it follows from Lemma 1.3 that the mass of a G-module is no less than its mass as an $FG(n)$-module and hence that Lemma 1.3 is also true for the tensor product of two simple $FG(n)$-modules.

2. THE MAIN CASE

Let us begin by defining a little more notation. Given a module V for G or $FG(n)$, we define its *radical* rad V to be the intersection of its maximal submodules and we define its *socle* to be its maximal semisimple submodule. Then we set $\text{rad}^i V = \text{rad}(\text{rad}^{i-1} V)$ and we define $\text{soc}^i V$ to be the full preimage in V of $\text{soc}(V/\text{soc}^{i-1} V)$. The sequence $V/\text{rad}\, V$, $\text{rad}\, V/\text{rad}^2 V, \ldots$ is called the radical series of V and the first term $V/\text{rad}\, V$ is called the *head* and denoted by hd V.

We make heavy use of the following statements about "generic" cohomology.

LEMMA 2.1. *Let V be a finite-dimensional rational G-module over F.*

(a) *For sufficiently large values of n (depending on V) the restriction map*

$$H^1(G, V) \to H^1(G(n), V)$$

is an isomorphism

(b) *For any natural number i, the map*

$$H^1(G, V) \to H^1(G, V_i)$$

induced by twisting by τ^i is an isomorphism.

(c) *For any positive integer n and λ, $\mu \in X_{\tau^n}$ the restriction map*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \to \mathrm{Ext}^1_{G(n)}(L(\lambda), (L(\mu))$$

is injective.

Proof. Part (a) is a special case of [CPSK, Thm. 7.1] if n is even. To prove (b) we fix i and choose an even value of n which is bigger than i and big enough so that (a) holds for both V and V_i. Then we have a commutative diagram

$$\begin{array}{ccc} H^1(G, V) & \xrightarrow{\tau^i} & H^1(G, V_i) \\ \text{res}\downarrow & & \downarrow\text{res} \\ H^1(G(n), V) & \xrightarrow{\tau^i} & H^1(G(n), V_i) \end{array}$$

in which the vertical arrows are isomorphisms given by (a) and the bottom arrow is an isomorphism because $\tau_{|G(n)}$ is an automorphism. This proves (b) and combining this with the main theorem of [Avrunin] we obtain (a) for odd values of n too.

If n is even, (c) is a special case of [CPSK, Thm. 7.4], but we shall give a proof which includes the odd case as well, based on [Andersen, Prop. 2.7]. We use some standard facts about the induced modules $H^0(\lambda) = \mathrm{ind}_B^G(\lambda)$ for which we refer to [Jantzen, pp. 199–207]. Suppose we have a nonsplit G-module extension V of $L(\lambda)$ by $L(\mu)$. Since $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Ext}^1_G(L(\mu, (L(\lambda))$ and nonsplit extensions can only exist when one of λ or μ lies strictly below the other in the usual partial ordering of X, we may assume $\lambda \lneqq \mu$. Then it is a standard fact that V embeds into $H^0(\mu)$. Let ρ_n be the highest weight of S_N, where $N = \{0, ..., n-1\}$. Then it is easy to check using Weyl's formula that $S_N = L(\rho_n) = H^0(\rho_n)$. We have $\mathrm{Hom}_B(\mu, \rho_n \otimes H^0(\rho_n - \mu)) \neq 0$, since μ is a minimal weight in the tensor product. Thus, there is a nonzero G-map

$$H^0(\mu) \to H^0(\rho_n) \otimes H^0(\rho_n - \mu) \cong L(\rho_n) \otimes H^0(\rho_n - \mu),$$

which must be injective since its restriction to $\operatorname{soc}(H^0(\mu)) \cong L(\mu)$ is not zero.

Thus it suffices to show that

$$\begin{aligned}\operatorname{Hom}_{G(n)}(L(\lambda), L(\rho_n) \otimes H^0(\rho_n - \mu))\\ \cong \operatorname{Hom}_{G(n)}(L(\rho_n), L(\lambda) \otimes H^0(\rho_n - \mu)) = 0.\end{aligned}$$

We define a linear function $f: X \otimes_{\mathbf{Z}} \mathbf{R} \to \mathbf{R}$ by $f(\lambda_4) = 1$, $f(\lambda_1) = \sqrt{2}$, $f(\lambda_3) = a$ and $f(\lambda_2) = a\sqrt{2}$, where $1/(2-\sqrt{2}) < a < 2$. Then one can check that f takes positive values on all positive roots. The weights of $L(\lambda) \otimes H^0(\rho_n - \mu)$ lie strictly below ρ_n, since $\lambda \lneq \mu$. Therefore, from Steinberg's Tensor Product Theorem and the fact that $f(\tau v) > f(v)$ for all $v \in X_+$, it follows that any $G(n)$-composition factor $L(\omega)$, $\omega \in X_{\tau^n}$, of the above tensor product must satisfy $f(\omega) < f(\rho_n)$. This completes the proof of (c).

Many of our arguments involve repeated applications of the following principle, which follows directly from the long exact sequence of cohomology for G.

LEMMA 2.2. *Let V and W be simple G-modules and let $d = \dim_F \operatorname{Ext}^1_G(V, W)$. Let X be a G-module with $\operatorname{soc} X \cong W$ and $X/\operatorname{soc} X$ isomorphic to a direct sum of d copies of V. Let Y be a simple module and let Z be a simple submodule of $Y \otimes V$. Suppose that the natural map*

$$\operatorname{Hom}_G(Z, Y \otimes W) \hookrightarrow \operatorname{Hom}_G(Z, Y \otimes X)$$

is an isomorphism. Then $d \leqslant \dim_F \operatorname{Ext}^1_G(Z, Y \otimes W)$.

When we apply this result, the module $Y \otimes W$ is simple and not isomorphic to Z, so the hypothesis of the lemma becomes equivalent to the triviality of $\operatorname{Hom}_G(Z, Y \otimes X) \cong \operatorname{Hom}_G(Z \otimes Y^*, X)$. Clearly, the last group is trivial if W is not isomorphic to any composition factor of $Z \otimes Y^*$, which will be the case if, for example $m(W) > m(Z \otimes Y^*)$. Another simple but useful criterion is given by the next lemma.

LEMMA 2.3. *Let λ be dominant weight written as*

$$\lambda = \sum_{i=r_0}^{r_1} \tau^i v_i, \qquad v_i \in X_\tau,\ v_{r_0}, v_{r_1} \neq 0$$

and let $\lambda' = \sum_{j=s_0}^{s_1} \tau^j v'_j$ and $\lambda'' = \sum_{k=t_0}^{t_1} \tau^k v''_k$ be expressions of the same kind for the dominant weights λ' and λ''. Suppose $s_0 < t_0$ and that $L(v'_{s_0})^{\tau^{s_0}} \ncong L(v_{r_0})^{\tau^{r_0}}$. Then $L(\lambda)$ is not a composition factor of the G-module $L(\lambda') \otimes L(\lambda'')$.

