The critical group of a graph

Peter Sin

Texas State U., San Marcos, March 21th, 2014.

Critical groups of graphs

Outline

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- The Paley graphs are a very important class of strongly regular graphs arising from finite fields.
- We'll say something about the computation of their critical groups, which involves groups, characters and number theory.

Critical groups of graphs

Pierre-Simon Laplace (1749-1827)

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- $\operatorname{rank}(L) = |V| - 1$.

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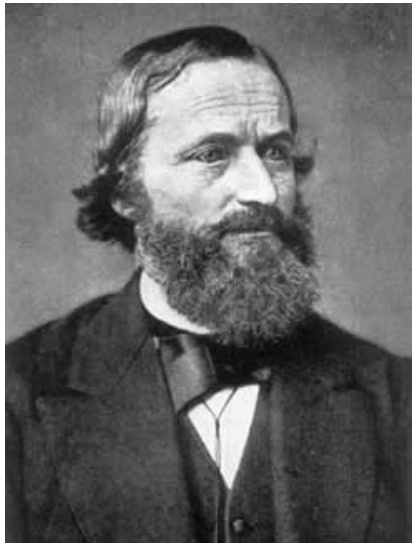
- Let $\varepsilon : \mathbf{Z}^V \to \mathbf{Z}$, $\sum_{v \in V} a_v v \mapsto \sum_{v \in V} a_v$.
- $L(\ker(\varepsilon)) \subseteq \ker(\varepsilon)$, and $K(\Gamma) \cong \mathrm{Ker}(\varepsilon)/L(\mathrm{Ker}(\varepsilon))$

Kirchhoff's Matrix-Tree Theorem



Gustav Kirchhoff (1824-1887)

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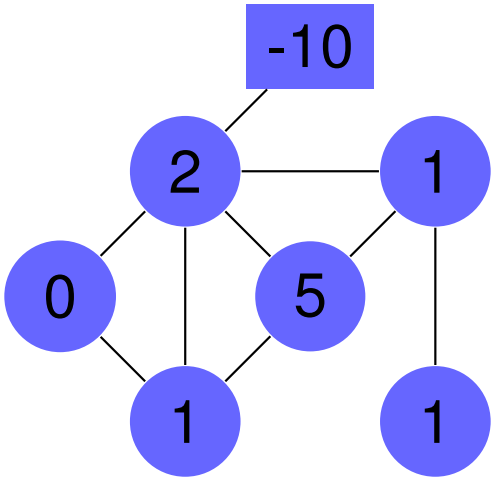
Also, $\det(\tilde{L}) = |K(\Gamma)| = \frac{1}{|V|} \prod_{j=2}^{|V|} \lambda_j$.

Critical groups of graphs

Rules



- A configuration is an assignment of a nonnegative integer $s(v)$ to each round vertex v and $-\sum_v s(v)$ to the square vertex.

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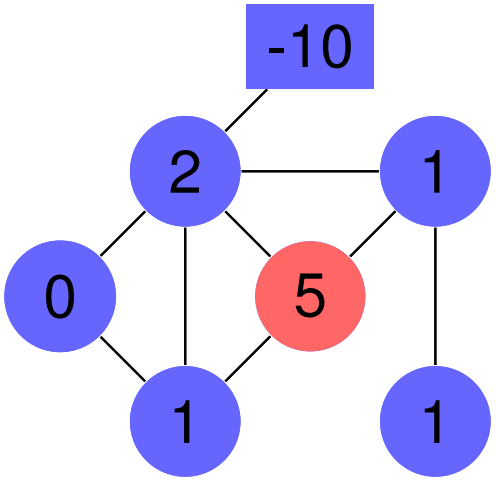

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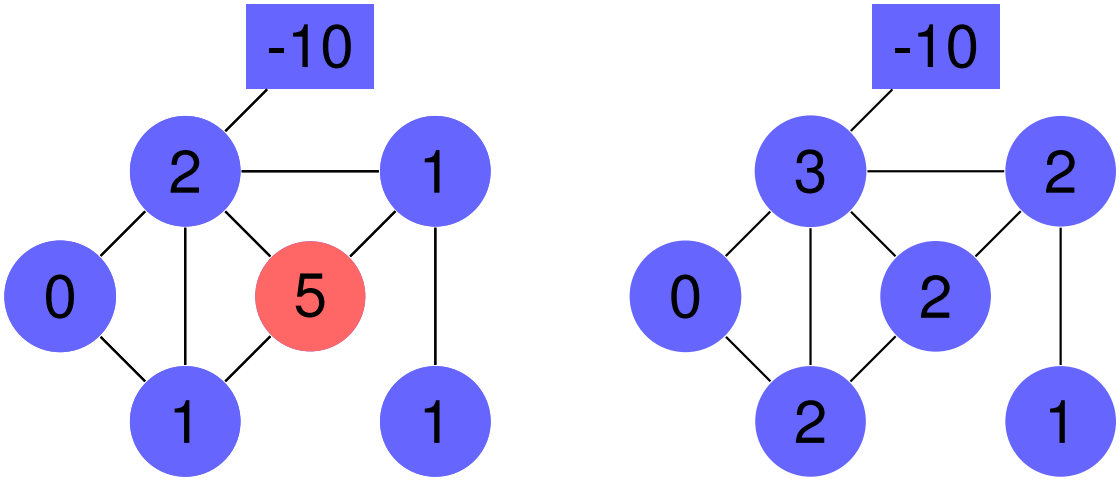

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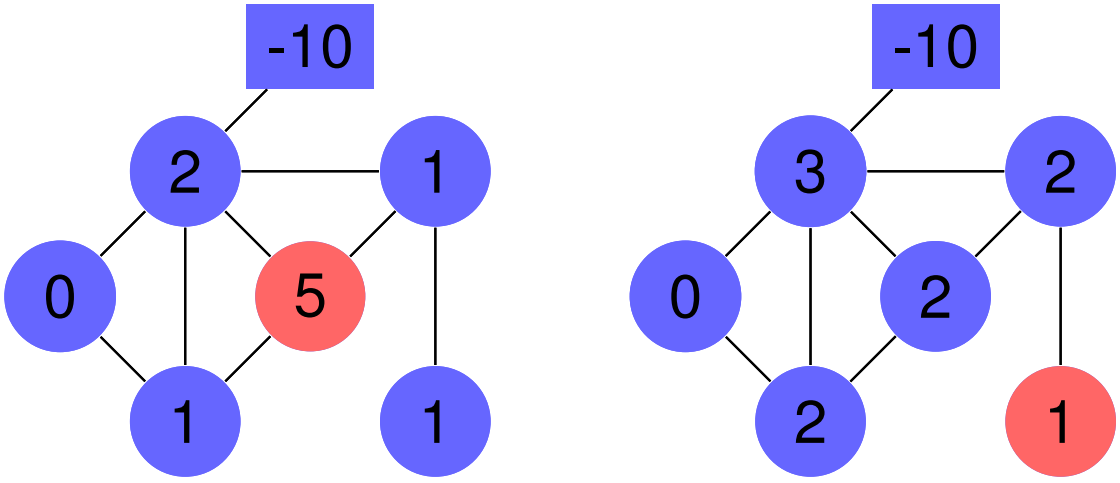

