First Name: ...................................... Last Name: ......................................

“On my honor, I have neither given nor received unauthorized aid in doing this assignment.”

Signature:.................................................................

Directions: Submit solutions to any 4 of the following 6 problems, and clearly indicate on the front page which 4 you would like graded.

No books, no notes, no tablets, no calculators, no computers, no phones!
Write your solutions clearly and legibly for full credit.

Good luck!

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Problem 1. (10 points) Let $P$ be a projector.
   (a) Find all eigenvalues of $P$.
   (b) Show $\text{Null} (P) = \text{Col} (I - P)$
   (c) Show $\text{Null} (I - P) = \text{Col} (P)$

Problem 2. (10 points) Let $A \in \mathbb{C}^{m \times n}$.
   (a) Show the matrix 2-norm is invariant under unitary transformation: $\|AV\|_2 = \|A\|_2$ and $\|UA\|_2 = \|A\|_2$ for unitary matrices $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$.
   (b) Prove or give a counterexample: $\|A\|_2 \leq \sqrt{\|A\|_\infty \|A\|_1}$. If you prove this, make sure to justify each nontrivial step.
   (c) Prove or give a counterexample: $\|A\|_F \leq \|A\|_2$, where $\|A\|_F$ is the Frobenius norm of $A$. If you prove this, make sure to justify each nontrivial step.

Problem 3. (10 points) Let $A = U\Sigma V^*$ be the singular value decomposition (SVD) of $A \in \mathbb{C}^{m \times n}$ with rank $(A) = p \leq n \leq m$.
   (a) Show $\{u_1, u_2, \ldots, u_p\}$ is a basis for $\text{Col} (A)$, where $u_1, \ldots, u_p$ are the first $p$ columns of $U$.
   (b) Show $\{u_{p+1}, u_{p+2}, \ldots, u_m\}$ is a basis for $\text{Null} (A^*)$.
   (c) Show $\|A\|_2 = \sigma_1$, the first singular value of $A$.

Problem 4. (10 points)
   (a) Prove that every square matrix $A$ has a Schur decomposition.
   (b) Prove that if square matrix $A$ is both normal and upper triangular then it is diagonal.

Problem 5. (10 points)
   (a) Let $w \in \mathbb{C}^n$. Determine all eigenvalues of the Householder reflector $H(w)$, including their multiplicities. Justify your answer.
   (b) Given $A \in \mathbb{C}^{m \times n}$ with $m \geq n$, show that $A^*A$ is nonsingular if and only if $A$ has full rank.

Problem 6. (10 points) Define the matrices $A$ and $B$ by

$$A = \begin{pmatrix}
1 & 2 & 0 \\
1 & 2 & 0 \\
1 & 2 & 0 \\
1 & 2 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 & 1 \\
0 & 2 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.$$  

   (a) Find both full and economy singular value decompositions of $A$.
   (b) Find both full and economy QR decompositions of $B$. 