Proof. Since twisting by τ commutes with taking tensor products, we see using (1.2) that every composition factor of $L(\lambda') \otimes L(\lambda'')$ is of the form $L(\nu_{s_0}) \otimes L(\mu)^{\tau^{s_0+1}}$ for some $\mu \in X_+$. The lemma is now clear from (1.2).

Remark. Lemma 2.3 does not hold for the finite groups $G(n)$, because τ induces a partial order on the set of simple G-modules but not on the simple $FG(n)$-modules. This is a major factor in making calculations for G much simpler than for the finite groups.

The next lemma is used to choose the module Z in many applications of Lemma 2.2.

LEMMA 2.4. *The module S is isomorphic to a direct summand of $E \otimes M$. Consequently, the simple module E embeds into $M \otimes S$ and M embeds into $E \otimes S$.*

Proof. From Table II we see $[E \otimes M : S] = 1$. We have $S \cong V(\lambda_3 + \lambda_4)$ and so since $\lambda_3 + \lambda_4$ is the maximal weight of $E \otimes M$ and Weyl modules do not extend simple modules of lower weight [Jantzen, Prop. 2.14, p. 207], there is a submodule of $E \otimes M$ isomorphic to S. Then the self-duality of $E \otimes M$ and the uniqueness of S as a composition factor imply that S is isomorphic to a direct summand. The last part follows from the isomorphisms

$$\mathrm{Hom}_G(S, E \otimes M) \cong \mathrm{Hom}_G(E, S \otimes M) \cong \mathrm{Hom}_G(M, E \otimes S).$$

Let us introduce the following notations, suggested by Lemma 2.2. For two simple modules $L(\lambda)$ and $L(\mu)$, let $d(L(\lambda), L(\mu)) = \dim_F \mathrm{Ext}^1_G(L(\lambda), L(\mu))$ and let $X(L(\lambda), L(\mu))$ be a G-module with socle isomorphic to $L(\mu)$ and the quotient by its socle isomorphic to a direct sum of $d(L(\lambda), L(\mu))$ copies of $L(\lambda)$. The injective hull of $L(\mu)$ has a submodule fitting this description and any other is isomorphic to it.

We now come to the two principal results of this section.

LEMMA 2.5. *Let I and J be disjoint finite sets of natural numbers. Then*

$$\mathrm{Ext}^1_G(E_I \otimes M_J, S_{I \cup J}) = 0.$$

Proof. If $I \cup J = \varnothing$, then $\mathrm{Ext}^1_G(F, F) = 0$, so we assume $I \cup J \neq \varnothing$. By Lemma 2.1(b), we can assume that the smallest element of $I \cup J$ is 0. Let $n-1$ be the largest element of this set and define $N = \{0, ..., n-1\}$ and $K = N \setminus (I \cup J)$. Then by Lemma 2.1(c), we have

$$\mathrm{Ext}^1_G(E_I \otimes M_J \otimes S_K, S_N) \hookrightarrow \mathrm{Ext}^1_{G(n)}(E \otimes M_J \otimes S_K, S_N) = 0,$$

by the injectivity of the $FG(n)$-module S_N. Thus the lemma is proved if we show

$$d(E_I \otimes M_J, S_{I \cup J}) \leqslant d(E_I \otimes M_J \otimes S_K, S_N).$$

Now every element of K is greater than 0 and neither E nor M is isomorphic to S, so Lemma 2.3 shows that $S_{I \cup J}$ is not a composition factor of $(E_I \otimes M_J \otimes S_K) \otimes S_K$. Therefore the desired inequality follows from Lemma 2.2.

Remark. This lemma could be formulated in a much broader setting, since its proof requires only general properties of modular representations of semisimple groups.

PROPOSITION 2.6. *Let I, J, and K be disjoint finite sets of natural numbers with $|K| \geqslant 1$ or $|I \cup J| \geqslant 4$. Then $H^1(G, E_I \otimes M_J \otimes S_K) = 0$.*

Proof. First suppose $I \cup J = \varnothing$. If $|K| = 1$, we can apply Lemma 2.1(b) in conjunction with the isomorphism $S \cong V(\lambda_3 + \lambda_4)$ to deduce $H^1(G, S_K) = 0$. If $|K| \geqslant 2$, we choose n large enough so that $K \subseteq N = \{0, ..., n-1\}$. Then for any subset $L \subseteq N \backslash K$ and any element $r \in N \backslash (K \cup L)$, a comparison of masses using Lemma 1.3 reveals that $S_{K \cup L}$ is not a composition factor of $(S_r \otimes S_r) \otimes S_L \cong S_r \otimes S_{L \cup \{r\}}$. Therefore, Lemma 2.2 yields

$$d(S_L, S_{K \cup L}) \leqslant d(S_{L \cup \{r\}}, S_{K \cup L \cup \{r\}}).$$

It follows that $d(F, S_K) \leqslant d(S_{N \backslash K}, S_N)$, which is zero by Lemma 2.1(c) and the injectivity of S_N for $FG(n)$.

We may assume from now on that $I \cup J$ is not empty. Let a be its smallest element. Suppose $b \in I \cup J \backslash \{a\}$. Lemma 2.4 shows that if $b \in I$, then $S_b \otimes (E_I \otimes M_J) = (S_b \otimes E_b) \otimes E_{I \backslash \{b\}} \otimes M_J$ has a submodule isomorphic to $E_{I \backslash \{b\}} \otimes M_{J \backslash \{b\}}$, while if $b \in J$ then $S_b \otimes (E_I \otimes M_J)$ has a submodule isomorphic to $E_{I \cup \{b\}} \otimes M_{J \backslash \{b\}}$. Let (I', J') be $(I \backslash \{b\}, J \cup \{b\})$ or $(I \cup \{b\}, J \backslash \{b\})$ accordingly. Since $b > a$ and $a \in I' \cup J' = I \cup J$, Lemma 2.3 tells us that $[S_b \otimes (E_{I'} \otimes M_{J'}) : S_K] = 0$. Therefore, by Lemma 2.2,

$$d(E_I \otimes M_J, S_K) \leqslant d(E_{I'} \otimes M_{J'}, S_{K \cup \{b\}}).$$

Now if $c \in I' \cup J' \backslash \{a, b\}$, the same reasoning leads to

$$d(E_{I'} \otimes M_{J'}, S_{K \cup \{b\}}) \leqslant d(E_{I''} \otimes M_{J''}, S_{K \cup \{b, c\}}),$$

where (I'', J'') is obtained from (I', J') in the same way that the latter was obtained from (I, J). Continuing in this manner, we eventually arrive at

$$d(E_I \otimes M_J, S_K) \leqslant d(E_{I^*} \otimes M_{J^*}, S_{(I \cup J \cup K) \backslash \{a\}}), \tag{2.6.1}$$

where $I^* \cup J^* = I \cup J$ and $S_a \otimes (E_{I^*} \otimes M_{J^*})$ has a submodule isomorphic to $E_J \otimes M_I$.

We now consider the cases $|K| \geqslant 1$ and $|I \cup J| \geqslant 4$ separately.