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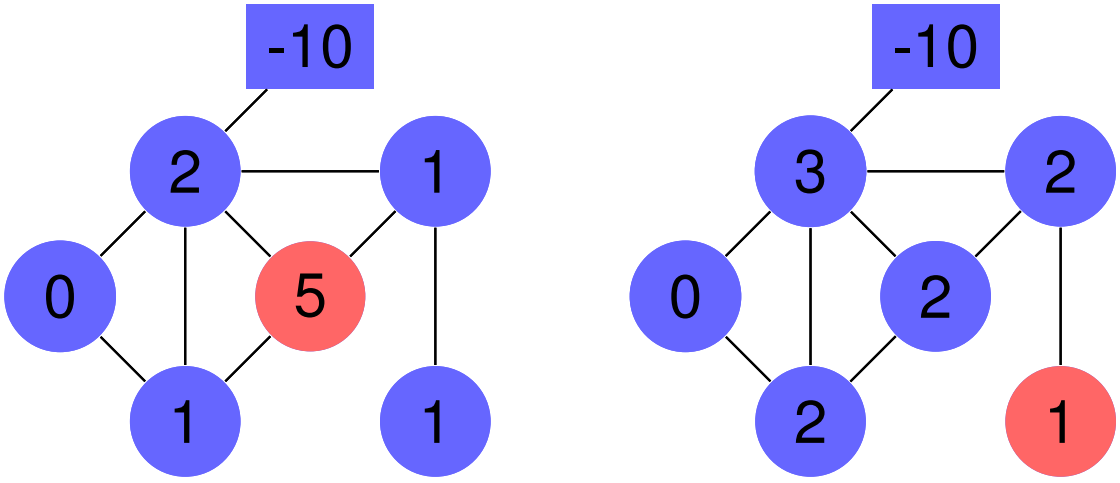

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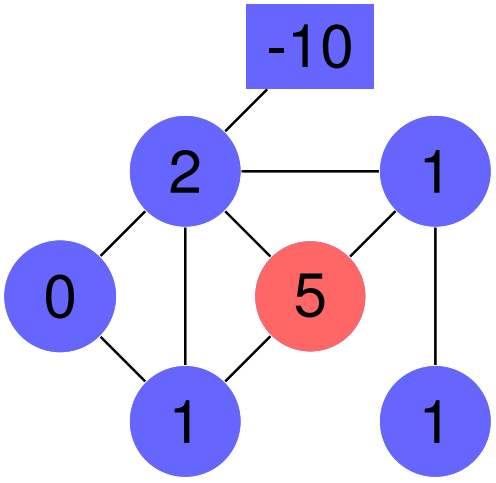

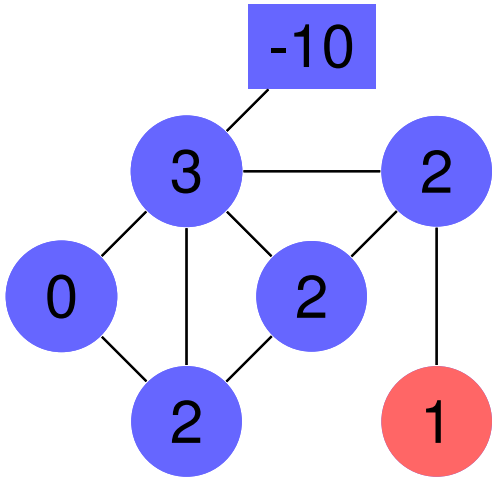

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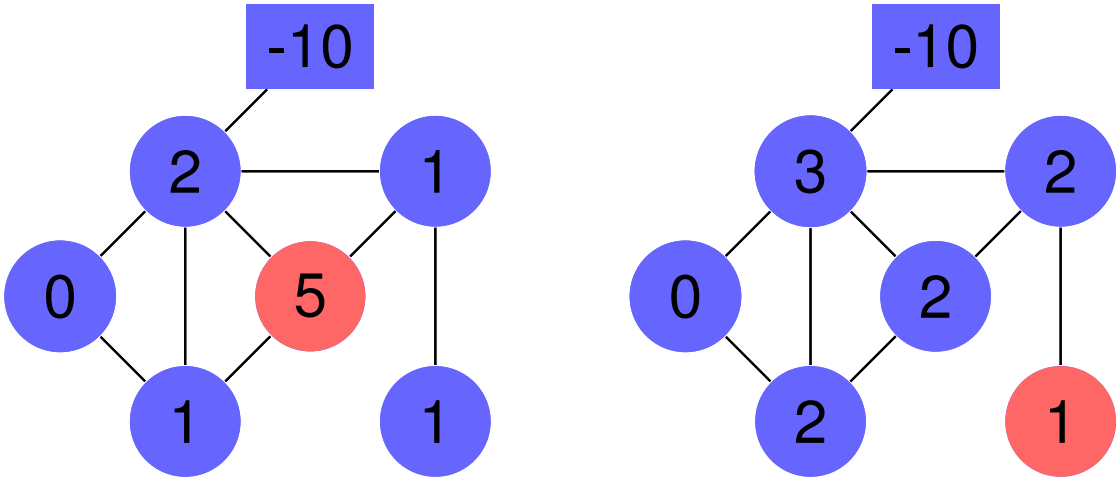

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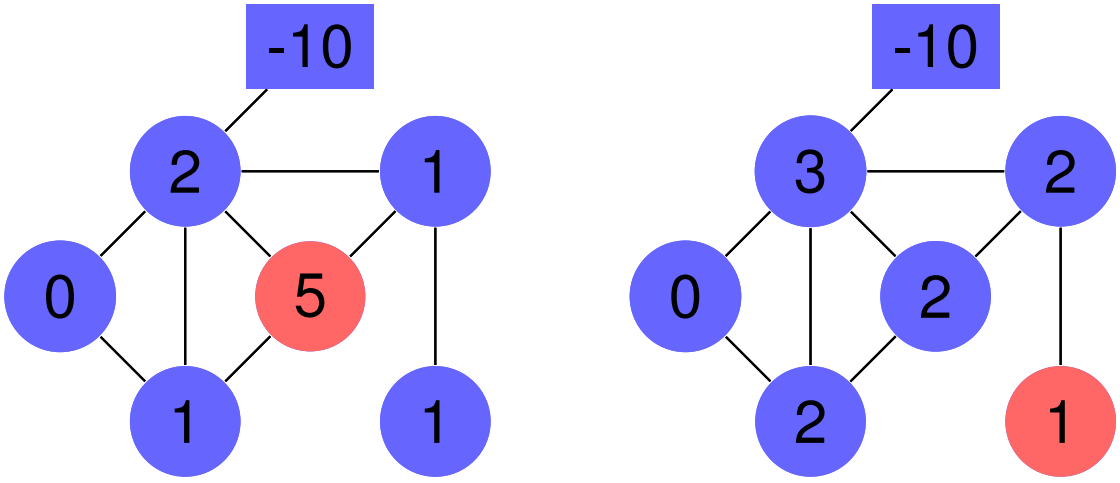

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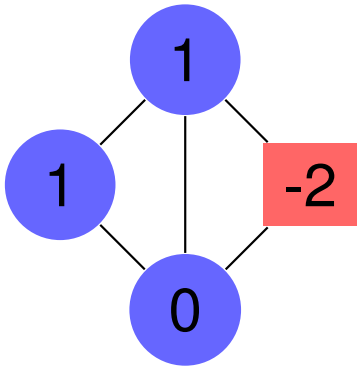
- A configuration is *stable* if no round vertex can be fired.
- A configuration is *recurrent* if there is a sequence of firings that lead to the same configuration.
- A configuration is *critical* if it is both recurrent and stable.

