Smith normal forms of matrices associated with the Grassmann graphs of lines in $\mathrm{PG}(n-1, q)$

Peter Sin, U. of Florida

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in honor of Robert Griess Jr.'s 71st birthday
Tapei, August 23rd, 2016

Smith normal forms

Critical groups of graphs

Graphs from lines

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Cross-characteristics

Defining characteristic

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Γ the complementary graph, is the skew lines graph.

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Definition

A strongly regular graph with parameters (v, k, λ, μ) is a k-regular graph such that

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Γ is also a SRG. So we have two families of SRGs paramtrized by n, p and t.

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Survey article on SNFs in combinatorics by R. Stanley (JCTA 2016).

Critical groups of graphs

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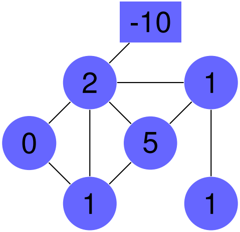

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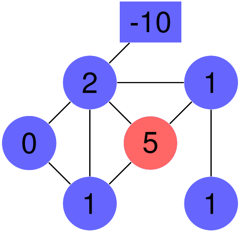

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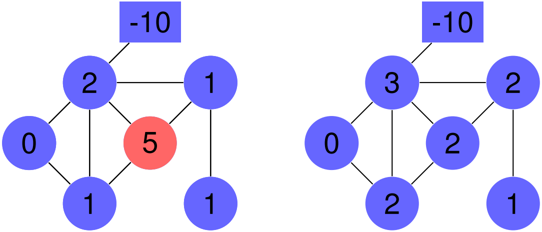

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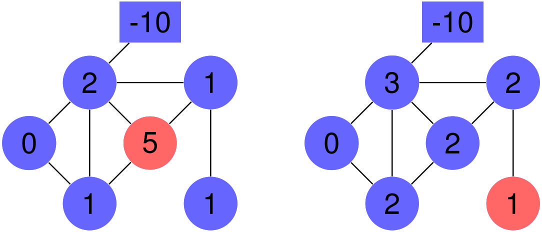

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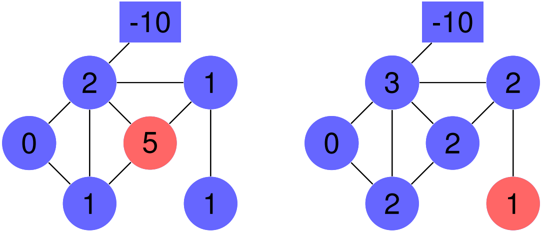

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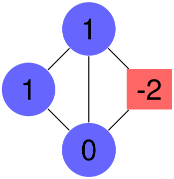
The square vertex is fired only when no others can be fired.

Example game



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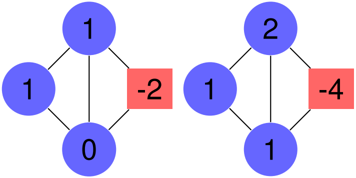

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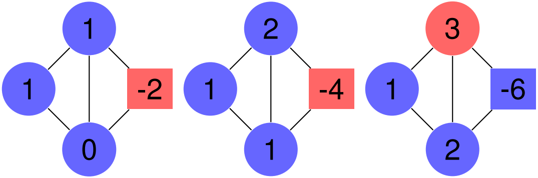

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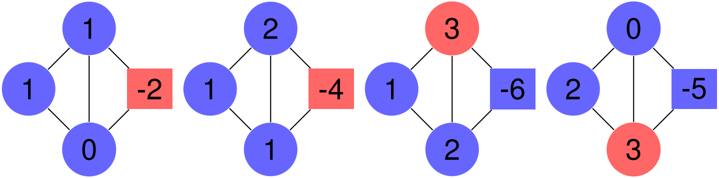

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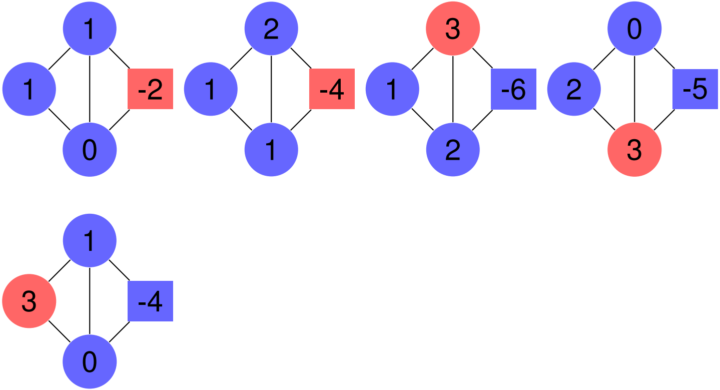

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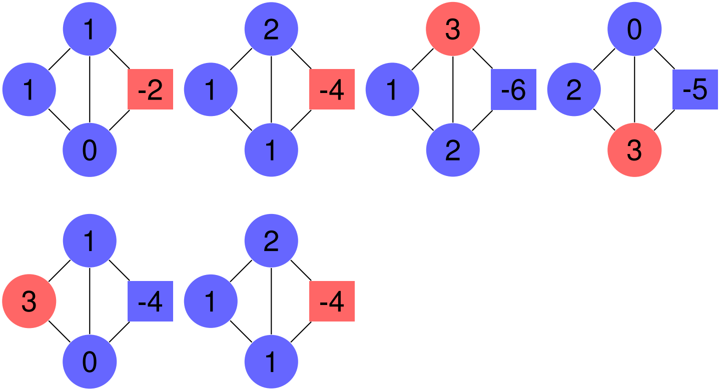

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A configuration is *stable* if no round vertex can be fired, *recurrent* if there is a sequence of firings leading back to the same configuration, *critical* if recurrent and stable.

