# The critical group of a graph

Peter Sin, U. of Florida

James Madison University, February 8th, 2016.

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- We'll consider the problem of computing the critical group for families of graphs, using the Paley graphs as an example.
- Techniques involve groups, characters and number theory.

# Critical groups of graphs

Pierre-Simon Laplace (1749-1827)

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- $\mathrm{rank}(L) = |V| - 1$.

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- $L(\ker(\varepsilon)) \subseteq \ker(\varepsilon)$, and $K(\Gamma) \cong \operatorname{Ker}(\varepsilon)/L(\operatorname{Ker}(\varepsilon))$

# Kirchhoff's Matrix-Tree Theorem

Gustav Kirchhoff (1824-1887)

## Kirchhoff's Matrix Tree Theorem

*For any connected graph* $\Gamma$*, the number of spanning trees is equal to* $\det(\tilde{L})$*, where* $\tilde{L}$ *is obtained from L be deleting the row and column corrresponding to any chosen vertex.*

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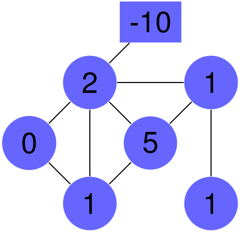
Also, $\det(\tilde{L}) = |K(\Gamma)| = \frac{1}{|V|} \prod_{j=2}^{|V|} \lambda_j$.

# Critical groups of graphs

# Rules




- A *configuration* is an assignment of a nonnegative integer $s(v)$ to each round vertex $v$ and $-\sum_v s(v)$ to the square vertex.

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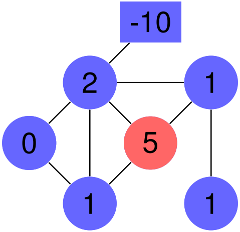


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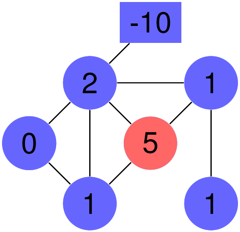


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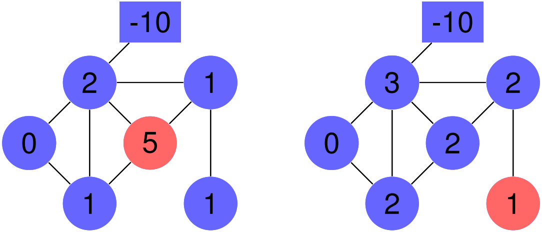


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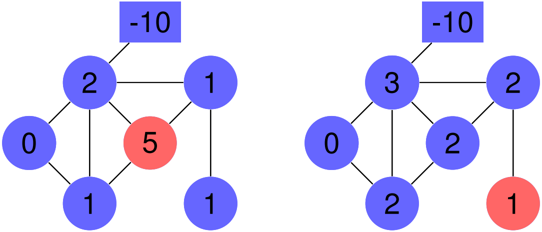


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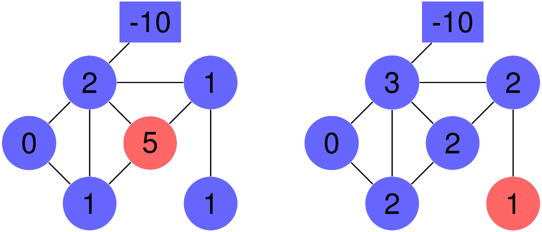


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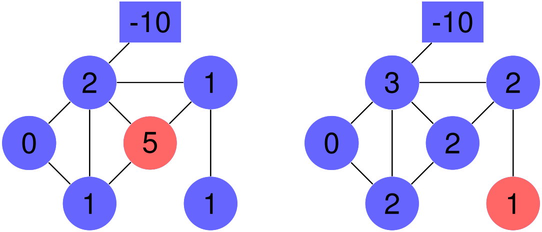


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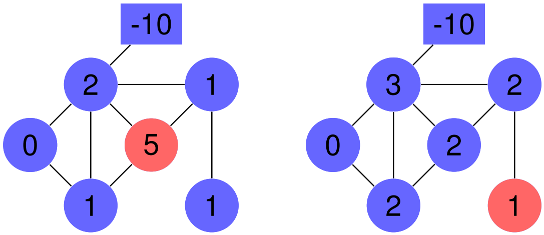


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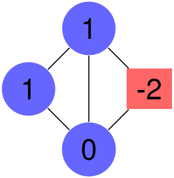
- A configuration is *stable* if no round vertex can be fired.
- A configuration is *recurrent* if there is a sequence of firings that lead to the same configuration.
- A configuration is *critical* if it is both recurrent and stable.