Sample game 1



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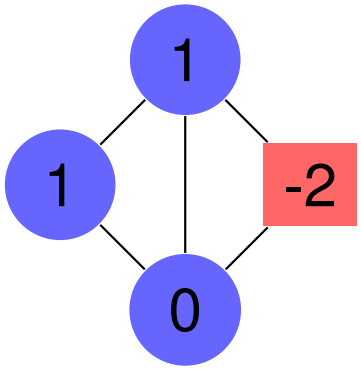

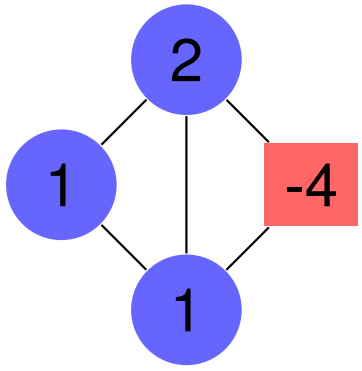

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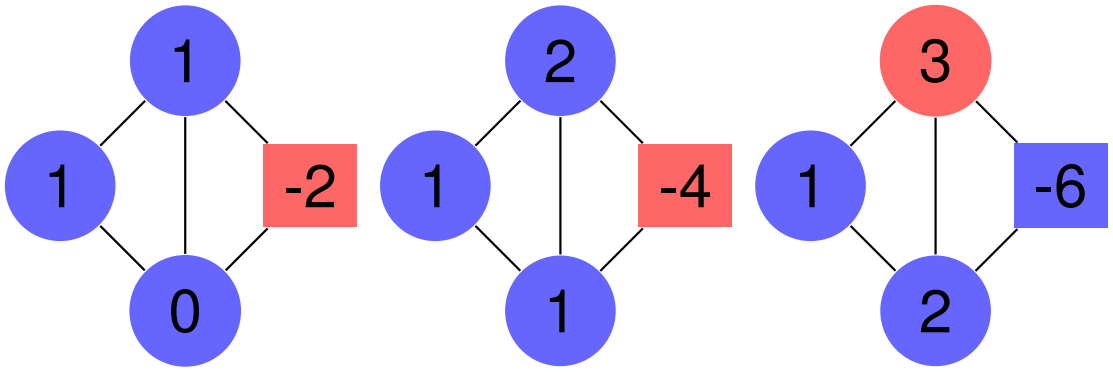

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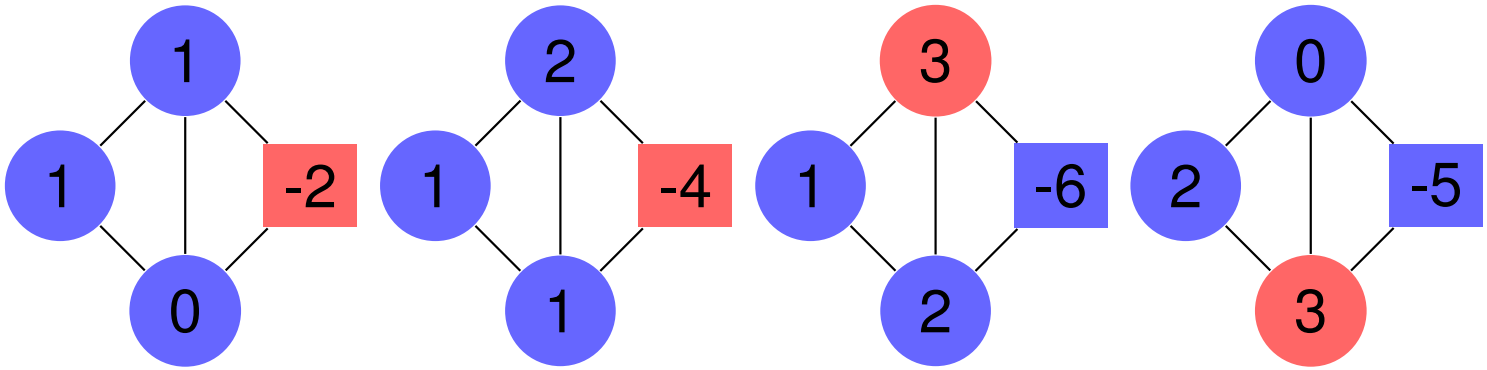

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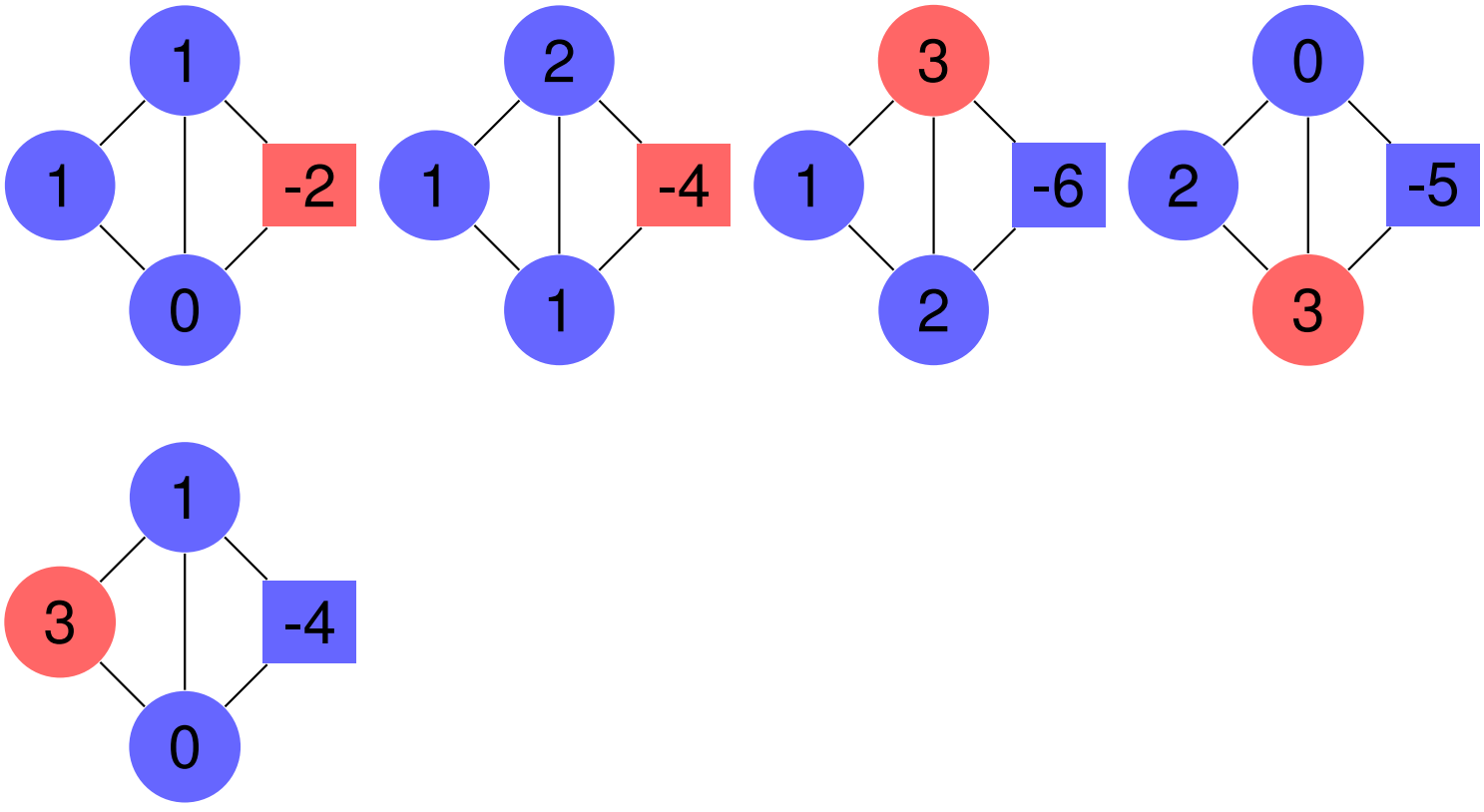

