1.4.12. **Show that a subset $W$ of a vector space $V$ is a subspace if and only if** $\text{Span}(W) = W$.

Suppose first that $\text{Span}(W) = W$. Then by Theorem 1.5 $\text{Span}(W)$ is a subspace, so $W$ is a subspace.

Conversely, suppose that $W$ is a subspace. Then, by definition of subspace, $W$ is non-empty, and $W$ is closed under addition and scalar multiplication. It follows that every linear combination of vectors of $W$ lies in $W$. Thus, $\text{Span}(W) \subseteq W$. On the other hand, for any $w \in W$, we have $w = 1 \cdot w$, so $w \in \text{Span}(W)$. This shows that $W \subseteq \text{Span}(W)$. We have proved that $W = \text{Span}(W)$.

1.4.15. **Let $S_1$ and $S_2$ be subsets of a vector space $V$. Prove that** $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1) \cap \text{Span}(S_2)$. **Give an example in which** $\text{Span}(S_1 \cap S_2)$ and $\text{Span}(S_1) \cap \text{Span}(S_2)$ **are equal and one in which they are unequal.**

Let $v \in \text{Span}(S_1 \cap S_2)$. Then, by definition of span, $v$ can be expressed as a linear combination $v = a_1 v_1 + \cdots + a_m v_m$, where $m$ is a nonnegative integer, $v_1, \ldots, v_m$ are elements of $S_1 \cap S_2$, and $a_1, \ldots, a_m$ are scalars. The elements $v_i$ lie in $S_1$, so $v \in \text{Span}(S_1)$, and they also lie in $S_2$, so $v \in \text{Span}(S_2)$ as well. Thus, $v \in \text{Span}(S_1) \cap \text{Span}(S_2)$. Since $v$ was an arbitrary element of $\text{Span}(S_1 \cap S_2)$, we have proved that $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1) \cap \text{Span}(S_2)$.

Example of equality. Obviously, if $S_1 = S_2$, then $\text{Span}(S_1 \cap S_2) = \text{Span}(S_1) = \text{Span}(S_2) = \text{Span}(S_1) \cap \text{Span}(S_2)$. (As a concrete example, we could take $V = \mathbb{R}$, $S_1 = S_2 = \emptyset$.)

Unequal example. Let $V = \mathbb{R}$, $S_1 = \{1\}$ and $S_2 = \{2\}$, Then $S_1 \cap S_2 = \emptyset$, so $\text{Span}(S_1 \cap S_2) = \text{Span}(\emptyset) = \{0\}$, while $\text{Span}(S_1) = \mathbb{R} = \text{Span}(S_2) = \text{Span}(S_1) \cap \text{Span}(S_2)$. 