Linear similarity of graphs

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New Directions in Combinatorics Workshop, National University Singapore, May 26th, 2016.

Outline

Introduction

Paley and Peisert graphs

Matrix similarity over rings of algebraic integers

ℓ-local similarity, for $\ell \neq p$

p-local similarity

Jacobi sums

Matrix invariants

Γ simple graph, A its $0-1$ adjacency matrix.

A is symmetric so *similar* (by orthogonal matrices) to a diagonal matrix

$$D = PAP^{-1}$$

A is integral, so is *equivalent* (by unimodular matrices) to its *Smith Normal Form*

$$E = UAV$$

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If Γ' is another graph, we can ask if A and A' are both similar (graphs cospectral) and equivalent.

Many examples exist, e.g. the saltire pair.

- But there may be some $c \in \mathbb{Z}$ such that $A + cI$ and $A' + cI$ are not equivalent.

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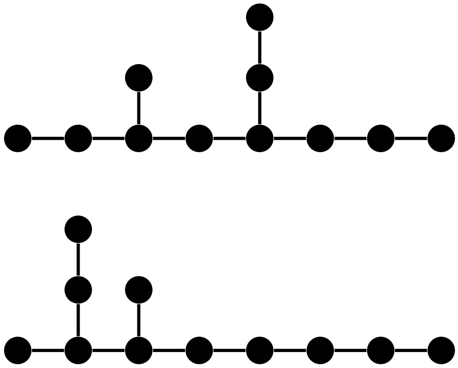
Example from T. Hall on MathOverflow

http://mathoverflow.net/questions/52169/adjacency-matrices-of-graphs/



Hall showed that the adjacency matrices A and A' are similar by a unimodular integral matrix.

Hence for any integers a, b, $aA + bI$ and $aA' + bI$ are both equivalent and similar.

But $A + J$ is not equivalent to $A' + J$, where J is the matrix whose entries are all equal to 1.

Question

Do there exist nonisomorphic graphs Γ and Γ' such that for all $a, b, c \in \mathbb{Z}$, the matrices $aA + bI + cJ$ and $aA' + bI + cJ$ are similar and equivalent?

These integral combinations are called *generalized adjacency matrices* and include the adjacency matrix of the complementary graph, the $(-1, 1, 0)$-adjacency matrix, and (for regular graphs) the Laplacian matrices.

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Strongly regular graphs

The adjacency matrix A of a strongly regular graph $SRG(v, k, \lambda, \mu)$ satisfies

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J$$

Thus if Γ and Γ' are SRGs with the same parameters, and $\mu \neq 0$, any invertible matrix C transforming A to A' must fix J and conjugate $aA + bI + cJ$ to $aA' + bI + cJ$.

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A family of examples

The rest of this talk is to give an infinite sequence of pairs of graphs such that for all integers a, b, c, the matrices $aA + bI + cJ$ and $aA' + bI + cJ$ are both similar and equivalent. The examples come from Paley graphs and Peisert graphs over fields of order p^2, $p \equiv 3 \pmod 4$. I stumbled across them in the process of computing critical groups (Smith Normal forms of Laplacians). Techniques I'll describe for proving equivalence grew out work a paper of Chandler-S-Xiang (2014) computing the critical groups of Paley graphs.

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Both graphs can be defined easily as Cayley graphs.

Let $q \equiv 1 \pmod 4$, $S = {\mathbb{F}_q^\times}^2$. The *Paley graph* $\Gamma(q)$ is the Cayley graph based on the group $(\mathbb{F}_q, +)$ with generating set S.

Let $q = p^{2e}$, $p \equiv 3 \pmod 4$. and β a generator of $\mathbb{F}_q^\times$. Set $S' = {\mathbb{F}_q^\times}^4 \cup \beta {\mathbb{F}_q^\times}^4$. The *Peisert graph* $\Gamma'(q)$ is the Cayley graph based on the group $(\mathbb{F}_q, +)$ with generating set S'.

When both are defined $\Gamma(q)$ and $\Gamma'(q)$ are strongly regular graphs with the same parameters $(q, \frac{(q-1)}{2}, \frac{(q-5)}{4}, \frac{(q-1)}{4})$. Hence they are cospectral.

Peisert (2001) showed that $\Gamma(q) \not\cong \Gamma'(q)$ if $q \neq 9$.

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Theorem

(Guralnick (1980), Taussky(1979), Dade(1963), Reiner-Zassenhaus (1971)) Let D be the ring of algebraic integers in a number field K. Suppose that B and B' are square matrices with entries in D Then the following are equivalent.

(i) *B and B' are similar over D_P for every prime ideal P of D.*

(ii) *B and B' are similar over some finite integral extension of D.*

(iii) *There is a finite extension L of K, such that for each for each prime P of D, there is a prime Q of the ring E of integers of L, with $Q \supseteq P$, such that B and B' are similar over the local ring E_Q.*

Note that the SNF is locally determined.

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Discrete Fourier transform

X, complex character table of $(\mathbb{F}_q, +)$ with elements ordered in the same way as for the rows and columns of $A(q)$.

X is invertible as a matrix in the ring $\mathbb{Z}[\zeta][\frac{1}{p}]$, ζ a complex primitive p-th root of unity.