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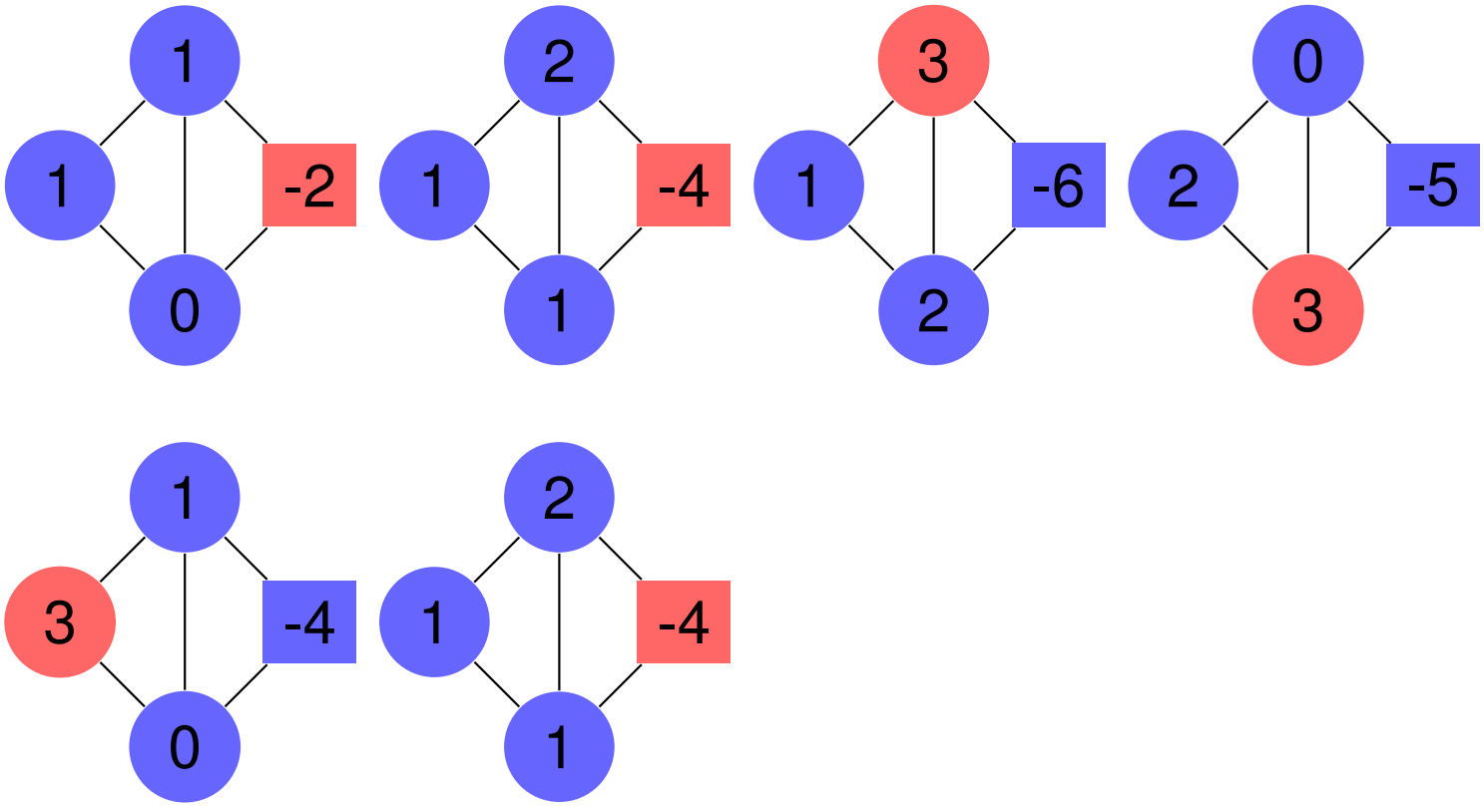

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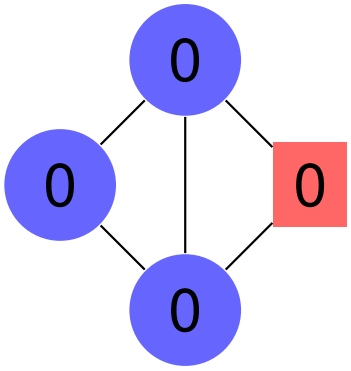

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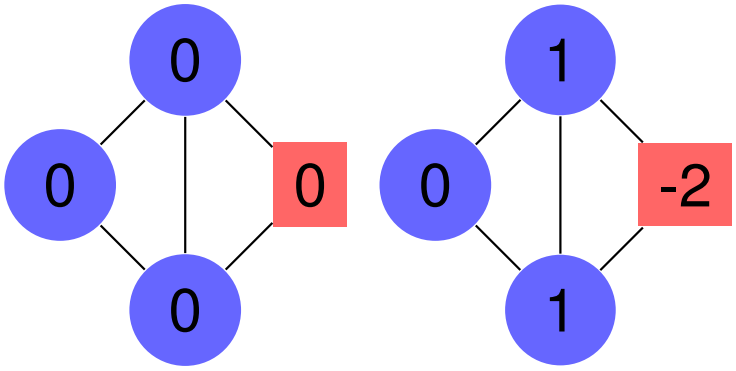

