The critical group of a graph

Peter Sin, U. of Florida

Gainesville International Number Theory Conference,
March 20th, 2016
in honor of Krishna Alladi's 60th birthday

Critical groups of graphs

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We'll discuss the general problem of computing the critical group for families of graphs, and the specific case of the Paley graphs.

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The finite group $K(\Gamma)$ is called the *critical group* of Γ.

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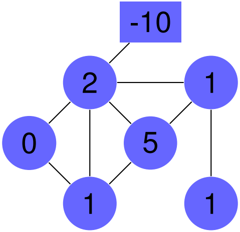
Also, $\det(\tilde{L}) = |K(\Gamma)| = \frac{1}{|V|}\prod_{j=2}^{|V|}\lambda_j$.

Rules



A *configuration* is an assignment of a nonnegative integer $s(v)$ to each round vertex v and $-\sum_v s(v)$ to the square vertex.

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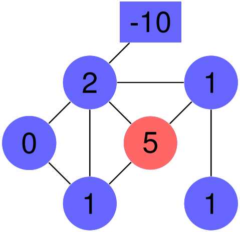

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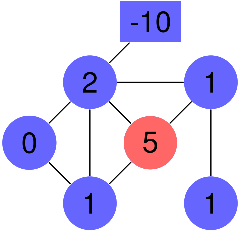

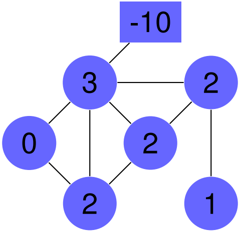

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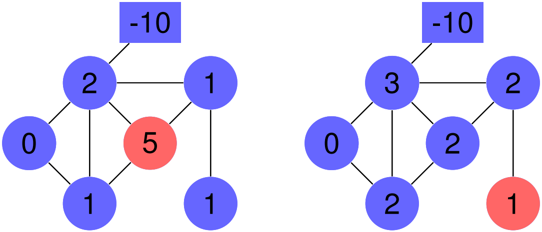

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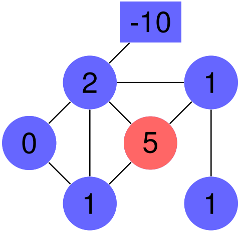

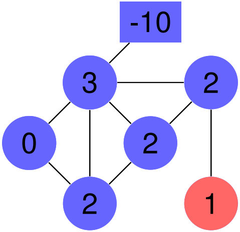

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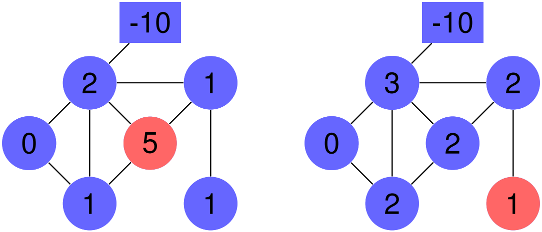

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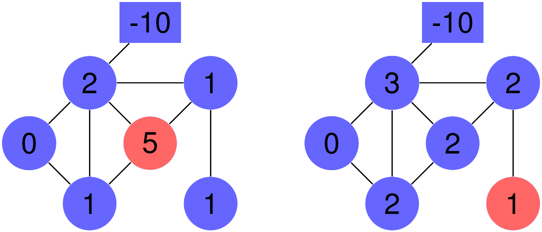

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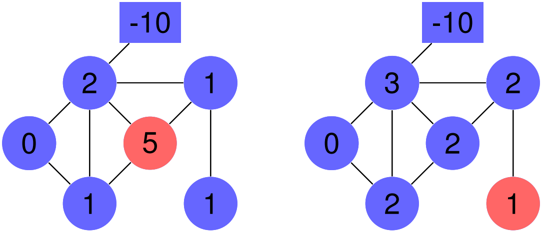

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A configuration is *critical* if it is both recurrent and stable.