To begin with, we suppose $|K| \geqslant 2$. By Lemma 1.3 we have

$$m(S_{(I \cup J \cup K) \setminus \{a\}}) \geqslant 5\,|I \cup J| + 5 > m(E_J \otimes M_I)),$$

so by Lemma 2.2 and (2.6.1) we have

$$d(E_I \otimes M_J, S_K) \leqslant d(E_J \otimes M_I, S_{I \cup J \cup K}).$$

Pick $N = \{0, ..., n-1\}$ to contain $I \cup J \cup K$. Then for any $L \subseteq N \setminus (I \cup J \cup K)$ and $r \in N \setminus (I \cup J \cup K \cup L)$ we have

$$m(S_{I \cup J \cup K \cup L}) \geqslant 5\,|L| + 5\,|I \cup J| + 10 > m((S_r \otimes S_r) \otimes E_J \otimes M_I \otimes S_L),$$

hence $[S_r \otimes (E_J \otimes M_I \otimes S_{L \cup \{r\}}) : S_{I \cup J \cup K \cup L}] = 0$. Then by Lemma 2.2, we have

$$d(E_J \otimes M_I \otimes S_L, S_{I \cup J \cup K \cup L}) \leqslant d(E_J \otimes M_I \otimes S_{L \cup \{r\}}, S_{I \cup J \cup K \cup L \cup \{r\}}).$$

It follows that

$$d(E_J \otimes M_I, S_K) \leqslant d(E_J \otimes M_I \otimes S_{N \setminus (I \cup J \cup K)}, S_N),$$

and the last term is zero by Lemma 2.1(c).

We next consider the case when K is a singleton set $\{k\}$, say. Suppose first that $|I \cup J| \geqslant 2$. Then we can see from Table II that

$$m(S_{(I \cup J \cup K) \setminus \{a\}}) \geqslant 5\,|I \cup J| \geqslant m(E_J \otimes M_I) + 2\,|I \cup J| \geqslant m(E_J \otimes M_I) + 4,$$

with equality if and only if $J = \emptyset$ and $|I| = 2$. Moreover, Lemma 1.3 and Table II show that $m(S_a \otimes (E_J \otimes M_I)) < m(E_J \otimes M_I) + 4$ unless $a \in I$. Therefore $S_{(I \cup J \cup K) \setminus \{a\}}$ is certainly not a composition factor of $S_a \otimes E_J \otimes M_I$ unless $J = \emptyset$ and $|I| = 2$. We now show that it is not a composition factor even in the case. Let $I = \{a, b\}$. We have $m(S_{\{k, b\}}) = 10$. Using Lemma 1.3 and Table II, we see that any composition factor of $S_a \otimes M_{\{a,b\}} = (S_a \otimes M_a) \otimes M_b$ of mass 10 must be a composition factor of $(E_{a+1} \otimes S_a) \otimes M_b$, since $Z_{a+1} \otimes S_a$ is the only composition factor of $S_a \otimes M_a$ of mass 7 or more. If $a + 1 \neq b$ then $(E_{a+1} \otimes S_a) \otimes M_b$ is simple and different from $S_{\{k,b\}}$, while if $a + 1 = b$, the only composition factor of $(E_b \otimes S_a) \otimes M_b$ of mass 10 is $S_{\{a,b\}}$, which is also different from $S_{\{k,b\}}$. Thus, we have shown

$$[S_a \otimes (E_J \otimes M_I) : S_{(I \cup J \cup K) \setminus \{a\}}] = 0.$$

Therefore by Lemma 2.2 and (2.6.1) we have

$$d(E_I \otimes M_J, S_K) \leqslant d(E_J \otimes M_I, S_{I \cup J \cup K}). \qquad (2.6.2)$$

We now argue as in the case $|K| \geqslant 2$ that the right hand side is zero. Choose $N = \{0, ..., n-1\}$ to contain $I \cup J \cup K$. For any $L \subseteq N \backslash (I \cup J \cup K)$ and $r \in N \backslash (I \cup J \cup K \cup L)$, we have by our assumptions on $|I \cup J|$ and $|K|$,

$$\begin{aligned} m(S_{I \cup J \cup K \cup L}) &\geqslant 5 + 5\,|I \cup J| + 5\,|L| \geqslant 5 + 2\,|I \cup J| + m(E_J \otimes M_I \otimes S_L) \\ &\geqslant 9 + m(E_J \otimes M_I \otimes S_L) > m(S_r \otimes (E_J \otimes M_I \otimes S_{L \cup \{r\}})), \end{aligned}$$

the last inequality being due to the fact that $m(S_r \otimes S_r) = 8$, from Table II. Then repeated applications of Lemma 2.2 lead to

$$d(E_J \otimes M_I, S_{I \cup J \cup K}) \leqslant d(E_J \otimes M_I \otimes S_{N \backslash (I \cup J \cup K)}, S_N),$$

and the right hand side is zero by Lemma 2.1(c). This completes the proof in the case $|K| = 1$, $|I \cup J| \geqslant 2$.

We turn to the situation where $K = \{k\}$ and $I \cup J = \{a\}$. Let b be the smaller of a and k and c the larger of the two and define $P = \{b, b+1, ..., c\}$ and $L = P \backslash \{a, k\}$.

Suppose first that $a \in J$. Then because $[S_a \otimes E_a : S_k] = 0$, Lemma 2.2 implies $d(M_a, S_k) \leqslant d(E_a, S_{\{a,k\}})$. Further, since Lemma 2.3 shows that $S_{\{a,k\}} = S_{\{b,c\}}$ is not a composition factor of $S_L \otimes (E_a \otimes S_L)$, we have $d(E_a, S_{\{a,k\}}) \leqslant d(E_a \otimes S_L, S_P)$. Let $N = \{0, ..., c-b\}$. Then Lemma 2.1(b) and (c) show that

$$d(E_a \otimes S_L, S_P) = d(E_{a-b} \otimes S_{N \backslash \{0, c-b\}}, S_N) = 0.$$

Now suppose $a \in I$ and that $k \neq a+1$. Then we have $[S_a \otimes E_a : S_k] = 0$ from Table II, which leads via Lemma 2.2 to

$$d(E_a, S_k) \leqslant d(M_a, S_{\{a,k\}}). \qquad (2.6.3)$$

We show that (2.6.3) also holds when $k = a+1$, but let us assume it for the time being and show how to finish the case $a \in I$. By Lemma 2.3, $[S_L \otimes (M_a \otimes S_L) : S_{\{a,k\}}] = 0$, so as in the previous paragraph, we obtain

$$d(M_a, S_{\{a,k\}}) \leqslant d(M_a \otimes S_L, S_P) = 0.$$

It remains to prove (2.6.3) for $k = a+1$. We wish to apply Lemma 2.2 by showing that $\mathrm{Hom}_G(M_a \otimes S_a, X(E_a, S_{a+1})) = 0$. Since S_a is a quotient of $M_a \otimes E_a$, by Lemma 2.4, it will suffice to show $\mathrm{Hom}_G(M_a \otimes M_a \otimes E_a, X(E_a, S_{a+1})) = 0$. Now the only composition factors of $M_a \otimes M_a$ whose tensor products with E_a have a composition factor S_{a+1} are $E_a \otimes M_{a+1}$ and $E_{a+1} \otimes M_a$. We have

$$\mathrm{Hom}_G(E_a \otimes M_{a+1} \otimes E_a, E_a) \cong \mathrm{Hom}_G(E_a \otimes M_{a+1}, E_a \otimes E_a) = 0$$

and

$$\mathrm{Hom}_G(E_a \otimes M_{a+1} \otimes E_a, S_{a+1}) \cong \mathrm{Hom}_G(E_a \otimes M_{a+1}, E_a \otimes S_{a+1}) = 0,$$

which shows $\mathrm{Hom}_G(E_a \otimes M_{a+1} \otimes E_a, X(E_a, S_{a+1})) = 0$, with a similar easy calculation to show $\mathrm{Hom}_G(E_{a+1} \otimes M_a \otimes E_a, X(E_a, S_{a+1})) = 0$. Therefore we have proved (2.6.3) in this remaining case.