# Sample game 1



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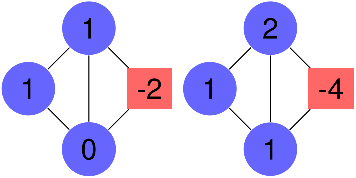

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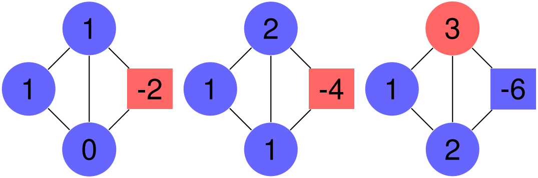

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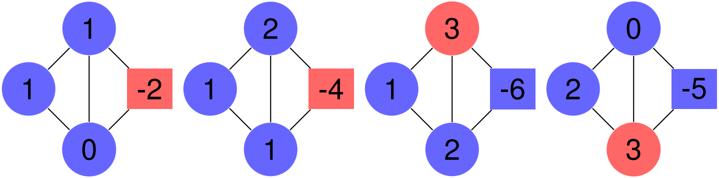

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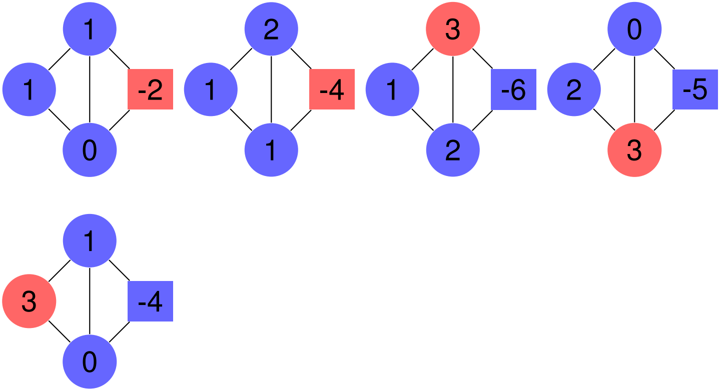

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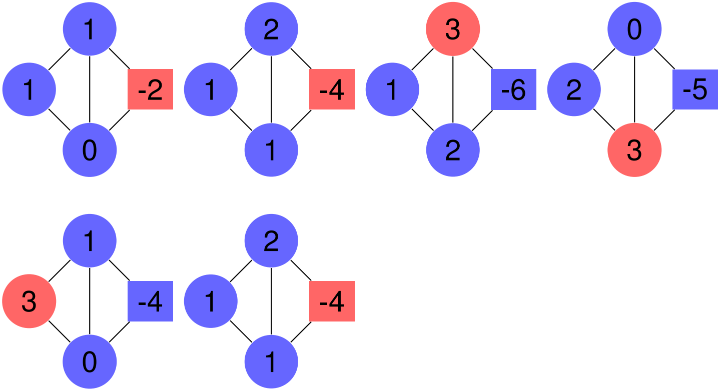

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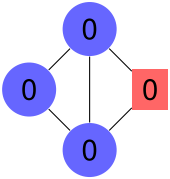

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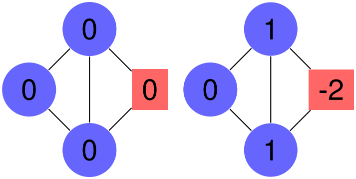

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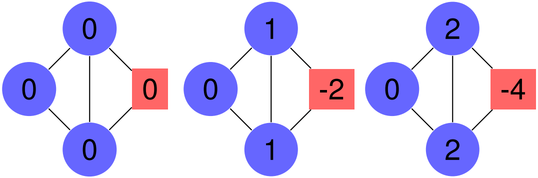

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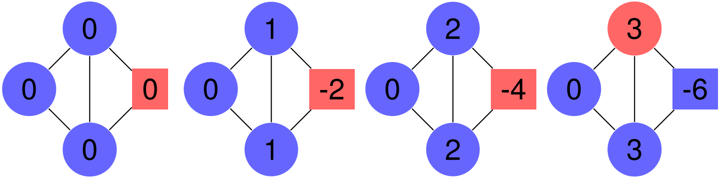

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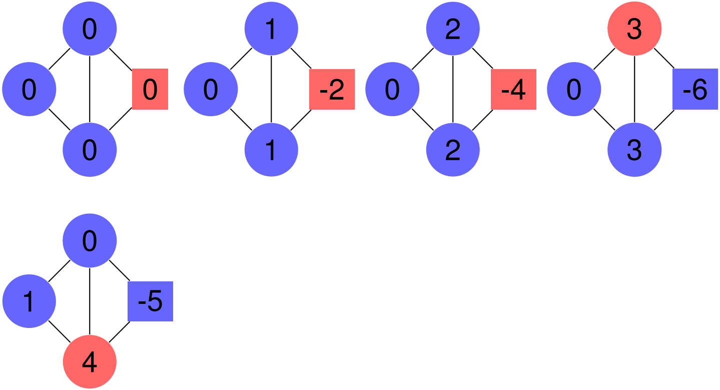