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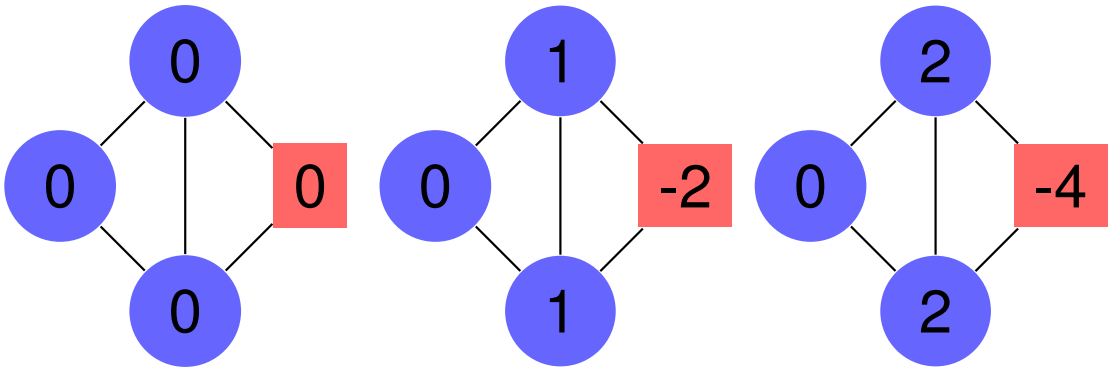

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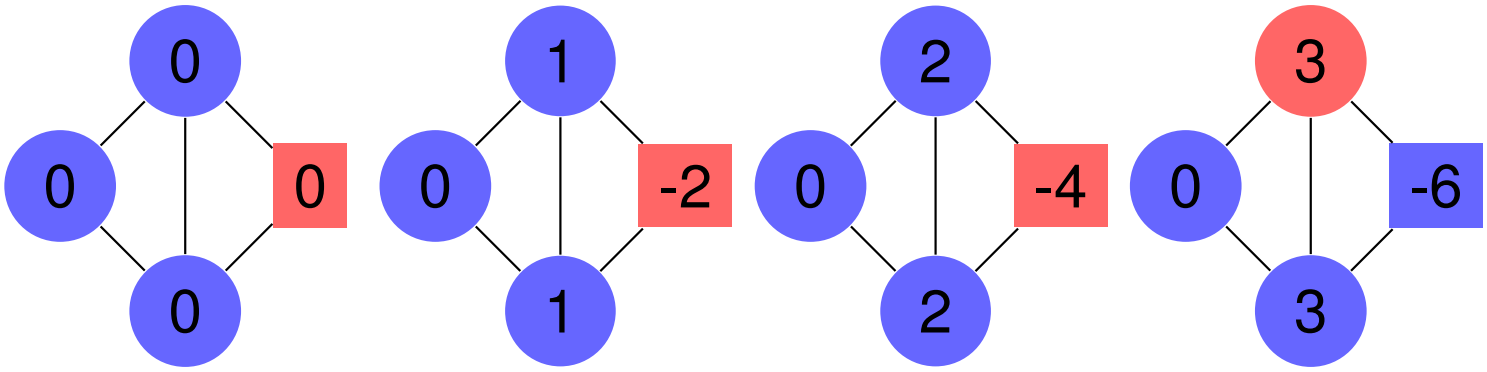

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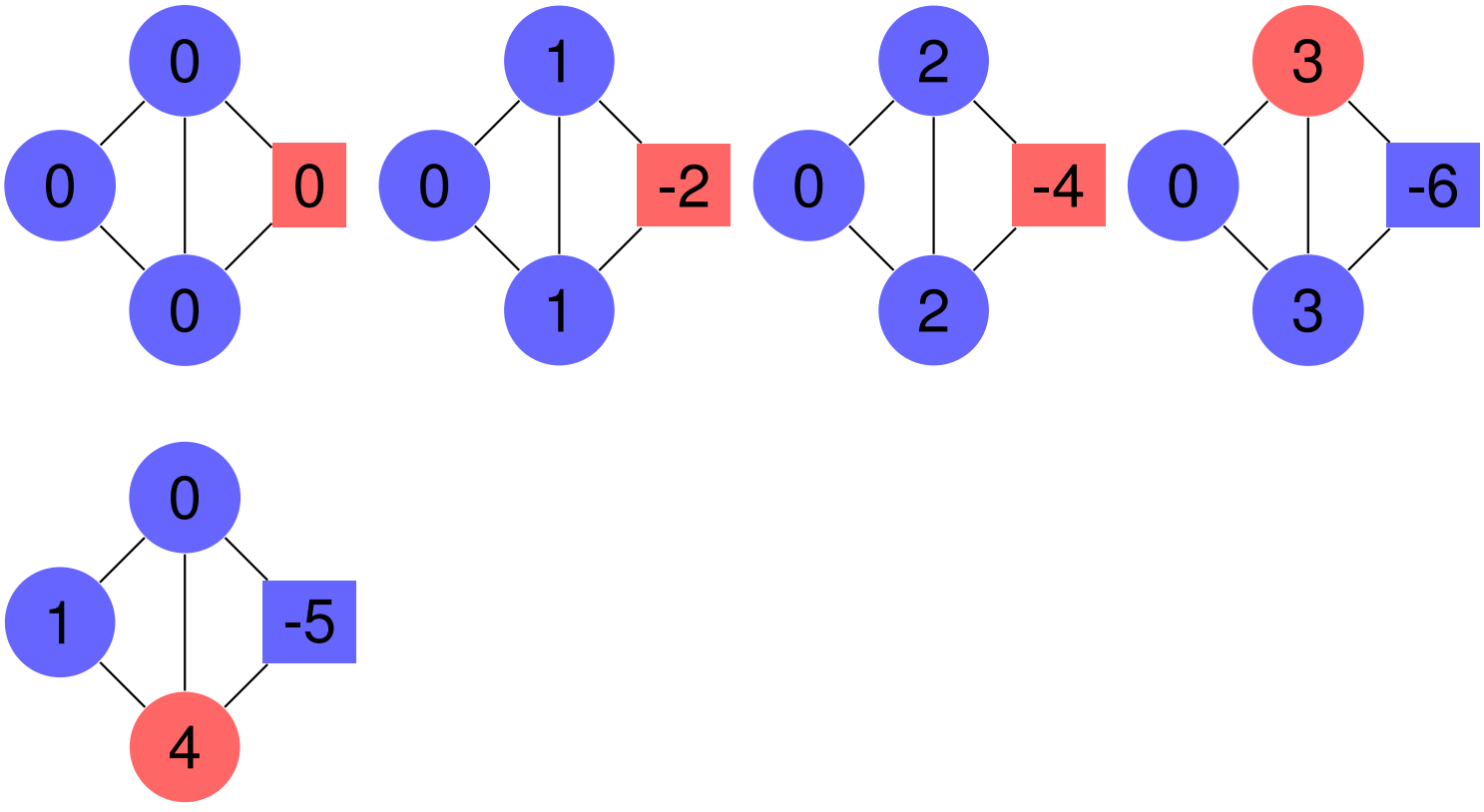