This completes the proof of the proposition for $|K| \geqslant 1$, so we now consider the case $|I \cup J| \geqslant 4$, assuming, as we may, that $K = \varnothing$.

We show (in the notation of (2.6.1)) that

$$d(S_{(I \cup J) \setminus \{a\}}, E_{I^*} \otimes M_{J^*}) \leqslant d(S_{I \cup J}, E_J \otimes M_I). \tag{2.6.4}$$

Then we are finished, because Lemma 2.5 shows the right hand side to be zero and together with (2.6.1) this establishes the vanishing of $H^1(G, E_I \otimes M_J)$. Suppose first that $|I \cup J| \geqslant 5$. Then by Lemma 1.3 and Table II we have

$$m(S_{(I \cup J) \setminus \{a\}}) \geqslant m(E_J \otimes M_I) + 5 > m(S_a \otimes (E_J \otimes M_I)),$$

so (2.6.4) is an immediate consequence of Lemma 2.2. This leaves us with the case $|I \cup J| = 4$. Let $I \cup J = \{a, b, c, d\}$, with $a < b < c < d$. In order to prove (2.6.4), it is enough to show that $[S_a \otimes (E_J \otimes M_I) : S_{\{b,c,d\}}] = 0$. Now, $m(S_{\{b,c,d\}}) = 15$ whereas by Lemma 1.3 and Table II we have

$$m(S_a \otimes E_J \otimes M_I) \leqslant \begin{cases} 3 + 3\,|I| + 2\,|J| & \text{if } a \in J \\ 4 + 3\,|I| + 2\,|J| & \text{if } a \in I \end{cases}.$$

The right hand side is less than 15 unless $|J| \leqslant 1$ and unless $a \in I$ in the case $|J| = 1$. Therefore, we may assume $a \in I$. Then

$$m(S_a \otimes E_J \otimes M_I) = m((S_a \otimes M_a) \otimes E_J \otimes M_{I \setminus \{a\}}) \leqslant 7 + 9 = 16.$$

The only composition of $S_a \otimes M_a$ of mass 6 or more are $S_a \otimes E_{a+1}$, $M_{\{a, a+1\}}$ and $E_{\{a, a+1, a+2\}}$. Thus, the only composition factors of $S_a \otimes E_J \otimes M_I$ of mass at least 15 are composition factors of

$$E_{a+1} \otimes (E_J \otimes M_{I \setminus \{a\}} \otimes S_a), \qquad M_{a+1} \otimes (E_J \otimes M_I),$$

or

$$E_{\{a+1, a+2\}} \otimes (E_{J \cup \{a\}} \otimes M_{I \setminus \{a\}}).$$

Each of these three modules has the form of a tensor product of two simple modules, exactly one of which has a tensor factor E_a, M_a or S_a. Therefore, Lemma 2.3 shows that $S_{\{b,c,d\}}$ is not a composition factor of any of the three, and (2.6.4) follows.

This completes the proof of the proposition.

3. Calculations of $H^1(G, E_I \otimes M_J)$ for $|I \cup J| \leqslant 3$

To begin with, we summarize some facts about Weyl modules and their duals, the modules $H^0(\lambda)$. Our notation follows that of [Jantzen].

LEMMA 3.1. *Let $\lambda, \mu \in X_+$.*

(a) *If $\mu < \lambda$, then*

$$\mathrm{Ext}^1_G(L(\lambda), L(\mu)) \cong \mathrm{Hom}_G(\mathrm{rad}\, V(\lambda), L(\mu))$$
$$\cong \mathrm{Hom}_G(L(\mu), H^0(\lambda)/\mathrm{soc}\, H^0(\lambda)).$$

(b) $\mathrm{Ext}^1_G(V(\lambda), H^0(\mu)) = 0$.

(c) *$V(\lambda) \otimes V(\mu)$ has a descending filtration*

$$V = V^0 \supset V^1 \supset \cdots \supset V^r = 0,$$

with $V^i/V^{i+1} \cong V(v_i)$ for some $v_i \in X_+$, $(i = 0, ... r-1)$, and such that whenever $v_i > v_j$ we have $i > j$. In particular, $V(\lambda) \otimes V(\mu)$ has a submodule isomorphic to $V(\lambda + \mu)$.

Proof. Part (a) is Prop. 2.14, p. 207 of [Jantzen] and has already been used in this paper. Part (b) is a special case of Prop. 4.13, p. 236 in [Jantzen] and (c) is a theorem of Donkin [Donkin, 7.3.11].

In addition to these facts, we also use the Jantzen sum formula [Jantzen, Prop. 8.19, p. 314] in one or two places.

We divide our analyis of the groups $H^1(G, E_I \otimes M_J)$ according to $|I \cup J|$. The case $I \cup J = \varnothing$ has already been covered in Lemma 2.5. If $|I \cup J| = 1$, then by Lemma 2.1(b), we need only compute $H^1(G, E)$, $H^1(G, S)$ and $H(G, M)$. We have seen earlier that E and S are simple Weyl modules, so the first two are zero. A very easy computation shows that the composition factors of rad $V(\lambda_3)$ are E and F. Since $H^1(G, E) = 0$, the radical must be semisimple. Thus, $H^1(G, M) \cong F$ by Lemma 3.1(a).

Now suppose $|I \cup J| > 1$. By Lemma 2.1(b), we are reduced to the case where $0 \in I \cup J$. Define (I', J') to be $(J \cup \{0\}, I \backslash \{0\})$ if $0 \in I$ and to be $(J \backslash \{0\}, I \cup \{0\})$ if $0 \in J$. Then by Lemma 2.4, $S_{(I \cup J) \backslash \{0\}} \otimes (E_I \otimes M_J)$ has a submodule isomorphic to $E_{I'} \otimes M_{J'}$. Also, since $0 \in I' \cup J'$, Lemma 2.3 shows that $[S_{(I \cup J) \setminus \{0\}} \otimes (E_{I'} \otimes M_{J'}) : F] = 0$. Therefore, Lemma 2.2 yields

$$d(E_I \otimes M_J, F) \leqslant d(E_{I'} \otimes M_{J'}, S_{(I \cup J) \setminus \{0\}}). \tag{3.2}$$

Also, we note that $S \otimes (E_{I'} \otimes M_{J'})$ has a submodule isomorphic to $E_J \otimes M_I$. All of this leads to the following statement which will provide the basis for a large part of the remaining calculations.