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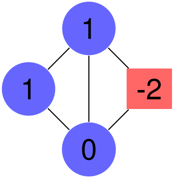

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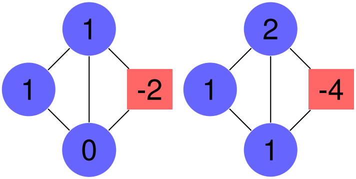

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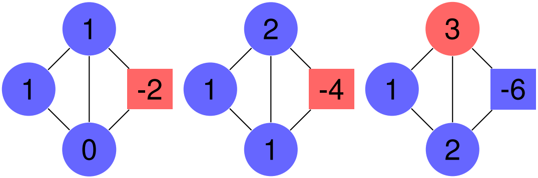

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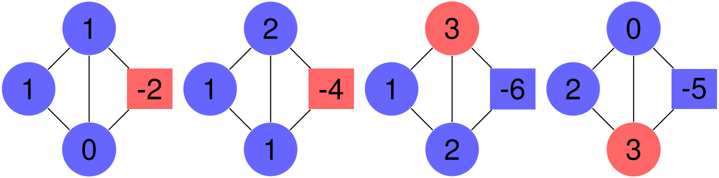

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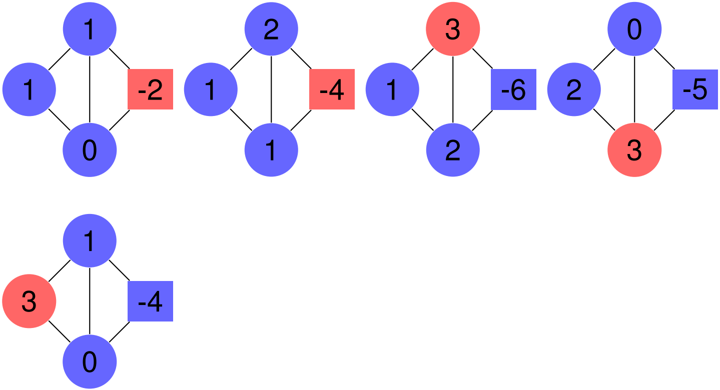

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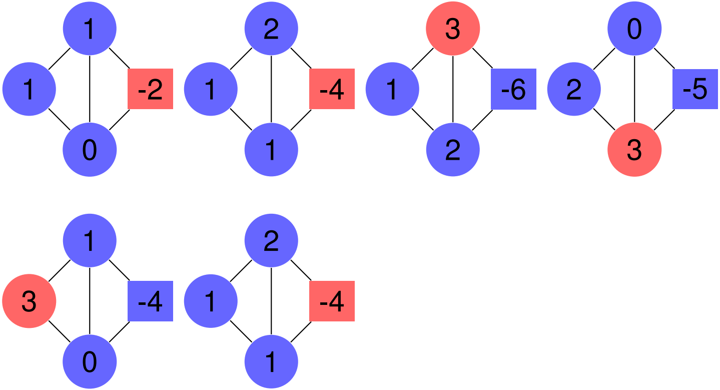

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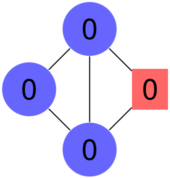

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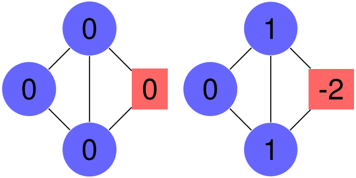

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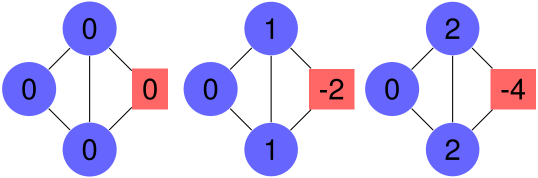

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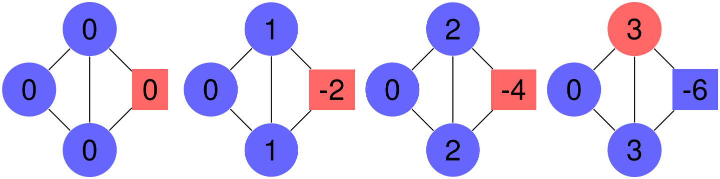