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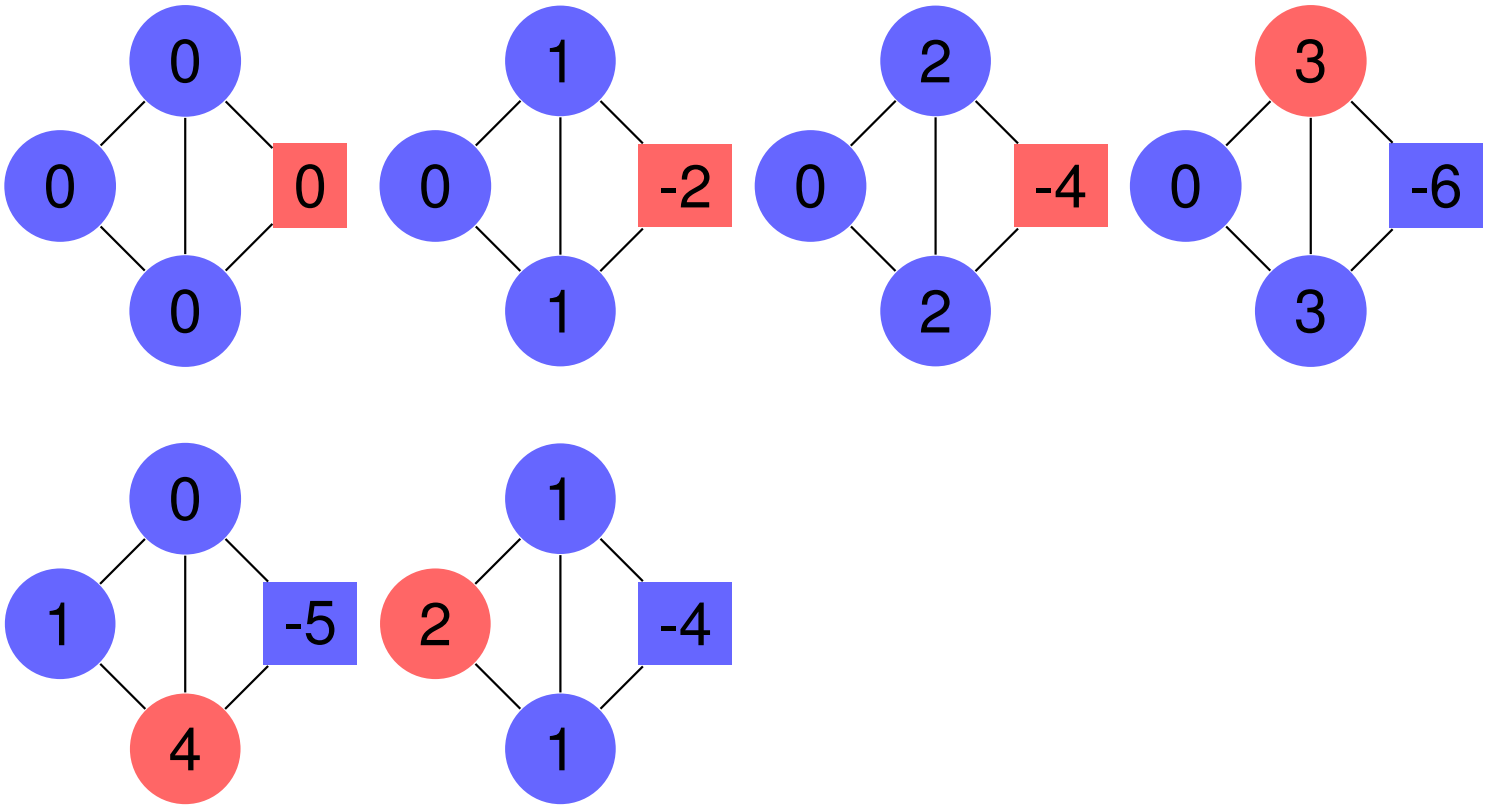

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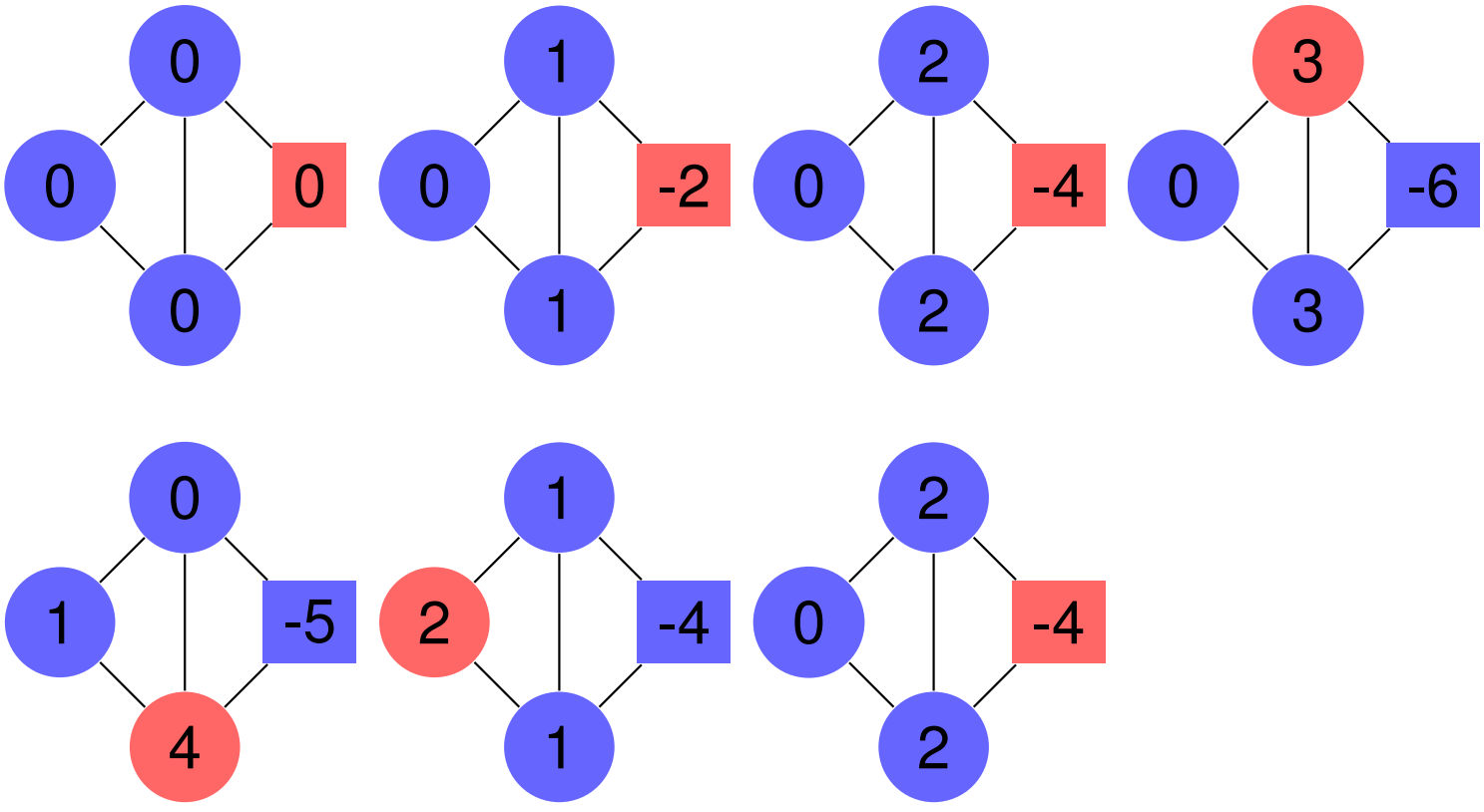

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Theorem

The set of critical configurations has a natural group operation making it isomorphic to the critical group $K(\Gamma)$.