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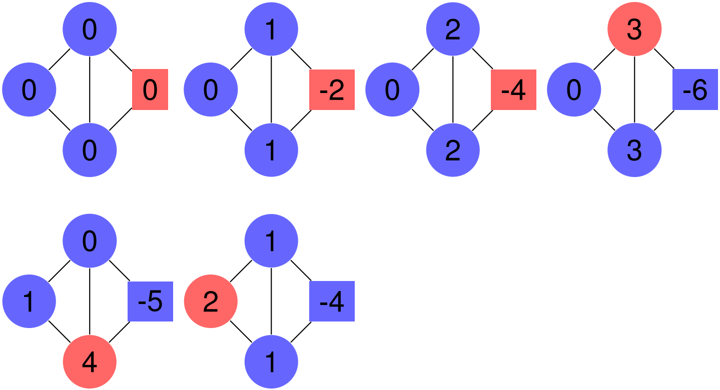

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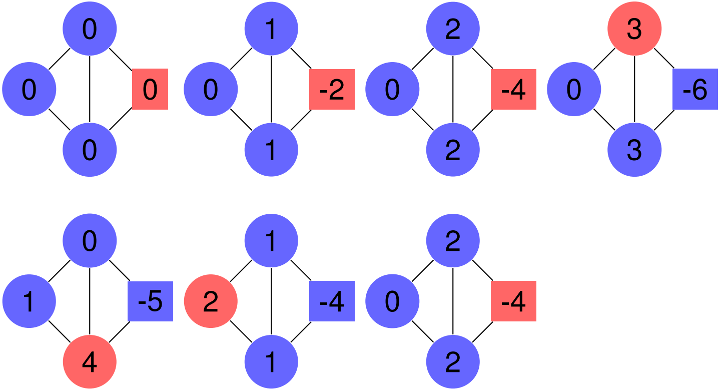

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**Theorem**

*The set of critical configurations has a natural group operation making it isomorphic to the critical group $K(\Gamma)$.*

# Critical groups of graphs

# Smith normal form

Henry John Stephen Smith (1826-1883)

- Given an integer matrix $X$, there exist unimodular integer matrices $P$ and $Q$ such that

$$PXQ = \left[\begin{array}{c:c} Y & 0 \\ \hdashline 0 & 0 \end{array}\right], \quad Y = \mathrm{diag}(s_1, s_2, \ldots s_r), \quad s_1 | s_2 | \cdots | s_r.$$

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- the SNF of the Laplacian gives the structure of the critical group.

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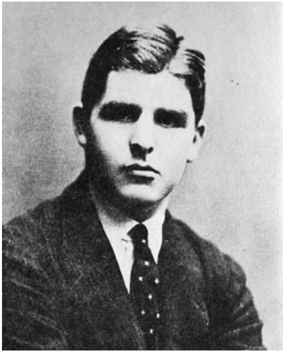

Raymond E. A. C. Paley (1907-33)

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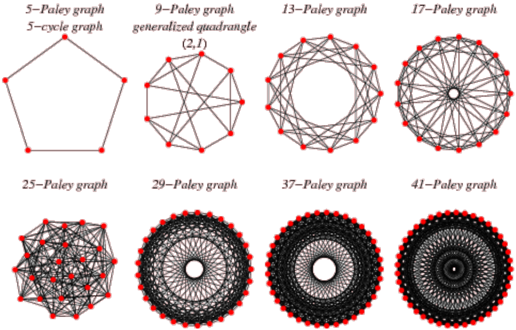
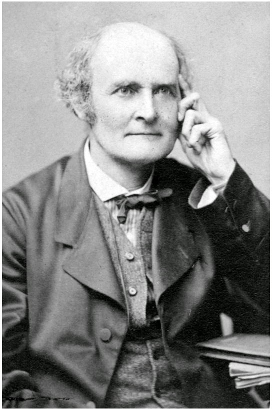
- Vertex set is $\mathbb{F}_q$, $q = p^t \equiv 1 \pmod 4$
- $S =$ set of nonzero squares in $\mathbb{F}_q$
- two vertices $x$ and $y$ are joined by an edge iff $x - y \in S$.