LEMMA 3.3. *If* $[S \otimes E_J \otimes M_I : S_{(I \cup J) \setminus \{0\}}] = 0$ *then* $H^1(G, E_I \otimes M_J) = 0$.

Proof. Our hypothesis and Lemma 2.2 show that

$$d(E_{I'} \otimes M_{J'}, S_{(I \cup J) \setminus \{0\}}) \leqslant d(E_J \otimes M_I, S_{I \cup J})$$

and the right hand side is zero by Lemma 2.5. The lemma follows from (3.2).

A related result is the following lemma.

LEMMA 3.4. *Suppose* I *and* J *are partitioned as* $I = A \cup B$ *and* $J = C \cup D$. *Assume one of the following*:

(a) $0 \in A$ *and* $\mathrm{Hom}_G(S \otimes (E_{A \setminus \{0\}} \otimes M_{C \cup \{0\}}), X(E_A \otimes M_C, E_B \otimes M_D)) = 0$, *or*

(b) $0 \in C$ *and* $\mathrm{Hom}_G(S \otimes (E_{A \cup \{0\}} \otimes M_{C \setminus \{0\}}), X(E_A \otimes M_C, E_B \otimes M_D)) = 0$.

Then $H^1(G, E_I \otimes M_J) = 0$.

Proof. The arguments for the two parts are similar, so we prove (a) only. By Lemma 2.4, the module $E_{A \setminus \{0\}} \otimes M_{C \cup \{0\}}$ embeds into $S \otimes (E_A \otimes M_C)$. Therefore the hypothesis and Lemma 2.2 give

$$\begin{aligned} d(E_I \otimes M_J, F) &= d(E_A \otimes M_C, E_B \otimes M_D) \\ &\leqslant d(E_{A \setminus \{0\}} \otimes M_{C \cup \{0\}}, E_B \otimes M_D \otimes S) \\ &= d(E_{I \setminus \{0\}} \otimes M_{J \cup \{0\}}, S). \end{aligned}$$

Then since by Lemma 2.3, $S_{(I \cup J) \setminus \{0\}} \otimes (E_{I \setminus \{0\}} \otimes M_{J \cup \{0\}})$ has no composition factor S and since it has a submodule isomorphic to $E_J \otimes M_I$, by Lemma 2.4, we may apply Lemma 2.2 and Lemma 2.5 to obtain

$$d(E_{I \setminus \{0\}} \otimes M_{J \cup \{0\}}, S) \leqslant d(E_J \otimes M_I, S_{I \cup J}) = 0.$$

At this point, we simplify our notation a little by omitting set braces on subscripts, writing $E_{1,2}$ for $E_{\{1,2\}}$, etc.

The Case $|I \cup J| = 3$. Let $I \cup J = \{0, a, b\}$ with $0 < a < b$ and let $\{c, d\} = \{a, b\}$.

(a) $|I| = 0$. We shall show $H^1(G, M_{0,a,b}) = 0$. By Lemma 3.3 it suffices to show that $S_{a,b}$ is not a composition factor of $S \otimes E_{0,a,b} = (S \otimes E) \otimes E_{a,b}$, which follows immediately from a comparison of masses using Lemma 1.3 and Table II.

(b) $|I| = 1$. We subdivide this case into (i) $0 \in I$ and (ii) $0 \in J$.

(i) We prove $H^1(G, E \otimes M_{a,b}) = 0$. By Lemma 3.3, we need only show that $[(S \otimes M) \otimes E_{a,b} : S_{a,b}] = 0$. Since $m(S_{a,b}) = 10$, we may restrict our attention to those composition factors of $S \otimes M$ of mass 6 or more, these being $E_1 \otimes S$, $M_{0,1}$ and $E_{0,1,2}$. We note that each of these has a tensor factor E, M or S, so by Lemma 2.3 $S_{a,b}$ cannot be a composition factor of the tensor product of any of them with $E_{a,b}$. This completes (i).

(ii) We wish to show $H^1(G, E_c \otimes M_{0,d}) = 0$, so to apply Lemma 3.3 we want $[(S \otimes E) \otimes M_c \otimes E_d : S_{a,b}] = 0$. Since $m(M_c \otimes E_d) = 5$, we need only to consider the tensor product of $M_c \otimes E_d$ with those composition factors of $S \otimes E$ of mass 5 or more. These are S, $E_1 \otimes S$, $E \otimes M_1$ and $E_2 \otimes M$. As in (i), Lemma 2.3 gives the desired result. This completes (ii) and with it, case (b)

(c) $|I| = 2$. Here we again split the analysis into the subcases (i) $0 \in J$ and (ii) $0 \in I$.

(i) We aim to show $H^1(G, E_{a,b} \otimes M) = 0$. By Lemma 3.3, it will suffice to prove $[(S \otimes E) \otimes M_{a,b} : S_{a,b}] = 0$. Arguing as we did for (a) and (b), we need only consider those composition factors of $S \otimes E$ which have mass at least 4 and which in addition have no tensor factors E, M or S. This leaves only $E_{1,2}$. It is easy to see that $S_{a,b}$ will be not a composition factor of $E_{1,2} \otimes M_{a,b}$ unless $a = 1$ and $b = 2$, so with this exception our argument is finished. The proof of

$$H^1(G, E_{1,2} \otimes M) = 0 \tag{1}$$

is postponed until near the end of Section 3 when we deal with several similar calculations left over from the present discussion.

(ii) In order to show $H^1(G, E_{0,c} \otimes M_d) = 0$ using Lemma 3.3 we must show $[(S \otimes M) \otimes M_c \otimes E_d : S_{a,b}] = 0$. The sole composition factor of $S \otimes M$ having mass at least 5 and not having a tensor factor E, M or S is $E_2 \otimes M_1$. One easily sees that $[(E_2 \otimes M_1) \otimes (E_d \otimes M_c) : S_{a,b}] = 0$ except for when $d = a = 1$ and $c = b = 2$. Thus, it remains to show $H^1(G, E_{0,2} \otimes M_1) = 0$. We apply Lemma 3.4, with $A = \{0\}$, $B = \{2\}$, $C = \varnothing$ and $D = \{1\}$, by showing

$$\mathrm{Hom}_G(M \otimes S, X(E, E_2 \otimes M_1)) = 0.$$

Since S is an homomorphic image of $M\otimes E$, we may replace $M\otimes S$ by $(M\otimes M)\otimes E$ in the equation above. It is not hard to see that the only composition factor of $M\otimes M$ whose tensor product with E has $E_2\otimes M_1$ as a composition factor is $E\otimes M_1$. Thus, the equation follows from

$$\mathrm{Hom}_G(E\otimes M_1\otimes E, E)\cong \mathrm{Hom}_G(E\otimes M_1, E\otimes E)=0$$

and

$$\mathrm{Hom}_G(E\otimes M_1\otimes E, E_2\otimes M_1)\cong \mathrm{Hom}_G(E\otimes M_1, E_{0,2}\otimes M_1)=0.$$

This finishes case (c).