Critical groups of graphs

Equivalence and Smith normal form

Henry John Stephen Smith (1826-1883)

Given an integer matrix X, there exist unimodular integer matrices P and Q such that

$$PXQ = \left[\begin{array}{c|c} Y & 0 \\ \hline 0 & 0 \end{array}\right], \quad Y = \mathrm{diag}(s_1, s_2, \ldots s_r), \quad s_1 | s_2 | \cdots | s_r.$$

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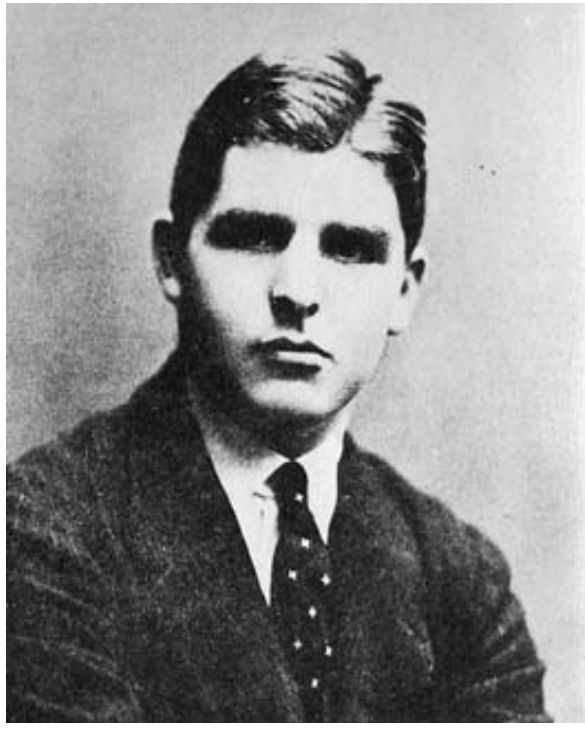
- Conference graphs on a square-free number of vertices (Lorenzini).

Critical groups of graphs



Raymond E. A. C. Paley (1907-33)

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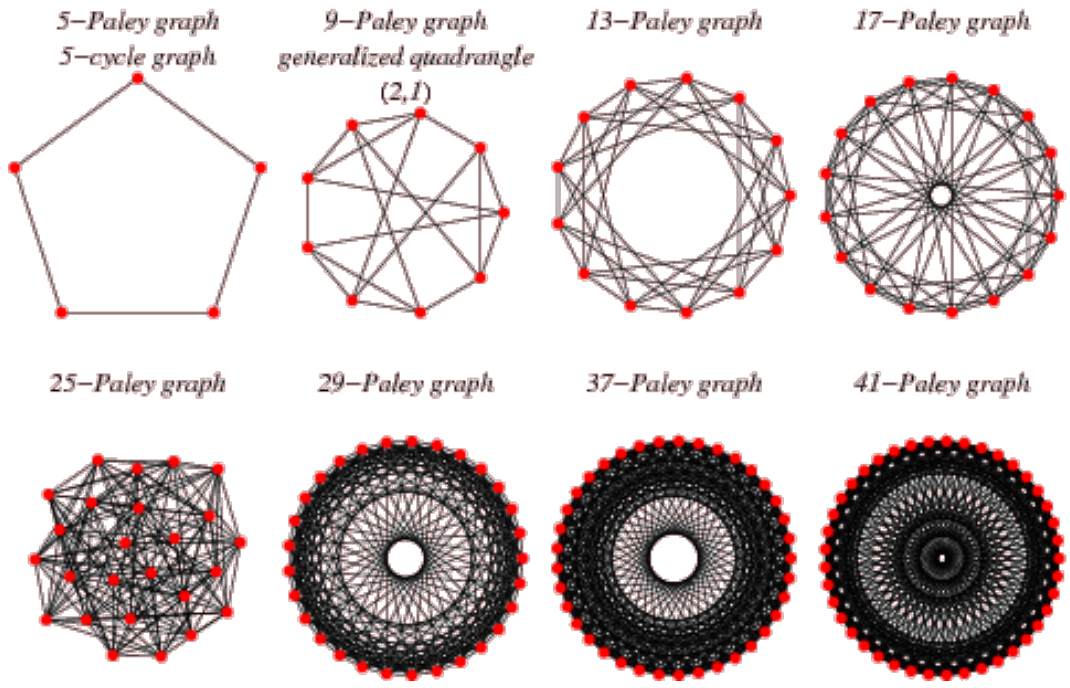
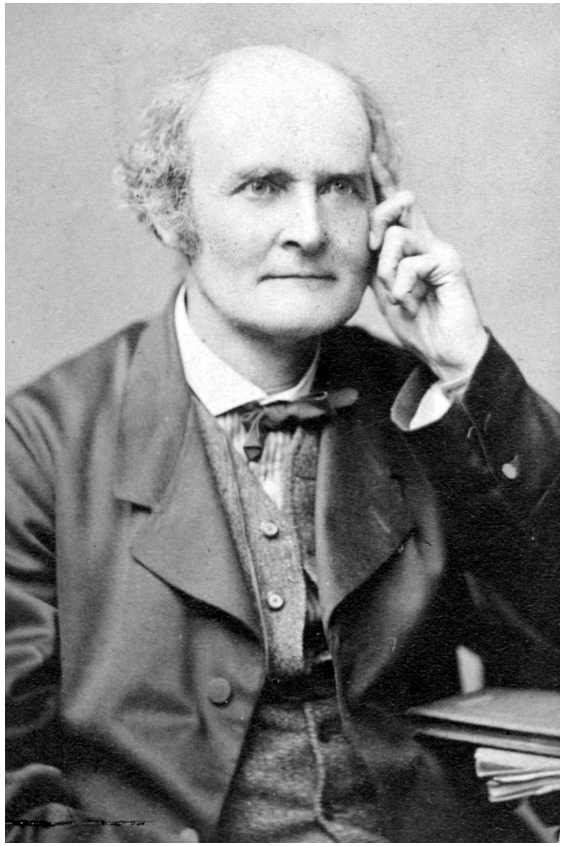
- $S =$ set of nonzero squares in $\mathbb{F}_q$
- two vertices x and y are joined by an edge iff $x - y \in S$.