Some Paley graphs (from Wolfram Mathworld)

# Paley graphs are Cayley graphs

We can view $\mathrm{P}(q)$ as a Cayley graph on $(\mathbb{F}_q, +)$ with connecting set $S$



Arthur Cayley (1821-95)

# Paley graphs are strongly regular graphs

It is well known and easily checked that $\mathrm{P}(q)$ is a *strongly regular graph* and that its eigenvalues are $k = \frac{q-1}{2}$, $r = \frac{-1+\sqrt{q}}{2}$ and $s = \frac{-1-\sqrt{q}}{2}$, with multiplicities 1, $\frac{q-1}{2}$ and $\frac{q-1}{2}$, respectively.

# Critical groups of graphs

David Chandler and Qing Xiang

# Symmetries

- $$|K(\mathrm{P}(q))| = \frac{1}{q}\left(\frac{q+\sqrt{q}}{2}\right)^k \left(\frac{q-\sqrt{q}}{2}\right)^k = q^{\frac{q-3}{2}}\mu^k,$$

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  where $\mu = \frac{q-1}{4}$.
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# $p'$-part


Joseph Fourier (1768-1830)

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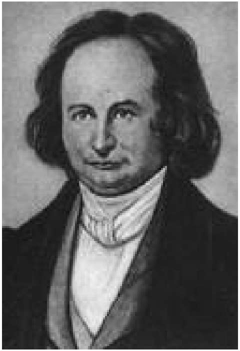
- $$\frac{1}{q} X L \overline{X}^t = \mathrm{diag}(k - \psi(S))_\psi, \tag{1}$$
- Interpret this as $PLQ$-equivalence over suitable local rings of integers.

**Theorem**
$K(\mathrm{P}(q))_{p'} \cong (\mathbf{Z}/\mu\mathbf{Z})^{2\mu}$, *where* $\mu = \frac{q-1}{4}$.

# The $p$-part



Carl Gustav Jacob Jacobi (1804-51)

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$$e_i = \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1})[x].$$

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# Jacobi Sums

The *Jacobi sum* of two nontrivial characters $T^a$ and $T^b$ is

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**Lemma**

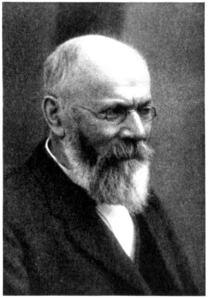
(i) $L(\mathbf{1}) = 0$.

(ii) $L(e_k) = \frac{1}{2}(\mathbf{1} - q([0] - e_k))$.

(iii) $L([0]) = \frac{1}{2}(q[0] - e_k - \mathbf{1})$.

### Corollary

*The Laplacian matrix L is equivalent over R to the diagonal matrix with diagonal entries $J(T^{-i}, T^k)$, for $i = 1, \ldots, q-2$ and $i \neq k$, two 1s and one zero.*



Carl Friedrich Gauss (1777-1855)  Ludwig Stickelberger (1850-1936)

# Gauss and Jacobi

Gauss sums: If $1 \neq \chi \in \mathrm{Hom}(\mathbb{F}_q^\times, R^\times)$,

$$g(\chi) = \sum_{y \in \mathbb{F}_q^\times} \chi(y)\zeta^{\mathrm{tr}(y)},$$

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## Lemma

*If $\chi$ and $\psi$ are nontrivial multiplicative characters of $\mathbb{F}_q^\times$ such that $\chi\psi$ is also nontrivial, then*

$$J(\chi, \psi) = \frac{g(\chi) g(\psi)}{g(\chi\psi)}.$$

# Stickelberger's Congruence

**Theorem**

*For $0 < a < q-1$, write $a$ $p$-adically as*

$$a = a_0 + a_1 p + \cdots + a_{t-1} p^{t-1}.$$

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**Theorem**

*Let $a, b \in \mathbf{Z}/(q-1)\mathbf{Z}$, with $a, b, a + b \not\equiv 0 \pmod{q-1}$. Then number of times that $p$ divides $J(T^{-a}, T^{-b})$ is equal to the number of carries in the addition $a + b \pmod{q-1}$ when $a$ and $b$ are written in $p$-digit form.*

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- Reformulate as a count of closed walks on a certain directed graph.
- Transfer matrix method yields the generating function for our counting problem from the adjacency matrix of the digraph.

## Theorem

*Let $q = p^t$ be a prime power congruent to 1 modulo 4. Then the number of p-adic elementary divisors of $L(\mathrm{P}(q))$ which are equal to $p^\lambda,\ 0 \le \lambda < t$, is*

$$f(t,\lambda) = \sum_{i=0}^{\min\{\lambda, t-\lambda\}} \frac{t}{t-i} \binom{t-i}{i} \binom{t-2i}{\lambda - i} (-p)^i \left(\frac{p+1}{2}\right)^{t-2i}.$$

*The number of p-adic elementary divisors of $L(\mathrm{P}(q))$ which are equal to $p^t$ is $\left(\frac{p+1}{2}\right)^t - 2$.*

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Thank you for your attention!