(d) $|I|=3$. Our goal is to show $H^1(G, E_{0,a,b})=0$. Bearing in mind Lemma 3.3, we are interested in determining when $S_{a,b}$ is a composition factor of $(S\otimes M)\otimes M_{a,b}$. The usual considerations of masses and τ-restricted tensor factors show that the critical composition factors of $S\otimes M$ are $E_{1,2}$, S_1 and $E_2\otimes M_1$. By Lemma 2.3, it is clear that in order for the tensor product of any of these with $M_{a,b}$ to have $S_{a,b}$ as a composition factor we must have $a=1$, so we assume this. Then, we have $[E_{1,2}\otimes M_{1,b}:S_{1,b}]=0$ unless $b=2$. Next consider $S_1\otimes M_{1,b}=(S_1\otimes M_1)\otimes M_b$. Out of all the composition factors of $S_1\otimes M_1$ only $E_2\otimes S_1$ has the property that its tensor product with M_b could possibly have $S_{1,b}$ as a composition factor, and then only when $b=2$. Finally, consider $E_2\otimes M_1\otimes M_{1,b}=(M_1\otimes M_1)\otimes E_2\otimes M_b$. Since $m(E_2\otimes M_b)=5$, we are interested in the composition factors of $M_1\otimes M_1$ of mass at least 5, namely, $M_1\otimes E_2$ and $M_2\otimes E_1$. Lemma 2.3 shows that $S_{1,b}$ is a composition factor neither of $E_2\otimes M_1\otimes E_2\otimes M_b$ nor of $E_1\otimes M_2\otimes E_2\otimes M_b$. Thus we have shown that $H^1(G, E_{0,a,b})=0$ unless $a=1$ and $b=2$. The proof of

$$H^1(G, E_{0,1,2})=0 \tag{2}$$

is given later.

The Case $|I\cup J|=2$. Let $I\cup J=\{0, a\}$.

(a) $|I|=0$. We would like to prove $H^1(G, M_{0,a})=0$. By Lemma 3.3, we know this to be true for those values of a for which $[S\otimes E_{0,a}:S_a]=0$. The composition factors of $S\otimes E$ which are relevant are those which have no tensor factor E, M or S and which have mass no less than 3. These are $E_{1,2}$, M_1 and M_2. Therefore it is clear that for $a>2$, the multiplicity in question is zero, and we are left to show

$$H^1(G, M_{0,1})=0 \tag{3}$$

and

$$H^1(G, M_{0,2}) = 0. \tag{4}$$

This is done later.

(b) $|I| = 1$. There are two subcases: (i) $0 \in I$ and (ii) $0 \in J$.

(i) We prove $H^1(G, E \otimes M_a) = 0$. It is straightforward to check that $[S \otimes M \otimes E_a : S_a] = 0$ unless $a = 1$ or 2, so Lemma 3.3 gives the result for all values of a except for these. We leave the proof of

$$H^1(G, E \otimes M_1) = 0 \tag{5}$$

until later but we give the argument for the triviality of $H^1(G, E \otimes M_2)$ here, using Lemma 3.4, with $A = \{0\}$, $B = C = \varnothing$ and $D = \{2\}$. Thus, we aim to show

$$\mathrm{Hom}_G(S \otimes M, X(E, M_2)) = 0.$$

By Lemma 2.4, it is enough to prove this with $E \otimes M \otimes M$ in place of $S \otimes M$. Now, the only composition factors of $M \otimes M$ whose tensor products with E have M_2 as a composition factor are $E \otimes M_1$, $E_{0,2}$ and $E_1 \otimes M$. A little thought shows

$$\mathrm{Hom}_G(E \otimes E \otimes M_1, E) = 0 \qquad \text{and} \qquad \mathrm{Hom}_G(E \otimes E \otimes M_1, M_2) = 0.$$

Hence $\mathrm{Hom}_G(E \otimes E \otimes M_1, X(E, M_2)) = 0$. Similarly, one can also prove $\mathrm{Hom}_G(E \otimes E_{0,2}, X(E, M_2)) = 0$ and $\mathrm{Hom}_G(E \otimes E_1 \otimes M, X(E, M_2)) = 0$, so Lemma 3.4 can be applied to yield the desired result.

(ii) According to the statement of the main theorem, we must try to show $H^1(G, E_a \otimes M) \cong 0$ if $a \neq 2$ and $H^1(G, E_2 \otimes M) \cong F$. If $a > 3$ then it is easily checked that $[S \otimes E \otimes M_a : S_a] = 0$, and the result then follows from Lemma 3.3. Next, we recall that $E_3 \cong L(2\lambda_1)$, $E_1 \cong L(\lambda_1)$ and $M = L(\lambda_3)$. We have $\lambda_1 < \lambda_3 < 2\lambda_1$. Thus, by Lemma 3.1(a) and the isomorphism $\mathrm{rad}\, V(\lambda_3) \cong E \oplus F$ proved earlier (See the case $|I \cup J| = 1$), we see that $H^1(G, M \otimes E_1) \cong \mathrm{Ext}^1_G(L(\lambda_3), L(\lambda_1)) = 0$. Also, a routine calculation reveals that $L(\lambda_3)$ is not even a composition factor of $V(2\lambda_1)$, so certainly $H^1(G, E_3 \otimes M) \cong \mathrm{Ext}^1_G(L(2\lambda_1), L(\lambda_3)) = 0$. The isomorphism

$$H^1(G, E_2 \otimes M) \cong F \tag{6}$$

is established later.

(c) $|I| = 2$. We wish to show $H^1(G, E_{0,a}) = 0$ for $a \neq 1$ and $H^1(G, E_{0,1}) = 0$. The latter follows immediately from the fact that $\mathrm{rad}\, V(\lambda_1) \cong L(\lambda_4)$, which can be verified by a very simple character calculation. Next, if $a > 4$ then it can easily be seen that $[S \otimes M_{0,a} : S_a] = 0$, so

Lemma 3.3 yields the result. For $a=3$ or 4, we apply Lemma 3.4 with $A=\{0\}$, $B=\{a\}$, $C=D=\varnothing$. Since S is a quotient of $E\otimes M$, we are able to apply Lemma 3.4 if we show

$$\mathrm{Hom}_G((E\otimes M)\otimes M, X(E, E_a))=0.$$

It is not hard to see that the composition factors of $M\otimes M$ whose tensor products with E have E_a as a composition factor are $E_{0,1}$, $E_{0,2}$, $E_1\otimes M$ and $E\otimes M_1$. Taking the first of these, we have

$$\mathrm{Hom}_G(E\otimes E_{0,1}, E)\cong \mathrm{Hom}_G(E_{0,1}, E\otimes E)=0$$

and

$$\mathrm{Hom}_G(E\otimes E_{0,1}, E_a)\cong \mathrm{Hom}_G(E_{0,1}, E\otimes E_a)=0,$$

so $\mathrm{Hom}_G(E\otimes E_{0,1}, X(E, E_a))=0$. Similar calculations show that there are no nonzero G-maps from the tensor products of E with the other composition factors in our list into $X(E, E_a)$, so the hypotheses of Lemma 3.4 are established and $H^1(G, E_{0,a})=0$ when $a=3$ or 4. We are left to show

$$H^1(G, E_{0,2})=0. \tag{7}$$

Completion of the Proof. At this point, all that remains is to prove (1)–(7) above. Parts (6) and (7) follow directly from the following lemma.