Theorem

(Dhar, Björner-Lovász, Biggs, Gabrielov,...) Consider the chip-firing game on a connected graph $\mathcal{G}$.

Any starting configuration leads to a unique critical configuration.

The set of critical configurations has a natural group operation making it isomorphic to the critical group $K(\mathcal{G})$.

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- $$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r = \mathrm{Ker}(\alpha) \supseteq 0.$$

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All quotients $\overline{M}_a / \overline{M}_{a+1}$ are $\mathbb{F}_\ell \, \mathrm{GL}(n, q)$-modules, so the number of nonzero e_i is at most the composition length of $\overline{M}$ as a $\mathbb{F}_\ell \, \mathrm{GL}(n, q)$-module.

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Based on James results, it is easy to work out the submodule structure of $\overline{M}$ in all cases.

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We'll look at the case $c = 0$ and $b = 0$. Then $v_\ell(r) = v_\ell(s) = a$ and $v_\ell(|S(\Gamma)|) = af + ag + 2a$

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$\overline{A'}(\overline{Y})$ is nonzero, so it has D_1 as a composition factor.

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This shows that $(A' - (r+s)I)(Y) \subseteq M_{2a} \cap Y$, so as $\ell \mid (r+s)$, $\overline{A'}(\overline{Y}) \subseteq \overline{M_{2a} \cap Y}$.

$\overline{A'}(\overline{Y})$ is nonzero, so it has D_1 as a composition factor.

$\dim \overline{M_{2a} \cap Y} \geq f - 1$ and so $\dim \overline{M}_{2a} \geq f$.

Example: $\ell \mid q+1$, n even, $v_\ell(r) = v_\ell(s) = a$

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Further analysis of A' and structure of $\overline{M}$ shows $\dim \overline{M}_a \geq= g + 2$.

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- $$\begin{aligned} a(f+g) + 2a = v_\ell(|S(\Gamma)|) &= \sum_{i \geq 0} i e_i \\ &\geq \sum_{a \leq i < 2a} i e_i + \sum_{i \geq 2a} i e_i \\ &\geq a \sum_{a \leq i < 2a} e_i + 2a \sum_{i \geq 2a} e_i \\ &\geq a(\dim \overline{M}_a - \dim \overline{M}_{2a}) + 2a \dim \overline{M}_{2a} \\ &\geq a(g+2) + af. \end{aligned}$$

Therefore, equality holds throughout, and it follows that $e_0 = f - 1$, $e_a = g - f + 2$, $e_{2a} = f$ (and $e_i = 0$ otherwise).

Critical groups of graphs

Graphs from lines

Smith normal forms

Chip-firing game

Cross-characteristics

Defining characteristic

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- The skew lines matrices A and L are much harder but much of the difficulty was handled in earlier work for the case $n = 4$ (Brouwer-Ducey-S.)
- Note $A \equiv -L \mod (p^{4t})$, so just consider A.

Example

Table: The elementary divisors of the incidence matrix of lines vs. lines in $\mathrm{PG}(3,9)$, where two lines are incident when skew.

Elem. Div.	1	3	3^2	3^4	3^5	3^6	3^8
Multiplicity	361	256	6025	202	256	361	1

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For general n, we have

$$\begin{aligned} A(A + q(\left[\begin{smallmatrix} n-2 \\ 1 \end{smallmatrix}\right]_q - 2)I) \\ = q^3 \left[\begin{smallmatrix} n-3 \\ 1 \end{smallmatrix}\right]_q I + q^3 \frac{\left[\begin{smallmatrix} n-3 \\ 1 \end{smallmatrix}\right]_q \Big(\left[\begin{smallmatrix} n-1 \\ 1 \end{smallmatrix}\right]_q - (q+2) \Big)}{q+1} J, \end{aligned}$$

Reductions using SRG equation

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  4. Once we have e_i $0 \leq i < t$ we have them all.

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Nontrivial to relate p-elementary divisors of $A_{r,1}$ and $A_{1,s}$ to those of $A_{r,1}A_{1,s}$.

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  $d(\vec{s})$ is the dimension of a $\mathrm{GL}(n, q)$ composition factor of $\mathbb{F}_q^{\mathcal{L}_1}$.

For nonnegative integers α, β, define the subsets of $\mathcal{H}$

$$\mathcal{H}_\alpha(s) = \left\{(s_0, \ldots, s_{t-1}) \in \mathcal{H} \,\middle|\, \sum_{i=0}^{t-1} \max\{0, s - s_i\} = \alpha \right\}$$

and

$$\begin{aligned} {}_\beta\mathcal{H}(r) &= \{(n - s_0, \ldots, n - s_{t-1}) \,|\, (s_0, \ldots, s_{t-1}) \in \mathcal{H}_\beta(r)\} \\ &= \left\{(s_0, \ldots, s_{t-1}) \in \mathcal{H} \,\middle|\, \sum_{i=0}^{t-1} \max\{0, s_i - (n - r)\} = \beta \right\}. \end{aligned}$$

General formula for $e_i(A_{r,1}A_{1,s})$

Theorem

Let $E_i = e_i(A_{r,1}A_{1,s})$ denote the multiplicity of p^i as a p-adic elementary divisor of $A_{r,1}A_{1,s}$.

$E_{t(r+s)} = 1$.

For $i \neq t(r+s)$,

$$E_i = \sum_{\vec{s} \in \Gamma(i)} d(\vec{s}),$$

where

$$\Gamma(i) = \bigcup_{\substack{\alpha+\beta=i \\ 0\le\alpha\le t(s-1) \\ 0\le\beta\le t(r-1)}} {}_{\beta}\mathcal{H}(r) \cap \mathcal{H}_{\alpha}(s).$$

Thank you for your attention!