Some Paley graphs (from Wolfram Mathworld)

Paley graphs are Cayley graphs

We can view $\mathrm{P}(q)$ as a Cayley graph on $(\mathbb{F}_q, +)$ with connecting set S



Arthur Cayley (1821-95)

Paley graphs are strongly regular graphs

It is well known and easily checked that $\mathrm{P}(q)$ is a *strongly regular graph* and that its eigenvalues are $k = \frac{q-1}{2}$, $r = \frac{-1+\sqrt{q}}{2}$ and $s = \frac{-1-\sqrt{q}}{2}$, with multiplicities 1, $\frac{q-1}{2}$ and $\frac{q-1}{2}$, respectively.

Critical groups of graphs

David Chandler and Qing Xiang

Symmetries

- $$|K(\mathrm{P}(q))| = \frac{1}{q}\left(\frac{q+\sqrt{q}}{2}\right)^k \left(\frac{q-\sqrt{q}}{2}\right)^k = q^{\frac{q-3}{2}}\mu^k,$$

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- Use $\mathbb{F}_q$-action to help compute p'-part.

Symmetries

- $$|K(\mathrm{P}(q))| = \frac{1}{q}\left(\frac{q+\sqrt{q}}{2}\right)^k\left(\frac{q-\sqrt{q}}{2}\right)^k = q^{\frac{q-3}{2}}\mu^k,$$

  where $\mu = \frac{q-1}{4}$.
- $\mathrm{Aut}(\mathrm{P}(q)) \geq \mathbb{F}_q \rtimes S$.
- $K(\mathrm{P}(q)) = K(\mathrm{P}(q))_p \oplus K(\mathrm{P}(q))_{p'}$
- Use $\mathbb{F}_q$-action to help compute p'-part.
- Use S-action to help compute p-part.

p'-part

Joseph Fourier (1768-1830)

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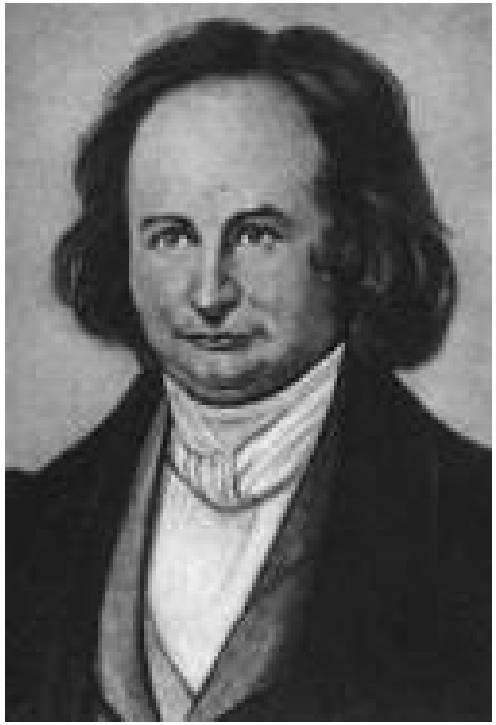
- $$\frac{1}{q} X L \overline{X}^t = \mathrm{diag}(k - \psi(S))_\psi, \tag{1}$$
- Interpret this as PLQ-equivalence over suitable local rings of integers.

Theorem

$K(\mathrm{P}(q))_{p'} \cong (\mathbf{Z}/\mu\mathbf{Z})^{2\mu}$, *where* $\mu = \frac{q-1}{4}$.

The *p*-part



Carl Gustav Jacob Jacobi (1804-51)

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The *Jacobi sum* of two nontrivial characters T^a and T^b is

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Lemma

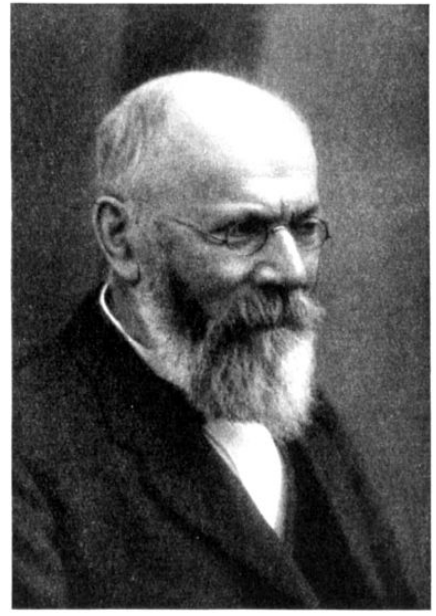
(i) $L(\mathbf{1}) = 0$.

(ii) $L(e_k) = \frac{1}{2}(\mathbf{1} - q([0] - e_k))$.

(iii) $L([0]) = \frac{1}{2}(q[0] - e_k - \mathbf{1})$.

Corollary

The Laplacian matrix L is equivalent over R to the diagonal matrix with diagonal entries $J(T^{-i}, T^k)$, for $i = 1, \ldots, q-2$ and $i \neq k$, two 1*s and one zero.*



Carl Friedrich Gauss (1777-1855) Ludwig Stickelberger (1850-1936)

Gauss and Jacobi

Gauss sums: If $1 \neq \chi \in \mathrm{Hom}(\mathbb{F}_q^\times, R^\times)$,

$$g(\chi) = \sum_{y \in \mathbb{F}_q^\times} \chi(y)\zeta^{\mathrm{tr}(y)},$$

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Lemma

If χ and ψ are nontrivial multiplicative characters of $\mathbb{F}_q^\times$ such that $\chi\psi$ is also nontrivial, then

$$J(\chi, \psi) = \frac{g(\chi)g(\psi)}{g(\chi\psi)}.$$

Stickelberger's Congruence

Theorem

For $0 < a < q - 1$, *write* a *p-adically as*

$$a = a_0 + a_1 p + \cdots + a_{t-1} p^{t-1}.$$

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Theorem

Let $a, b \in \mathbf{Z}/(q-1)\mathbf{Z}$, *with* a, b, $a + b \not\equiv 0 \pmod{q-1}$. *Then number of times that* p *divides* $J(T^{-a}, T^{-b})$ *is equal to the number of carries in the addition* $a + b \pmod{q-1}$ *when* a *and* b *are written in* p*-digit form.*

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- Reformulate as a count of closed walks on a certain directed graph.
- Transfer matrix method yields the generating function for our counting problem from the adjacency matrix of the digraph.

Theorem

Let $q = p^t$ be a prime power congruent to 1 modulo 4. Then the number of p-adic elementary divisors of $L(\mathrm{P}(q))$ which are equal to p^λ, $0 \leq \lambda < t$, is

$$f(t,\lambda) = \sum_{i=0}^{\min\{\lambda, t-\lambda\}} \frac{t}{t-i} \binom{t-i}{i} \binom{t-2i}{\lambda-i} (-p)^i \left(\frac{p+1}{2}\right)^{t-2i}.$$

The number of p-adic elementary divisors of $L(\mathrm{P}(q))$ which are equal to p^t is $\left(\frac{p+1}{2}\right)^t - 2$.

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$$K(\mathrm{P}(5^3)) \cong (\mathbf{Z}/31\mathbf{Z})^{62} \oplus (\mathbf{Z}/5\mathbf{Z})^{36} \oplus (\mathbf{Z}/25\mathbf{Z})^{36} \oplus (\mathbf{Z}/125\mathbf{Z})^{25}.$$

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Thank you for your attention!