A REMARK ON GRASSMANN AND VERONESE EMBEDDINGS OF $\mathbb{P}^3$ IN CHARACTERISTIC 2.

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ABSTRACT. We answer a question raised in a recent paper by I. Cardinali and A. Pasini. Over an algebraically closed field of characteristic 2, we show that a certain projection of $\mathbb{P}^9$ to $\mathbb{P}^8$ induces an isomorphism of algebraic varieties from the quadratic Veronese embedding of $\mathbb{P}^3$ to the standard embedding of the orthogonal Grassmanian of lines of a quadric in $\mathbb{P}^4$.

1. INTRODUCTION AND NOTATION

The purpose of this note is to settle a question left open in a recent paper [2] concerning a certain morphism between the quadratic Veronese embedding of $\mathbb{P}^3$ and another embedding constructed using a Grassmannian of lines. In odd characteristic the morphism is an isomorphism of varieties, induced by a linear isomorphism of the ambient projective spaces. In characteristic 2 the linear map inducing the morphism is not an isomorphism, so the question of whether the morphism is an isomorphism was not resolved. In this note we show that it is indeed an isomorphism. The embeddings are introduced in §§2-4, and the precise question is stated in §5. In view of our specific objective we work entirely in characteristic 2, although this assumption is not necessary for the general discussion of the Veronese and Grassmanian varieties. The hypothesis enters in an essential way when we consider the subvariety of the Grassmanian defined by isotropic lines of a quadratic form in §4. In [2] Cardinali and Pasini observe that in characteristic 2 only, this subvariety is contained in a hyperplane of the ambient projective space. Their question is directly related to this phenomenon.

In addition to [2], further background material can be found in the texts of Harris [3] and Borel [1]. Harris's book contains a detailed description of Grassmanian and Veronese varieties, and [1, §AG.11] treats the concepts of varieties and morphisms defined over a subfield of an algebraically closed field.

Let k be an algebraically closed field of characteristic 2. Let W be a 4-dimensional vector space over k carrying a nonsingular alternating bilinear form $B(-,-)$. Let e_0, e_1, e_2, e_3 be a symplectic basis of W, so that $B(e_i, e_j) = \delta_{j,3-i}$, and let x_0, x_1, x_2, x_3 be the corresponding symplectic coordinates.

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2. The quadratic Veronese embedding

Let $\mathrm{S}ym_2(W)$ denote the subspace of symmetric tensors in $W \otimes W$. If we take the basis consisting of the vectors $e_i \otimes e_i$ and $e_i \otimes e_j + e_j \otimes e_i$, then the *Veronese map*

$$ver : \mathbb{P}(W) \to \mathbb{P}(\mathrm{S}ym_2(W)), \quad \langle v \rangle \mapsto \langle v \otimes v \rangle. \tag{1}$$

is given in coordinates by

$$(a_0 : a_1 : a_2 : a_3) \mapsto (a_i a_j)_{i \leq j}. \tag{2}$$

We will use a slightly different basis, namely

(3) $e_0 \otimes e_3 + e_3 \otimes e_0 + e_1 \otimes e_2 + e_2 \otimes e_1, e_0 \otimes e_0, e_0 \otimes e_1 + e_1 \otimes e_0, e_1 \otimes e_1,$
$e_0 \otimes e_3 + e_3 \otimes e_0, e_0 \otimes e_2 + e_2 \otimes e_0, e_2 \otimes e_2, e_1 \otimes e_3 + e_3 \otimes e_1, e_2 \otimes e_3 + e_3 \otimes e_2, e_3 \otimes e_3.$

In these coordinates the map is given by

$$(a_0 : a_1 : a_2 : a_3) \mapsto (a_1a_2 : a_0^2 : a_0a_1 : a_1^2 : a_0a_3 - a_1a_2 : a_0a_2 : a_2^2 : a_1a_3 : a_2a_3 : a_3^2). \tag{4}$$

The image $\mathcal{V}$ of the Veronese map is called the quadratic Veronese variety.

