

HOMEWORK ASSIGNMENT # 5, DUE FEBRUARY 10, 2016

1) A large tank initially contains 200 gallons of saltwater solution in which 15 lbs. of salt are dissolved. Starting at $t=0$ a solution containing 4 lbs. of salt per gallon flows into the tank at the rate of 3 gallons per minute. The mixture is kept uniform by stirring and the well-stirred mixture leaves the tank at the rate of 4 gallons per minute.

a) How much salt is in the tank at the end of one hour?

b) How much salt is in the tank when the tank contains only 50 gallons of saltwater solution?

2) Newton's Law of cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and that of the medium in which it is situated. A body of temperature 80 degrees Fahrenheit is placed at time $t=0$ in a medium the temperature of which is maintained constantly at 40 degrees Fahrenheit. At the end of 10 minutes the body has cooled to a temperature of 70 degrees Fahrenheit.

a) What is the temperature of the body at the end of 20 minutes?

b) When will the temperature of the body be 50 degrees Fahrenheit?

3) A ship which weighs 32,000 tons starts from rest under the force of a constant propeller thrust of 150 tons. The magnitude of the resistance (in lbs.) of the water to the passage of the ship is numerically equal to $10,000 \times |v|$, where v is in feet per second.

a) Find the velocity of the ship as a function of time.

b) Find the limiting velocity (as $t \rightarrow \infty$).

c) Find how long it takes the ship to attain a velocity of 80% of the limiting velocity (i.e. the asymptotic velocity).

4) a) A *Riccati equation* is a differential equation of the form

$$y' = f(x) + g(x)y + h(x)y^2,$$

where the functions f, g, h are continuous in some interval (α, β) . Show that if $y_1(x)$ is any particular solution of the Riccati equation, then the change of variables $w(x) = \frac{1}{y(x) - y_1(x)}$ reduces the Riccati equation to a linear first-order ODE:

$$w' + [g(x) + 2y_1(x)h(x)]w = -h(x),$$

which you can solve. By inverting the transformation ($y(x) = y_1(x) + \frac{1}{w(x)}$), you can therefore solve the original Riccati equation.

b) A nice example of a concrete application where a Riccati equation arises is the following. In an article on the theory of the propagation of a single act in a large population (in other words, an act that is performed at most once in the lifetime of an individual, such as suicide), a researcher¹ derived the following Riccati equation:

$$y' = (1 - y)[x(t) + by].$$

¹A. Rapoport, *Bull. Math. Biophys.*, **14**, 159-169 (1952)

(In this application $y = y(t)$ is the fraction of the population who performed the act at time t , $x(t)$ represents the external stimulus, and b is a constant chosen so that $by(t)$ represents the imitation component.) With the choices $x(t) = -\frac{1}{t \ln t} - 0.03$ and $b = 0.03$, solve the corresponding equation using the idea of part (a).

Hint. *The function $y_1(t) \equiv 1$ is a particular solution to this Riccati equation.*

5) Solve $12x^2y + 4x^3 + (4x^3 + y^2)y' = 0$ with the initial condition $y(1) = 2$.

Also from the text:

Section 3.1: Problems 3, 5, 13, 19, 21, 25, 35

Section 3.2: Problems 15