Now You Try It (NYTI):

3. The equation \( x - z = \ln(yz) \) implicitly defines \( z \) as a function of \( x \) and \( y \) near \((1, 1, 1)\). Use implicit differentiation to compute \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) at \((1, 1, 1)\).

\[
\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{1 - \frac{y}{yz}} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{1 + z}
\]

\[
\left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = \frac{1}{2}
\]

\[
\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\frac{z}{yz}}{1 - \frac{y}{yz}} = \frac{1}{1 + \frac{1}{2}} = \frac{-2}{y^2 + y}
\]

\[
\left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} = \frac{-2}{1 + 1} = -\frac{1}{2}
\]

4. The equation \( x - z = \ln(yz) \) implicitly defines \( z \) as a function of \( x \) and \( y \) near \((1, 1, 1)\). Suppose now that \( x = x(t) \) and \( y = y(t) \) are differentiable and \( x(0) = y(0) = 1, \ x'(0) = 2 \) and \( y'(0) = 3 \). Find \( \frac{dz}{dt} \) at \( t = 0 \).

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}
\]

\[
\frac{dz}{dt}(0) = \frac{\partial z}{\partial x}(1,0,0) \cdot x'(0) + \frac{\partial z}{\partial y}(1,0,0) \cdot y'(0)
\]

\[
= \frac{1}{2} \cdot 2 + (-\frac{1}{2}) \cdot 3
\]

\[
= -\frac{1}{2}
\]
Consider \( f(x, y) = xe^y \) and find the following

1. Sketch some level curves

\[
\begin{align*}
f(x, y) &= k \\
k = 1 : &\quad xe^y = 1 \\
&\quad e^y = \frac{1}{x} \\
&\quad \ln (e^y) = \ln \left( \frac{1}{x} \right) \\
&\quad y = \ln \left( \frac{1}{x} \right)
\end{align*}
\]

Figure 1

Figure 2
2. Find the rate of change of $f$ at the point $P(2,0)$ in the direction from $P$ to $Q\left(\frac{1}{2}, 2\right)$.

(a) We first compute the gradient vector:
\[\nabla f(x, y) = \langle f_x, f_y \rangle = \langle z^3, xz^3 \rangle\]
\[\nabla f(2, 0) = \langle 1, 2 \rangle\]

The unit vector in the direction of $\overrightarrow{PQ} = \left\langle -\frac{3}{2}, 2\right\rangle$ is $u = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$, so the rate of change of $f$ in the direction from $P$ to $Q$ is
\[D_uf(2, 0) = \nabla f(2, 0) \cdot u = \langle 1, 2 \rangle \cdot \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle\]
\[= 1 \left(-\frac{3}{5}\right) + 2 \left(\frac{4}{5}\right) = 1\]

3. Plot $\overrightarrow{PQ}$ and $\nabla f(2,0)$ in the figure 1 and 2

4. In what direction does $f$ have the maximum rate of change?
What is this maximum rate of change?

(b) According to Theorem 15, $f$ increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2 \rangle$. The maximum rate of change is
\[|\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5}\]