1.4.12. **Show that a subset W of a vector space V is a subspace if and only if $\mathrm{Span}(W) = W$.**

Suppose first that $\mathrm{Span}(W) = W$. Then by Theorem 1.5 $\mathrm{Span}(W)$ is a subspace, so W is a subspace.

Conversely, suppose that W is a subspace. Then, by definition of subspace, W is non-empty, and W is closed under addition and scalar multiplication. It follows that every linear combination of vectors of W lies in W. Thus, $\mathrm{Span}(W) \subseteq W$. On the other hand, for any $w \in W$, we have $w = 1.w$, so $w \in \mathrm{Span}(W)$. This shows that $W \subseteq \mathrm{Span}(W)$. We have proved that $W = \mathrm{Span}(W)$.

1.4.15. **Let S_1 and S_2 be subsets of a vector space V. Prove that $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$. Give an example in which $\mathrm{Span}(S_1 \cap S_2)$ and $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$ are equal and one in which they are unequal.**

Let $v \in \mathrm{Span}(S_1 \cap S_2)$. Then, by definition of span, v can be expressed as a linear combination $v = a_1v_1 + \cdots + a_mv_m$, where m is a nonnegative integer, $v_1, \ldots, v_m$ are elements of $S_1 \cap S_2$, and $a_1, \ldots, a_m$ are scalars. The elements v_i lie in S_1, so $v \in \mathrm{Span}(S_1)$, and they also lie in S_2, so $v \in \mathrm{Span}(S_2)$ as well. Thus, $v \in \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$. Since v was an arbitrary element of $\mathrm{Span}(S_1 \cap S_2)$, we have proved that $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.

Example of equality. Obviously, if $S_1 = S_2$, then $\mathrm{Span}(S_1 \cap S_2) = \mathrm{Span}(S_1) = \mathrm{Span}(S_2) = \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$. (As a concrete example, we could take $V = \mathbb{R}$, $S_1 = S_2 = \emptyset$.)

Unequal example. Let $V = \mathbb{R}$, $S_1 = \{1\}$ and $S_2 = \{2\}$, Then $S_1 \cap S_2 = \emptyset$, so $\mathrm{Span}(S_1 \cap S_2) = \mathrm{Span}(\emptyset) = \{0\}$, while $\mathrm{Span}(S_1) = \mathbb{R} = \mathrm{Span}(S_2) = \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.