Linear similarity of graphs

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Introduction

Paley and Peisert graphs

Matrix similarity over rings of algebraic integers

$\ell$-local similarity, for $\ell \neq p$

$p$-local similarity

Jacobi sums
Matrix invariants

Γ simple graph, A its 0 – 1 adjacency matrix.

A is symmetric so similar (by orthogonal matrices) to a diagonal matrix

\[ D = PAP^{-1} \]

A is integral, so is equivalent (by unimodular matrices) to its Smith Normal Form

\[ E = UAV \]
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http://mathoverflow.net/questions/52169/adjacency-matrices-of-graphs/
Hall showed that the adjacency matrices $A$ and $A'$ are similar by a unimodular integral matrix. Hence for any integers $a$, $b$, $aA + bl$ and $aA' + bl$ are both equivalent and similar. But $A + J$ is not equivalent to $A' + J$, where $J$ is the matrix whose entries are all equal to 1.

Question
Do there exist nonisomorphic graphs $\Gamma$ and $\Gamma'$ such that for all $a$, $b$, $c \in \mathbb{Z}$, the matrices $aA + bl + cJ$ and $aA' + bl + cJ$ are similar and equivalent?

These integral combinations are called *generalized adjacency matrices* and include the adjacency matrix of the complementary graph, the $(-1, 1, 0)$-adjacency matrix, and (for regular graphs) the Laplacian matrices.
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These integral combinations are called *generalized adjacency matrices* and include the adjacency matrix of the complementary graph, the $(-1, 1, 0)$-adjacency matrix, and (for regular graphs) the Laplacian matrices.
The adjacency matrix $A$ of a strongly regular graph $\text{SRG}(v, k, \lambda, \mu)$ satisfies

\[ A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J \]

Thus if $\Gamma$ and $\Gamma'$ are SRGs with the same parameters, and $\mu \neq 0$, any invertible matrix $C$ transforming $A$ to $A'$ must fix $J$ and conjugate $aA + bl + cJ$ to $aA' + bl + cJ$. 
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The rest of this talk is to give an infinite sequence of pairs of graphs such that for all integers $a$, $b$, $c$, the matrices $aA + bl + cJ$ and $aA' + bl + cJ$ are both similar and equivalent. The examples come from Paley graphs and Peisert graphs over fields of order $p^2$, $p \equiv 3 \pmod{4}$. I stumbled across them in the process of computing critical groups (Smith Normal forms of Laplacians). Techniques I’ll describe for proving equivalence grew out work a paper of Chandler-S-Xiang (2014) computing the critical groups of Paley graphs.
A family of examples

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Paley graphs, Peisert graphs

Both graphs can be defined easily as Cayley graphs.

Let $q \equiv 1 \pmod{4}$, $S = \mathbb{F}_q^2$. The Paley graph $\Gamma(q)$ is the Cayley graph based on the group $(\mathbb{F}_q, +)$ with generating set $S$.

Let $q = p^{2e}$, $p \equiv 3 \pmod{4}$, and $\beta$ a generator of $\mathbb{F}_q^\times$. Set $S' = \mathbb{F}_q^\times 4 \cup \beta \mathbb{F}_q^\times 4$. The Peisert graph $\Gamma'(q)$ is the Cayley graph based on the group $(\mathbb{F}_q, +)$ with generating set $S'$.

When both are defined $\Gamma(q)$ and $\Gamma'(q)$ are strongly regular graphs with the same parameters $(q, \frac{(q-1)}{2}, \frac{(q-5)}{4}, \frac{(q-1)}{4})$. Hence they are cospectral.

Peisert (2001) showed that $\Gamma(q) \not\cong \Gamma'(q)$ if $q \neq 9$. 
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Theorem

(Guralnick (1980), Taussky(1979), Dade(1963), Reiner-Zassenhaus (1971)) Let $D$ be the ring of algebraic integers in a number field $K$. Suppose that $B$ and $B'$ are square matrices with entries in $D$ Then the following are equivalent.

(i) $B$ and $B'$ are similar over $D_P$ for every prime ideal $P$ of $D$.

(ii) $B$ and $B'$ are similar over some finite integral extension of $D$.

(iii) There is a finite extension $L$ of $K$, such that for each prime $P$ of $D$, there is a prime $Q$ of the ring $E$ of integers of $L$, with $Q \supseteq P$, such that $B$ and $B'$ are similar over the local ring $E_Q$.