LEMMA 3.5. *The radical series of $V(2\lambda_4)$ is*

$$L(2\lambda_4), \qquad L(\lambda_3)\oplus L(\lambda_1), \qquad L(\lambda_4).$$

Proof. By computing the composition factors of $V(2\lambda_4)$ and applying the Jantzen sum formula, it can be seen that $V(2\lambda_4)$ has a descending filtration whose subquotients have the composition factors as given in the statement. Therefore, since we have already proved $\mathrm{Ext}^1_G(L(\lambda_3), L(\lambda_1))=0$, the middle layer is semisimple. We claim that $\mathrm{rad}^2 V(2\lambda_4)\neq 0$. If not, we would have $\mathrm{Hom}_G(L(\lambda_1), V(2\lambda_4))\neq 0$, but this is impossible because by Lemma 3.1(c), $V(2\lambda_4)$ embeds into $V(\lambda_4)\otimes V(\lambda_4)\cong E\otimes E$ and clearly $\mathrm{Hom}_G(E_1, E\otimes E)=0$. The lemma follows from the claim.

We next prove (2) which is equivalent to the vanishing of $\mathrm{Ext}^1_G(L(\lambda_1+\lambda_4), L(2\lambda_4))$. A routine computation gives the composition factors of $V(\lambda_1+\lambda_4)$ as

$$L(\lambda_1+\lambda_4), \qquad L(2\lambda_4), \qquad L(\lambda_3),$$
$$L(\lambda_1)\ (\text{twice}), \qquad L(\lambda_4)\ (\text{twice}), \qquad \text{and} \qquad L(0).$$

We know by Lemma 3.1(c) that $V(\lambda_1+\lambda_4)$ embeds into $V(\lambda_1)\otimes V(\lambda_4)$ and we identify it with its image. Then since

$$\text{hd } V(\lambda_1+\lambda_4)\cong L(\lambda_1+\lambda_4)\cong L(\lambda_1)\otimes L(\lambda_4)\cong \text{hd } V(\lambda_1)\otimes V(\lambda_4),$$

it follows that

$$\text{rad } V(\lambda_1+\lambda_4)\subseteq \text{rad } V(\lambda_1)\otimes V(\lambda_4)\cong V(\lambda_4)\otimes V(\lambda_4),$$

where the isomorphism follows immediately from what we already know about $V(\lambda_1)$ and $V(\lambda_4)$. Now since rad $V(\lambda_1+\lambda_4)$ has a composition factor $L(2\lambda_4)$, it contains the unique one-dimensional $2\lambda_4$-weight space of $V(\lambda_4)\otimes V(\lambda_4)$, and hence it has a submodule isomorphic to $V(2\lambda_4)$. Then from the structure of $V(2\lambda_4)$ given in Lemma 3.5, we see that in any (ascending) composition series of $V(\lambda_4)\otimes V(\lambda_4)$ we must have a composition factor $L(\lambda_1)$ below the unique composition factor $L(2\lambda_4)$. Then the self-duality of $V(\lambda_4)\otimes V(\lambda_4)$ shows that the three composition factors $L(\lambda_1)$ (twice) and $L(2\lambda_4)$ must always appear in the order $L(\lambda_1)$, $L(2\lambda_4)$, $L(\lambda_1)$. Since the composition multiplicities of these factors are the same in the submodule rad $V(\lambda_1+\lambda_4)$ as they are in $V(\lambda_4)\otimes V(\lambda_4)$ it follows that these factors must appear in the given order in any composition series of rad $V(\lambda_1+\lambda_4)$, which proves (2).

We now prove (1), or rather the equivalent statement $\text{Ext}^1_G(L(\lambda_1+2\lambda_4), L(\lambda_3))=0$. By Lemma 3.1(c) we may identify $V(\lambda_1+2\lambda_4)$ with a submodule of $V(\lambda_1)\otimes V(2\lambda_4)$. From the structures of $V(\lambda_1)$ and $V(2\lambda_4)$ already described, we know that their tensor product has a submodule W isomorphic to $L(\lambda_4)\otimes L(\lambda_4)\cong V(\lambda_4)\otimes V(\lambda_4)$. It can be checked that

$$[V(\lambda_1)\otimes V(2\lambda_4):L(2\lambda_4)]=3=[V(\lambda_1+\lambda_4):L(2\lambda_4)],$$

and since $[W:L(2\lambda_4)]=1$, we must have $[W\cap \text{rad } V(\lambda_1+2\lambda_4):L(2\lambda_4)]=1$. It follows as in the preceding paragraph that rad $V(\lambda_1+2\lambda_4)$ has a submodule isomorhic to $V(2\lambda_4)$. By a routine computation, we obtain

$$[V(\lambda_1+2\lambda_4):L(\lambda_3)]=1=[V(2\lambda_4):L(\lambda_3)]$$

and now (1) is clear.

Next, we shall prove $\text{Ext}^1_G(L(2\lambda_3), L(\lambda_3))=0$, which is equivalent to (4). We have already shown that rad $V(\lambda_3)\cong L(\lambda_4)\oplus F$. We consider the image U of $V(2\lambda_3)$ under the composition

$$V(2\lambda_3)\subsetneq V(\lambda_3)\otimes V(\lambda_3)\to L(\lambda_3)\otimes L(\lambda_3)=M\otimes M,$$

where the embedding is given by Lemma 3.1(c) and the second map is induced by the natural map of $V(\lambda_3)$ onto its head. The composition factors of $V(2\lambda_2)$ are readily computed and it turns out that $L(\lambda_1+2\lambda_4)\cong$

$E_{1,2}$ is one. However, it is easy to see using Table II that the kernel of the above maps has no such composition factor. Thus, $[U : E_{1,2}] \neq 0$. Since $\mathrm{Hom}_G(E_{1,2}, M \otimes M)) \cong \mathrm{Hom}_G(E_1 \otimes M, E_2 \otimes M) = 0$ and $\mathrm{hd}\, U \cong L(2\lambda_3) = M_2$, we deduce that $U \not\subseteq \mathrm{soc}^2(M \otimes M)$. Now $[M \otimes M : M_2] = 1$, so $[\mathrm{soc}^2(M \otimes M) : M_2] = 0$. Then by the self-duality of $M \otimes M$ we obtain

$$[(M \otimes M)/\mathrm{rad}^2(M \otimes M) : M_2] = 0. \tag{3.6}$$

We use this in conjunction with Lemma 3.4 (taking $A = B = \varnothing$, $C = \{0\}$ and $D = \{2\}$), so we need to show

$$\mathrm{Hom}_G(S \otimes E, X(M, M_2)) = 0.$$

By Lemma 2.4, we may prove this with $M \otimes E \otimes E$ in place of $S \otimes E$. The only composition factor of $E \otimes E$ whose tensor product with M has a composition factor M_2 is M. Thus, we are reduced to proving $\mathrm{Hom}_G(M \otimes M, X(M, M_2)) = 0$, which is clear from (3.6). This completes the proof (4).