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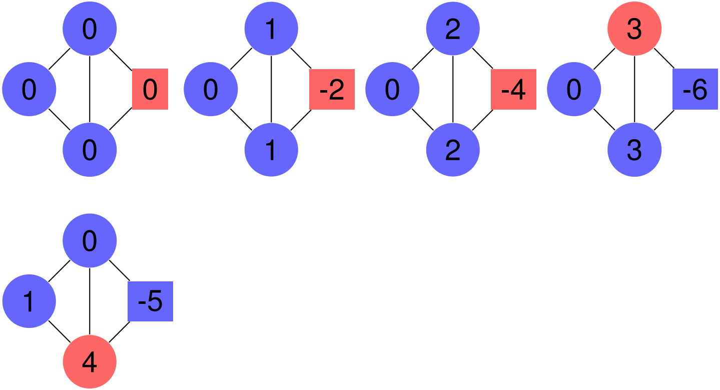

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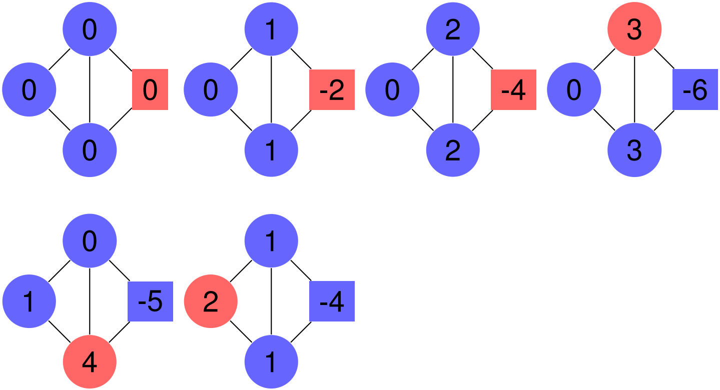

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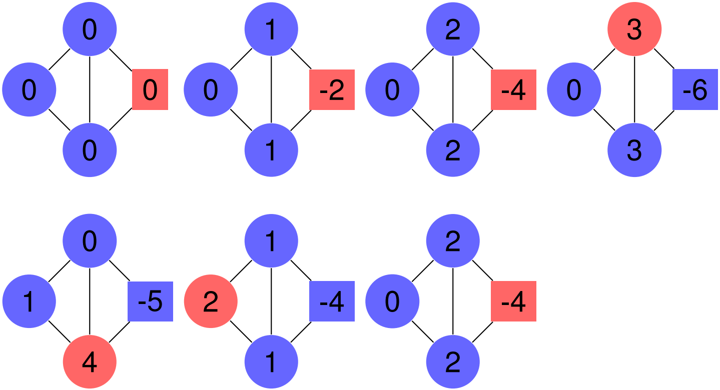

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Start with a configuration s and fire vertices in a sequence where each vertex v is fired $x(v)$ times, ending up with configuration s'.

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The set of critical configurations has a natural group operation making it isomorphic to the critical group $K(\Gamma)$.

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The SNF of the Laplacian gives the structure of the critical group.

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$\mathrm{P}(q)$ is a Cayley graph on $(\mathbb{F}_q, +)$ with connecting set S

Paley graphs are strongly regular graphs

$\mathrm{P}(q)$ is a *strongly regular graph*, self-complementary, with parameters $(v = q, k = \frac{(q-1)}{2}, \lambda = \frac{(q-5)}{4}, \mu = \frac{q-1}{4})$. Its eigenvalues are $k = \frac{q-1}{2}$, $r = \frac{-1+\sqrt{q}}{2}$ and $s = \frac{-1-\sqrt{q}}{2}$, with multiplicities 1, $\frac{q-1}{2}$ and $\frac{q-1}{2}$, respectively.

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$$\mathrm{Aut}(\mathrm{P}(q)) \geq \mathbb{F}_q \rtimes S.$$

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Use S-action to help compute p-part.

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This equation can be viewed as matrix similarity, hence equivalence, over suitable local rings of integers.

Theorem

$K(\mathrm{P}(q))_{p'} \cong (\mathbf{Z}/\mu\mathbf{Z})^{2\mu}$, *where* $\mu = \frac{q-1}{4}$.