3. The Klein quadric

Take the basis $e_i \wedge e_j$, $i < j$, for $\wedge^2(W)$, with coordinates p_{ij}. Each line $\langle v, w \rangle$ of $\mathbb{P}(W)$, determines a point $\langle v \wedge w \rangle$ of $\mathbb{P}(\wedge^2(W))$, so we get an embedding $\{\text{lines of } \mathbb{P}(W)\} \to \mathbb{P}(\wedge^2(W))$. If $v = (a_0 : a_1 : a_2 : a_3)$ and $w = (b_0 : b_1 : b_2 : b_3)$ then the image of the line is the point with coordinates $p_{ij} = a_i b_j - a_j b_i$. The image of the set of all lines is the Klein quadric $\mathcal{K}$, which is the set of points in $\mathbb{P}(\wedge^2(W))$ satisfying the equation

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.$$

Let V be the 5-dimensional subspace of $\wedge^2(W)$ defined by $p_{03} + p_{12} = 0$. In V we choose coordinates from the restrictions of the p_{ij} to V as follows: $X_0 = p_{01}$, $X_1 = p_{02}$, $X_2 = p_{03}$, $X_3 = p_{13}$, $X_4 = p_{23}$.

A line $\langle v, w \rangle$ is totally isotropic with respect to B if and only if its image lies in the intersection of $\mathcal{K}$ with the image of V in $\mathbb{P}(\wedge^2(W))$, so the set of totally B-isotropic lines in $\mathbb{P}(W)$ is mapped to the quadric $\mathcal{Q}$ in $\mathbb{P}(V)$ defined by the quadratic form

$$\chi := X_0X_4 - X_1X_3 - X_2^2. \tag{5}$$

4. The Grassmann embedding of lines in $\mathbb{P}(V)$

Under the Klein correspondence, we have seen that a totally isotropic line in $\mathbb{P}(W)$, defines a point of $\mathbb{P}(V)$. Also, given a point $\langle v_0 \rangle$ of $\mathbb{P}(W)$, we may consider the set of isotropic lines through that point. We claim that the images of these lines form a line in $\mathbb{P}(V)$. To see this, extend v_0 to a basis $\{v_0, v_1, v_2, v_3\}$ of W such that $v_0^\perp$ is spanned by $\{v_0, v_1, v_2\}$. Then it is easy to check that the line joining the two points $\langle v_0 \wedge v_1 \rangle$ and $\langle v_0 \wedge v_2 \rangle$ has as its points the images of the 2-dimensional isotropic subspaces of W containing v_0.

Hence we have a map from $\mathbb{P}(W)$ to the set of lines in $\mathbb{P}(V)$.

If $L = \langle A, B \rangle$ is a line in $\mathbb{P}(V)$, it defines a point $\langle A \wedge B \rangle$ in $\mathbb{P}(\wedge^2(V)) \cong \mathbb{P}^9$ and we have an embedding

$$gr : \mathbb{P}(W) \to \mathbb{P}(\wedge^2(V)). \tag{6}$$

Suppose L lies in $\mathcal{Q}$. If $A = (A_0 : A_1 : A_2 : A_3 : A_4)$ and $B = (B_0 : B_1 : B_2 : B_3 : B_4)$ then

$$A_0B_4 + A_4B_0 + A_1B_3 + A_3B_1 = 0, \tag{7}$$

since this is the symmetric bilinear form associated with the quadratic form χ defining $\mathcal{Q}$. Here, we have used the assumption of characteristic 2 and, as will become clear, it is the special form of (7) that makes characteristic 2 exceptional.

We take coordinates q_{ij}, $(0 \le i < j \le 4)$ on $\wedge^2(V)$ induced by our coordinates X_i on V. Then if L lies in $\mathcal{Q}$, the point $\langle A \wedge B\rangle$ lies in the image of the hyperplane H of $\wedge^2(V)$ defined by the equation $q_{04} + q_{13} = 0$, by (7). On H we take the same coordinates, but omit the q_{13} coordinate.