In order to prove (3) and (5) we take a closer look at the module $V(\lambda_2)$.

LEMMA 3.7. *We have*

$$V(\lambda_3) \otimes V(\lambda_4) \cong V(\lambda_3 + \lambda_4) \oplus Y,$$

where the module Y has a submodule W isomorphic to $V(\lambda_2)$.

Proof. A character calculation shows that there is a Weyl filtration of $V(\lambda_3) \otimes V(\lambda_4)$ as provided by Lemma 3.1(c), in which $V^{r-1} \cong V(\lambda_3 + \lambda_4)$ and $V^{r-2}/V^{r-1} \cong V(\lambda_2)$. Since $V(\lambda_3 + \lambda_4) \cong H^0(\lambda_3 + \lambda_4)$, the lemma follows from Lemma 3.1(b).

The following information was obtained by direct calculation.

LEMMA 3.8. *The composition factors of $V(\lambda_2)$ are*

M_1, $E_{0,1}$, E_2, M, E_1 *(twice)*, E, *and* F *(twice)*.

The Jantzen sum formula gives

$$\sum_{i>0} \mathrm{Ch}\, V(\lambda_2)^i = \mathrm{Ch}\, E_{0,1} + 2\,\mathrm{Ch}\, E_2 + 2\,\mathrm{Ch}\, E_1 + 4\,\mathrm{Ch}\, M + 3\,\mathrm{Ch}\, E + 4\,\mathrm{Ch}\, F.$$

(Here, $\{V(\lambda_2)^i\}_{i>0}$ is the Jantzen filtration; one has $V(\lambda_2)^1 = \mathrm{rad}\, V(\lambda_2)$.)

We can now prove (3) $\mathrm{Ext}^1_G(L(\lambda_2), L(\lambda_3)) = 0$. One can deduce from Lemma 3.8 that $V(\lambda_2)$ has a submodule whose composition factors are E_2, M, E, and F. Suppose for a contradiction that $\mathrm{Ext}^1_G(M_1, M) \cong$

$\mathrm{Hom}_G(\mathrm{rad}\ V(\lambda_2), M)$ is not zero. Then since $[V(\lambda_2) : M] = 1$, there is a submodule of $V(\lambda_2)$ whose composition factors are E_2, E, and F. Since we have already shown that $\mathrm{Ext}^1_G(E_2, E) = 0$ and $\mathrm{Ext}^1_G(E_2, F) = 0$, it follows that $\mathrm{Hom}_G(E_2, V(\lambda_2)) \neq 0$. But this contradicts Lemma 3.7 because $\mathrm{Hom}_G(E_2, V(\lambda_2) \otimes V(\lambda_4)) \cong \mathrm{Hom}_G(E_{0,2}, V(\lambda_3)) = 0$, from the structure of $V(\lambda_3)$. Thus, (3) is proved.

Finally, we must prove (5) $\mathrm{Ext}^1_G(M_1, E) = 0$. First we show $\mathrm{Ext}^1_G(M_1, E_{0,1}) = 0$. From Lemma 3.7, it follows that the first layer $L = V(\lambda)^1/V(\lambda)^2$ in the Jantzen filtration has composition factors E_1 (twice), $E_{0,1}$ (once), and possibly F (once). Since $[V(\lambda_2) : E_{0,1}] = 1$, it will suffice to show $\mathrm{Hom}_G(L, E_{0,1}) = 0$. Now, for groups of type F_4 the layers of the Jantzen filtration are known to be self-dual G-modules [Jantzen, pp. 312–313]. Therefore, if $E_{0,1}$ were a homomorphic image of L, it would be isomorphic to a direct summand, and then it would follow from the fact that $H^1(G, E_1) = 0$ that $\dim \mathrm{Hom}_G(L, E_1) \geqslant 2$. But this would contradict the fact that $\mathrm{Ext}^1_G(M_1, E_1) \cong \mathrm{Ext}^1_G(M, E) \cong F$, so we must have $\mathrm{Ext}^1_G(M_1, E_{0,1}) = 0$.

Next we claim that $\mathrm{Hom}_G(M \otimes E, E) = 0$. Consider the exact sequence

$$0 \to E \otimes M_1 \to V(\lambda_1) \otimes M_1 \to E_1 \otimes M_1 \to 0,$$

obtained using the structure of $V(\lambda_1)$. In the long exact sequence for $\mathrm{Ext}_G(E_1, -)$, we have

$$\mathrm{Hom}_G(E_1, V(\lambda_1) \otimes M_1) \cong \mathrm{Hom}_G(E_1 \otimes M_1, V(\lambda_1)) = 0$$

because $[E_1 \otimes M_1 : E] = 0$, obviously. Also, we proved above that

$$\mathrm{Ext}^1_G(E_1, E \otimes M_1) \cong \mathrm{Ext}^1_G(M_1, E_{0,1}) = 0.$$

Therefore,

$$\mathrm{Hom}_G(M_1 \otimes E_1, E_1) \cong \mathrm{Hom}_G(E_1, E_1 \otimes M_1) = 0,$$

and our claim follows.

With this information, we are ready to prove (5). By Lemma 2.4, we have

$$M \otimes E \cong S \oplus Q$$

for some nonzero module Q. We show that Q has a submodule isomorphic to M by eliminating all of the other composition factors of Q as possibilities for simple submodules. Since $[M \otimes E : S] = 1$, this is the same as showing that none of the composition factors of $M \otimes E$ other than S and M are isomorphic to submodules of $M \otimes E$. We showed in the last paragraph that there is no submodule isomorphic to E and it is not difficult to eliminate all of the remaining composition factors; for instance,

$$\mathrm{Hom}_G(E_{0,1}, M \otimes E) \cong \mathrm{Hom}_G(E_1 \otimes M, E \otimes E) = 0.$$

Thus, M is isomorphic to a submodule of Q, so there exists an exact sequence

$$0 \longrightarrow M_1 \xrightarrow{i} M_1 \otimes E_1 \longrightarrow (M_1 \otimes E_1)/i(M_1) \longrightarrow 0$$

Since $[M_1 \otimes E_1 : E] = 0$, the long exact sequence yields an injection of $\mathrm{Ext}^1_G(E, M_1) \cong \mathrm{Ext}^1_G(M_1, E)$ into $\mathrm{Ext}^1_G(E, M_1 \otimes E_1) \cong \mathrm{Ext}^1_G(M_1, E_{0,1}) = 0$, so (5) is proved.

The proof of the main theorem is now complete.

Note added in proof. All extensions of simple modules are now known. These calculations will appear later in this journal.

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