(McWilliams-Mann (1968))

$$\begin{aligned} XA(q)X^{-1} &= \mathrm{diag}(\psi(S))_\psi \\ &= U\,\mathrm{diag}(\psi(S'))_\psi U^{-1} = UXA'(q)X^{-1}U^{-1}. \end{aligned} \tag{1}$$

where ψ runs over the additive characters of $\mathbb{F}_q$ and $\psi(S) = \sum_{y \in S} \psi(y)$. Thus, the $\psi(S)$ are the eigenvalues of A.

Since A' and A are cospectral, we can extend the equation with some permutation matrix U.

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ℓ-local similarity

For any prime $\ell \neq p$, choose a prime ideal Λ of $\mathbb{Z}[\zeta]$ containing ℓ.

Equation (1) can be viewed as similarity over $\mathbb{Z}[\zeta]_\Lambda$.

$$XA(q)X^{-1} = UXA'(q)X^{-1}U^{-1}.$$

Proposition

Assume $q = p^{2e}$, $p \equiv 3 \pmod 4$. For each prime $\ell \neq p$, $A(q)$ is similar to $A'(q)$ over $\mathbb{Z}[\zeta]_\Lambda$, where Λ is a prime ideal containing ℓ.

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For convenience, replace A and A' by $K = 2A + I$ and $K' = 2A' + I$.

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The module $R^{\mathbb{F}_q}$

- $R_0 = Z[t]/\Phi_{q-1}(t) \cong \mathbb{Z}[\xi]$, ξ a primitive $(q-1)$-st root of unity.
- p is unramified in R_0, so if P is a prime ideal of R_0 containing p, then $R = (R_0)_P$ is a DVR with maximal ideal pR and $R/pR \cong \mathbb{F}_q$.
- $R^{\mathbb{F}_q}$ has basis elements $[x]$ for $x \in \mathbb{F}_q$.
- $\mu_K, \mu_{K'} : R^{\mathbb{F}_q} \to R^{\mathbb{F}_q}$, left multiplication.

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- $T : \mathbb{F}_q^\times \to R^\times$, $T(\beta^j) = \xi^j$, Teichmüller character, generates $\mathrm{Hom}(\mathbb{F}_q^\times, R^\times)$.
- $\mathbb{F}_q^\times$ acts on $R^{\mathbb{F}_q} = R[0] \oplus R^{\mathbb{F}_q^\times}$
- $R^{\mathbb{F}_q^\times}$ decomposes further into the direct sum of $\mathbb{F}_q^\times$-invariant components of rank 1, affording the characters T^i, $i = 0, \ldots, q-2$.
- The component affording T^i is spanned by

$$e_i = \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1})[x].$$

- New basis $\{e_i \mid i = 1, \ldots q-2\} \cup \{\mathbf{1}, [0]\}$,

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Next consider the action of the subgroup $H = \mathbb{F}_q^{\times 4}$ of fourth powers.

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Outline

Introduction

Paley and Peisert graphs

Matrix similarity over rings of algebraic integers

ℓ-local similarity, for $\ell \neq p$

p-local similarity

Jacobi sums

Jacobi Sums

Definition

Let θ and ψ be multiplicative characters of $\mathbb{F}_q^\times$ taking values in $R^\times$. The *Jacobi sum* is

$$J(\theta, \psi) = \sum_{x \in \mathbb{F}_q} \theta(x)\psi(1 - x).$$

(At $x = 0$, nonprinc. chars take value 0, princ. char takes value 1.)

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\end{aligned}$$

Notation

- Recall $r = \frac{(p^2-1)}{4}$.
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The matrix of μ_K on M_i is

$$K_i = \begin{bmatrix} 0 & J(i{+}2r,2r) & 0 & 0 \\ J(i,2r) & 0 & 0 & 0 \\ 0 & 0 & 0 & J(i{+}3r,2r) \\ 0 & 0 & J(i{+}r,2r) & 0 \end{bmatrix}$$

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$$K_0' \begin{bmatrix} q & 1 & -1 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J(3r,2r) \\ 0 & 0 & 0 & J(r,2r) & 0 \end{bmatrix}$$

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Outline of proof of R-similarity of K_i and K_i'

Proof of similarity of K_i' with K_i involves finding a new basis.

The definition of the new basis is not uniform for all i but depends on the p-adic valuations of the Jacobi sums appearing in these matrices.

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p-adic valuation of Jacobi Sums

Let $j \in \mathbb{Z}$ with $j \not\equiv 0 \pmod{(p^2-1)}$.

p-digit expresssion: $j = a_0 + a_1 p$, $0 \le a_i \le p-1$.

Set $s(j) = a_0 + a_1$.

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This valuation can be viewed as the number of carries, when adding the p-expansions of i and j, modulo p^2-1.

Finally, we also need the exact values (Berndt-Evans (1979))

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Concluding remarks

For all primes ℓ, $A(q)$ is similar to $A'(q)$ over $\mathbb{Z}_{(\ell)}$.

For all integers a,b,c the generalized adjacency matrices $aA(q) + bI + cJ$ and $aA'(q) + bI + cJ$ are cospectral and equivalent.

For which values of q are they are similar over $\mathbb{Z}$?

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Thank you for your attention!