Let $\mathcal{G} \subseteq \mathbb{P}(\wedge^2(V))$ denote the set of images in $\mathbb{P}(\wedge^2(V))$ of all lines in $\mathbb{P}(V)$ and let $\mathcal{G}_\chi \subset \mathbb{P}(H)$ denote the set of images in $\mathbb{P}(H)$ of all lines in $\mathcal{Q}$.

5. Isomorphism of the varieties $\mathcal{V}$ and $\mathcal{G}_\chi$

Let $T = \langle e_0{\otimes}e_3{+}e_3{\otimes}e_0{+}e_1{\otimes}e_2{+}e_2{\otimes}e_1\rangle \subseteq \mathrm{S}ym_2(W)$. The space of $\mathrm{Sp}(W)$-fixed points in $W{\otimes}W$ is one-dimensional, by the simplicity of W, and it is easy to check that this space equals T. In [2] it is shown that there is a $k\,\mathrm{Sp}(W)$-module map $\tilde{\pi} : \mathrm{S}ym_2(W) \to \wedge^2(V)$ whose kernel is T and whose image is H. (The existence and uniqueness of $\tilde{\pi}$ follow from fact that $\mathrm{S}ym_2(W)$ is isomorphic to a Weyl module whose highest weight appears also as the highest weight in $\wedge^2(V)$, with multiplicity one.) If we take coordinates on $\mathbb{P}(\mathrm{S}ym_2(W)) \cong \mathbb{P}^9$ with respect to the basis (3) above, and coordinates on $\mathbb{P}(H) \cong \mathbb{P}^8$ as described in the last section, the induced map $\pi : \mathbb{P}(\mathrm{S}ym_2(W)) \to \mathbb{P}(H)$ is simply projection relative to the first coordinate position.

In [2], it is shown that $\pi_{|\mathcal{V}}$ defines a bijection from $\mathcal{V}$ to $\mathcal{G}_\chi$, but the question of whether the map is an isomorphism of algebraic varieties is left open. (See p.102 in [2].) We shall prove that the map is indeed an isomorphism, by explicitly defining the inverse morphism.

We have defined the sets $\mathcal{V} \subseteq \mathbb{P}(\mathrm{S}ym_2(W)) \cong \mathbb{P}^9$ and $\mathcal{G}_\chi \subseteq \mathbb{P}(H) \cong \mathbb{P}^8$. Both $\mathcal{V}$ and $\mathcal{G}_\chi$ are projective varieties, and they are defined over $\mathbb{F}_2$.

The coordinates with respect to our basis (3) of the image in $\mathcal{V}$ of the point $p = (a_0 : a_1 : a_2 : a_3) \in \mathbb{P}(W)$ is given in (4).

Let us now compute the image of the point p under the Grassman embedding gr above. Using the same notation, we take $p = \langle v_0\rangle$, with $v_0 = (a_0, a_1, a_2, a_3)$, $v_1 = (b_0, b_1, b_2, b_3)$, $v_2 = (c_0, c_1, c_2, c_3)$ and $v_3 = (d_0, d_1, d_2, d_3)$ forming a symplectic basis. Then the points $\langle v_0 \wedge v_1\rangle$ and $\langle v_0 \wedge v_2\rangle$ of $\mathbb{P}(V)$ have $(X_0 : X_1 : X_2 : X_3 : X_4)$-coordinates

$$(a_0b_1 - a_1b_0 : a_0b_2 - a_2b_0 : a_0b_3 - a_3b_0 : a_1b_3 - a_3b_1 : a_2b_3 - a_3b_2),$$

and

$$(a_0c_1 - a_1c_0 : a_0c_2 - a_2c_0 : a_0c_3 - a_3c_0 : a_1c_3 - a_3c_1 : a_2c_3 - a_3c_2).$$

Next we consider the line joining these two points as a point of $\mathbb{P}(\wedge^2(V))$ with Plücker coordinates q_{ij} , $0 \leq i < j \leq 4$ induced by the coordinates X_i of V.