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A basis element for E_i is

$$e_i = \sum_{x \in \mathbb{F}_q^\times} T^i(x^{-1})[x].$$

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L maps each M_i to itself.

$$L(e_i) = \sum_{x\in\mathbb{F}_q^\times} T^i(x^{-1})L([x]).$$

Jacobi Sums

The *Jacobi sum* of two nontrivial characters T^a and T^b is

$$J(T^a, T^b) = \sum_{x \in \mathbb{F}_q} T^a(x) T^b(1 - x).$$

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Lemma

(i) $L(\mathbf{1}) = 0$.

(ii) $L(e_k) = \frac{1}{2}(\mathbf{1} - q([0] - e_k))$.

(iii) $L([0]) = \frac{1}{2}(q[0] - e_k - \mathbf{1})$.

Corollary

The Laplacian matrix L is equivalent over R to the diagonal matrix with diagonal entries $J(T^{-i}, T^k)$, for $i = 1, \ldots, q-2$ and $i \neq k$, two 1s and one zero.

Gauss and Jacobi sums

Gauss sums: If $1 \neq \chi \in \mathrm{Hom}(\mathbb{F}_q^\times, R^\times)$,

$$g(\chi) = \sum_{y \in \mathbb{F}_q^\times} \chi(y)\zeta^{\mathrm{tr}(y)},$$

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Lemma

If χ and ψ are nontrivial multiplicative characters of $\mathbb{F}_q^\times$ such that $\chi\psi$ is also nontrivial, then

$$J(\chi, \psi) = \frac{g(\chi)g(\psi)}{g(\chi\psi)}.$$

Stickelberger's Congruence

Theorem

For $0 < a < q - 1$, write a p-adically as

$$a = a_0 + a_1 p + \cdots + a_{t-1} p^{t-1}.$$

Then the number of times that p divides $g(T^{-a})$ is $a_0 + a_1 + \cdots + a_{t-1}$.

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Corollary

Let $a, b \in \mathbf{Z}/(q-1)\mathbf{Z}$*, with* $a, b, a+b \not\equiv 0 \pmod{q-1}$*. Then number of times that p divides* $J(T^{-a}, T^{-b})$ *is equal to the number of carries in the addition* $a+b \pmod{q-1}$ *when a and b are written in p-digit form.*

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Transfer matrix method yields the generating function for our counting problem from the adjacency matrix of the digraph.

Theorem (CSX, 2015)

Let $q = p^t$ be a prime power congruent to 1 modulo 4. Then the number of p-adic elementary divisors of $L(\mathrm{P}(q))$ which are equal to p^λ, $0 \leq \lambda < t$, is

$$f(t,\lambda) = \sum_{i=0}^{\min\{\lambda, t-\lambda\}} \frac{t}{t-i}\binom{t-i}{i}\binom{t-2i}{\lambda-i}(-p)^i\left(\frac{p+1}{2}\right)^{t-2i}.$$

The number of p-adic elementary divisors of $L(\mathrm{P}(q))$ which are equal to p^t is $\left(\frac{p+1}{2}\right)^t - 2$.

Example: $K(\mathrm{P}(5^3))$

$$f(3,0) = 3^3 = 27,\ f(3,1) = \binom{3}{1} \cdot 3^3 - \tfrac{3}{2}\binom{2}{1}\binom{1}{0} \cdot 5 \cdot 3 = 36.$$

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$$K(\mathrm{P}(5^3)) \cong (\mathbf{Z}/31\mathbf{Z})^{62} \oplus (\mathbf{Z}/5\mathbf{Z})^{36} \oplus (\mathbf{Z}/25\mathbf{Z})^{36} \oplus (\mathbf{Z}/125\mathbf{Z})^{25}.$$

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Theorem

Assume $q = p^2$

(a) *There is a number field K such that A and A^* are similar as matrices over $\mathcal{O}_K$. (Uses local-global principle for similarity of matrices (Guralnick).)*

(b) *For all $c \in \mathbf{Z}$, the matrices $A + cI$ and $A^* + cI$ have the same SNF.*

Thank you for your attention!