MAS4105

Second Exam (30 POINTS)

Time: 50 minutes. Read the questions very carefully. You may quote basic theorems from the first two chapters of the text as long as you state them clearly. Remember to write your arguments and calculations as clearly as possible. Bonus points may be awarded for clarity of exposition.

1.(10 points) Let V be the vector space of 2×2 matrices with real entries and let A be a fixed element of V.

(a) Show that the mapping $T : V \to V$ given by

$$T(X) = AX - XA, \quad X \in V,$$

is a linear mapping. (You may freely use rules of matrix algebra.)

(b) Let $A = \begin{pmatrix} 1 & 3 \\ -1 & 0 \end{pmatrix}$. Compute the matrix $[T]_\beta^\beta$ with respect to an ordered basis β of your choice.

(c) Find a nonzero element of $N(T)$ for the matrix A of (b).

2.(10 points) In $\mathbb{R}^2$, find the image of the triangle with vertices $(1, 2)$, $(2, 5)$ and $(3, 0)$ under the reflection in the line $y = 3x$. You may assume that this reflection is a linear map.

3.(10 points) Let $T : V \to W$ be a linear map.

(a) Prove directly from the definitions that if S is a generating set for V then $T(S) = \{T(s) \mid s \in S\}$ is a generating set for $R(T)$.

(b) Is it true that if S is a basis of V then $T(S)$ must be a basis of $R(T)$? Justify your answer with a proof or counterexample.