The $q_{0,1}$-coordinate of this line is therefore

$$
\begin{aligned}
(a_0b_1 - a_1b_0)(a_0c_2 - a_2c_0) - (a_0b_2 - a_2b_0)(a_0c_1 - a_1c_0)& \\
&= a_0^2b_1c_2 - a_0a_2b_1c_0 - a_0a_1b_0c_2 + a_1a_2b_0c_0 \\
&- a_0^2b_2c_1 + a_0a_1b_2c_0 + a_0a_2b_0c_1 - a_1a_2b_0c_0 \\
&= a_0 \begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix}.
\end{aligned} \tag{8}
$$

The $q_{0,4}$-coordinate of this line is,

$$
\begin{aligned}
&(a_0b_1 - a_1b_0)(a_2c_3 - a_3c_2) - (a_2b_3 - a_3b_2)(a_0c_1 - a_1c_0) \\
&= a_0a_2b_1c_3 - a_0a_2b_3c_1 + a_0a_3b_2c_1 - a_0a_3b_1c_2 + a_0a_1b_2c_3 - a_0a_1b_3c_2 \\
&- a_1a_2b_0c_3 + a_1a_2b_3c_0 - a_1a_3b_2c_0 + a_1a_3b_0c_2 - a_1a_0b_3c_2 + a_1a_0b_2c_3 \\
&= a_0 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + a_1 \begin{vmatrix} a_0 & a_2 & a_3 \\ b_0 & b_2 & b_3 \\ c_0 & c_2 & c_3 \end{vmatrix}.
\end{aligned} \tag{9}
$$

One can find similar expressions for all of the $q_{i,j}$-coordinates.

Now consider the vector $v_0 \wedge v_1 \wedge v_2 \in \wedge^3(W)$. It defines an element $f \in W^*$ by

$$v_0 \wedge v_1 \wedge v_2 \wedge w = f(w)e_0 \wedge e_1 \wedge e_2 \wedge e_3,$$

where the e_i are our standard symplectic basis. In coordinates, if $w = (r_0, r_1, r_2, r_3)$, we have

$$f(w) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ r_0 & r_1 & r_2 & r_3 \end{vmatrix} = r_0\Delta_{123} + r_1\Delta_{023} + r_2\Delta_{013} + r_3\Delta_{012}$$

where

$$\Delta_{ijk} = \begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix}.$$

So in the basis x_0, x_1, x_2, x_3 of W^*, we have $f = \Delta_{123}x_0 + \Delta_{023}x_1 + \Delta_{013}x_2 + \Delta_{012}x_3$. We have $f(v_0) = f(v_1) = f(v_2) = 0$ and since the v_i also form a symplectic basis, we have $f(v_3) = 1$. So $f(w) = B(w, v_0)$, which implies that $f = a_3x_0 + a_2x_1 + a_1x_2 + a_0x_3$. Hence we conclude that $\Delta_{012} = a_0$, $\Delta_{013} = a_1$, $\Delta_{023} = a_2$, and $\Delta_{123} = a_3$. Thus, the $q_{0,1}$ and $q_{0,4}$-coordinates of the line in question are a_0^2 and $a_0a_3 + a_1a_2$ respectively (when viewed as a point in either $\mathbb{P}(\wedge^2(V))$ or $\mathbb{P}(H)$).

By similar computations, we find that the image in $\mathcal{G}_\chi \subset \mathbb{P}(H)$ of $p \in \mathbb{P}(W)$ under the map $\mathrm{g}r$ has coordinates

$$(a_0^2 : a_0a_1 : a_1^2 : a_0a_3 + a_1a_2 : a_0a_2 : a_2^2 : a_1a_3 : a_2a_3 : a_3^2). \tag{10}$$

From (4) and (10) we see that π maps $\mathcal{V}$ bijectively to $\mathcal{G}_\chi$.

We will now construct the inverse morphism to $\pi_{|\mathcal{V}}$.