Note that the SNF is locally determined.
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Discrete Fourier transform

Let $X$, the complex character table of $(\mathbb{F}_q, +)$ with elements ordered in the same way as for the rows and columns of $A(q)$. $X$ is invertible as a matrix in the ring $\mathbb{Z}[\zeta][\frac{1}{\rho}]$, $\zeta$ a complex primitive $p$-th root of unity. (McWilliams-Mann (1968))

$$X A(q) X^{-1} = \text{diag}(\psi(S))_\psi$$
$$= U \text{diag}(\psi(S'))_{\psi} U^{-1} = UXA'(q)X^{-1}U^{-1}. \quad (1)$$

where $\psi$ runs over the additive characters of $\mathbb{F}_q$ and $\psi(S) = \sum_{y \in S} \psi(y)$. Thus, the $\psi(S)$ are the eigenvalues of $A$.

Since $A'$ and $A$ are cospectral, we can extend the equation with some permutation matrix $U$. 
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ℓ-local similarity

For any prime $\ell \neq p$, choose a prime ideal $\Lambda$ of $\mathbb{Z}[\zeta]$ containing $\ell$.

Equation (1) can be viewed as similarity over $\mathbb{Z}[\zeta]_\Lambda$.

$$XA(q)X^{-1} = UXA'(q)X^{-1}U^{-1}.$$ 

Proposition

Assume $q = p^{2e}$, $p \equiv 3 \pmod{4}$. For each prime $\ell \neq p$, $A(q)$ is similar to $A'(q)$ over $\mathbb{Z}[\zeta]_\Lambda$, where $\Lambda$ is a prime ideal containing $\ell$. 
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We wish to show that $A = A(p^2)$ is similar to $A' = A'(p^2)$ over the localization of some ring of algebraic integers at a prime containing $p$.

For convenience, replace $A$ and $A'$ by $K = 2A + I$ and $K' = 2A' + I$. 
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The module $R_{\mathbb{F}_q}$

- $R_0 = \mathbb{Z}[t]/\Phi_{q-1}(t) \cong \mathbb{Z}[\xi]$, $\xi$ a primitive $(q - 1)$-st root of unity.
- $p$ is unramified in $R_0$, so if $P$ is a prime ideal of $R_0$ containing $p$, then $R = (R_0)_P$ is a DVR with maximal ideal $pR$ and $R/pR \cong \mathbb{F}_q$.
- $R_{\mathbb{F}_q}$ has basis elements $[x]$ for $x \in \mathbb{F}_q$.
- $\mu_K, \mu_{K'} : R_{\mathbb{F}_q} \to R_{\mathbb{F}_q}$, left multiplication.
The module $R^\mathbb{F}_q$

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- $R_{F,q}$ has basis elements $[x]$ for $x \in F_q$.

- $\mu_K, \mu_{K'} : R_{F,q} \to R_{F,q}$, left multiplication.
The module $R^{\mathbb{F}_q}$

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- $R^{\mathbb{F}_q}$ has basis elements $[x]$ for $x \in \mathbb{F}_q$.
- $\mu_K, \mu_K' : R^{\mathbb{F}_q} \rightarrow R^{\mathbb{F}_q}$, left multiplication.
\( T : \mathbb{F}_q^\times \to R^\times \), \( T(\beta^j) = \xi^j \), Teichmüller character, generates \( \text{Hom}(\mathbb{F}_q^\times, R^\times) \).

- \( \mathbb{F}_q^\times \) acts on \( R_{\mathbb{F}_q} = R[0] \oplus R_{\mathbb{F}_q}^\times \)

- \( R_{\mathbb{F}_q}^\times \) decomposes further into the direct sum of \( \mathbb{F}_q^\times \)-invariant components of rank 1, affording the characters \( T^i, i = 0, \ldots, q - 2 \).

- The component affording \( T^i \) is spanned by

\[
e_i = \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1})[x].
\]

- New basis \( \{ e_i \mid i = 1, \ldots q - 2 \} \cup \{ 1, [0] \} \),
$T : \mathbb{F}_q^\times \to R^\times$, $T(\beta^j) = \xi^j$, Teichmüller character, generates $\text{Hom}(\mathbb{F}_q^\times, R^\times)$.

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\( T : \mathbb{F}_q^\times \to R^\times \), \( T(\beta^i) = \xi^i \), Teichmüller character, generates \( \text{Hom}(\mathbb{F}_q^\times, R^\times) \).

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\( R^{\mathbb{F}_q} \) decomposes further into the direct sum of \( \mathbb{F}_q^\times \)-invariant components of rank 1, affording the characters \( T^i, i = 0, \ldots, q - 2 \).

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$r := \frac{(q-1)}{4}$.

$T^i, T^{i+r}, T^{i+2r},$ and $T^{i+3r}$ are equal on $H$. For $i \notin \{0, r, 2r, 3r\}$ the elements $e_i, e_{i+r}, e_{i+2r}$ and $e_{i+3r}$ span the $H$-isotypic component

$$M_i = \{ m \in R^{\mathbb{F}_q} \mid ym = T^i(y)m, \forall y \in H \}$$

of $R^{\mathbb{F}_q}$ for $1 \leq i \leq \frac{q-5}{4}$.