We denote the coordinates of H by Z_i, $i = 1,\ldots,9$. and set $\tilde{U}_i = \{h \in \mathbb{P}(H) : Z_i(h) \neq 0\}$ and $U_i = \tilde{U}_i \cap \mathcal{G}_\chi$.

Then from (10) we see that

$$\mathcal{G}_\chi = U_1 \cup U_3 \cup U_6 \cup U_9,$$

since a_0, a_1, a_2 and a_3 cannot be simultaneously zero.

We will define morphisms $\phi_i : \tilde{U}_i \to \mathbb{P}(\mathrm{S}ym_2(W))$ for $i \in \{1,3,6,9\}$ with the properties that

(i) $\phi_{i|U_i}$ maps into $\mathcal{V}$;

(ii) $\phi_{i|U_i}$ and $\phi_{j|U_j}$ agree on $U_i \cap U_j$;

(iii) $\pi_{|\mathcal{V}} \circ \phi_{i|U_i}$ is the identity map of U_i;

(iv) $\phi_{i|U_i} \circ \pi_{|\mathcal{V}}$ is the identity map of ${\pi_{|\mathcal{V}}}^{-1}(U_i)$.

These properties mean that the morphisms $\phi_{i|U_i}$ constitute an inverse isomorphism ϕ to $\pi_{|\mathcal{V}}$.

On $\tilde{U}_1$, we take local affine coordinates Z_j/Z_1, $j = 2,\ldots, 9$, and define for $t = (1 : t_2 : t_3 : t_4 : t_5 : t_6 : t_7 : t_8 : t_9) \in \tilde{U}_1$

$$\phi_1(t) = (t_2t_5 : 1 : t_2 : t_3 : t_4 : t_5 : t_6 : t_7 : t_8 : t_9) \in \mathbb{P}(\mathrm{S}ym_2(W)). \tag{11}$$

On $\tilde{U}_3$, we take local affine coordinates Z_j/Z_3, $1 \leq j \leq 9$, $j \neq 3$, and define for $t = (t_1 : t_2 : 1 : t_4 : t_5 : t_6 : t_7 : t_8 : t_9) \in \tilde{U}_3$

$$\phi_3(t) = (t_4 + t_2t_7 : t_1 : t_2 : 1 : t_4 : t_5 : t_6 : t_7 : t_8 : t_9) \in \mathbb{P}(\mathrm{S}ym_2(W)). \tag{12}$$

On $\tilde{U}_6$, we take local affine coordinates Z_j/Z_6, $1 \leq j \leq 9$, $j \neq 6$, and define for $t = (t_1 : t_2 : t_3 : t_4 : t_5 : 1 : t_7 : t_8 : t_9) \in \tilde{U}_6$

$$\phi_6(t) = (t_4 + t_5t_8 : t_1 : t_2 : t_3 : t_4 : t_5 : 1 : t_7 : t_8 : t_9) \in \mathbb{P}(\mathrm{S}ym_2(W)). \tag{13}$$

On $\tilde{U}_9$, we take local affine coordinates Z_j/Z_9, $1 \leq j \leq 8$, and define for $t = (t_1 : t_2 : t_3 : t_4 : t_5 : t_6 : t_7 : t_8 : 1) \in \tilde{U}_9$

$$\phi_9(t) = (t_7t_8 : t_1 : t_2 : t_3 : t_4 : t_5 : t_6 : t_7 : t_8 : 1) \in \mathbb{P}(\mathrm{S}ym_2(W)). \tag{14}$$

In each of the four cases a point of the form (10) is mapped to the point (4) and the properties (i)-(iv) can be verified immediately. Thus $\pi_{|\mathcal{V}}$ and ϕ are inverse isomorphisms. Moreover, they are defined over $\mathbb{F}_2$. Thus, we have established the following statement.

Theorem 5.1. *The map $\pi_{|\mathcal{V}} : \mathcal{V} \to \mathcal{G}_\chi$ is an isomorphism of algebraic varieties, defined over $\mathbb{F}_2$.*

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