$M_0$, the submodule of $H$-fixed points in $R^{\mathbb{F}_q}$. Basis elements $1 = \sum_{x \in \mathbb{F}_q} x = e_0 + [0], [0], e_r, e_{2r}$ and $e_{3r}$.

$R^{\mathbb{F}_q} = M_0 \oplus \bigoplus_{i=1}^{q-5} 4 M_i$.

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$\mathbb{F}_q^\times 4$-decomposition

Next consider the action of the subgroup $H = \mathbb{F}_q^\times 4$ of fourth powers.

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Introduction

Paley and Peisert graphs

Matrix similarity over rings of algebraic integers

$\ell$-local similarity, for $\ell \neq p$

$p$-local similarity

Jacobi sums
Definition
Let $\theta$ and $\psi$ be multiplicative characters of $\mathbb{F}_q^\times$ taking values in $R^\times$. The Jacobi sum is

$$J(\theta, \psi) = \sum_{x \in \mathbb{F}_q} \theta(x)\psi(1 - x).$$

(At $x = 0$, nonprinc. chars take value 0, princ. char takes value 1.)

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$$= \sum_{z \in F_q} \sum_{x \in F_q^\times} \chi_S(z - x) T^i(x^{-1})[z].$$
Notation

- Recall \( r = \frac{(p^2-1)}{4} \).
- \( \eta = \xi^r, \alpha = \frac{(\eta-1)}{2}, \overline{\alpha} = \frac{(\eta+1)}{2} \)
- Write \( J(T^{-i}, T^{-j}) \) as \( J(i, j) \) for short.
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The matrix of $\mu_K$ on $M_i$ is

$$K_i = \begin{bmatrix}
0 & J(i+2r,2r) & 0 & 0 \\
J(i,2r) & 0 & 0 & 0 \\
0 & 0 & 0 & J(i+3r,2r) \\
0 & 0 & J(i+r,2r) & 0
\end{bmatrix}$$

The matrix of $\mu_{K'}$ on $M_i$ is

$$K'_i = \begin{bmatrix}
0 & 0 & \alpha J(i+r,3r) & \bar{\alpha} J(i+3r,r) \\
0 & 0 & \bar{\alpha} J(i+r,r) & \alpha J(i+3r,3r) \\
\bar{\alpha} J(i,r) & \alpha J(i+2r,3r) & 0 & 0 \\
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\end{bmatrix}$$
The matrix of $\mu_K$ on $M_0$ is

$$K_0' = \begin{bmatrix}
q & 1 & -1 & 0 & 0 \\
0 & 0 & q & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & J(3r, 2r) \\
0 & 0 & 0 & J(r, 2r) & 0
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$$K_0' = \begin{bmatrix}
q & 1 & -\alpha & 0 & -\bar{\alpha} \\
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Outline of proof of $R$-similarity of $K_i$ and $K'_i$

Proof of similarity of $K'_i$ with $K_i$ involves finding a new basis. The definition of the new basis is not uniform for all $i$ but depends on the $p$-adic valuations of the Jacobi sums appearing in these matrices. By close examination of Jacobi sums, we can reduce to just three cases, corresponding to whether $K_i$ has $p$-rank 1, 2, or 3.
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Let $j \in \mathbb{Z}$ with $j \not\equiv 0 \pmod{(p^2 - 1)}$.

$p$-digit expression: $j = a_0 + a_1 p$, $0 \leq a_i \leq p - 1$.

Set $s(j) = a_0 + a_1$.

$r = \frac{p^2 - 1}{4} = \frac{3(p-1)}{4} + \frac{p-3}{4} p$.

$3r = \frac{p^2 - 1}{4} = \frac{p-3}{4} + \frac{3(p-1)}{4} p$.

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$p$-adic valuation of Jacobi Sums

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More on Jacobi sums

By Stickelberger’s Theorem and relation between Gauss sums and Jacobi sums, we know that when \( i, j \) and \( i + j \) are not divisible by \( p^2 - 1 \) the \( p \)-adic valuation of \( J(i, j) \) is equal to

\[
c(i, j) := \frac{1}{p - 1} (s(i) + s(j) - s(i + j)),
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This valuation can be viewed as the number of carries, when adding the \( p \)-expansions of \( i \) and \( j \), modulo \( p^2 - 1 \). Finally, we also need the exact values (Berndt-Evans (1979))

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Concluding remarks

For all primes $\ell$, $A(q)$ is similar to $A'(q)$ over $\mathbb{Z}(\ell)$.

For all integers $a,b,c$ the generalized adjacency matrices $aA(q) + bl + cJ$ and $aA'(q) + bl + cJ$ are cospectral and equivalent.

For which values of $q$ are they similar over $\mathbb{Z}$?
Concluding remarks

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Thank you for your attention!