1. Evaluate \[ \int_0^1 x e^{x^3} \, dx \]

Integration by Parts + U-sub Problem

**First** do u-sub  \( u = x^3 \)
\[ du = 3x^2 \, dx \]

\[ \int_0^1 x e^{x^3} \, dx = \int_0^1 u e^u \left( \frac{1}{3} \, du \right) \]
\[ = \frac{1}{3} \int_0^1 u e^u \, du \]

\[ = \frac{1}{3} \left[ u e^u - e^u \right]_0^1 \]
\[ = \frac{1}{3} \left[ 1 e^1 - e^0 - (0 - 1) \right] \]
\[ = \frac{1}{3} \left[ e - e - (-1) \right] \]
\[ = \frac{1}{3} \]
EXTRA PROBLEM

2. \[ r_1 = 2 \cos(\theta) \quad \text{Circle wr Radius 1 tangent to y-axis} \]

\[ r_2^2 = 2 \cos(2\theta) \quad \text{lemniscate (petal of length } \sqrt{2} \text{) on x-axis} \]

Note \( r_1 \) hits the origin when \( \theta = \frac{\pi}{2} \)

and \( r_2 \) hits the origin when \( \theta = \frac{\pi}{4} \) and \( \frac{3\pi}{4} \)

So if we were to try to solve these algebraically we would have mistakenly put \( \) as the answer A. 0

Key: If they tell you to graph polar curves and count how many times the curves intersect each other then GRAPH IT.

Answer B. 1 point (the origin)
EXTRA PROBLEM

3. \[ \int_0^1 \frac{x}{x^2+2x+1} \, dx = \int_0^1 \frac{1}{(x+1)^2} \, dx \]

\[ B = \frac{x}{1} \bigg|_{x=-1} = -1 \]

\[ A = \frac{1}{x+1} + \frac{-1}{(x+1)^2} \]

\[ = \int_0^1 \frac{1}{x+1} + \frac{-1}{(x+1)^2} \, dx = \ln |x+1| + \frac{1}{x+1} \bigg|_0^1 \]

\[ = \ln (2) + \frac{1}{2} - \ln (1) - \frac{1}{1} \]

\[ = \ln (2) - \frac{1}{2} \]

\[ \text{Answer: } \ln (2) - \frac{1}{2} \]
Extra Problem 4

\[ x(t) = 2t - t^2 \quad x'(t) = 2 - 2t \]
\[ y(t) = 3t - t^3 \quad y'(t) = 3 - 3t^2 \]
\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)} = \frac{3-3t^2}{2-2t} \]

\[
\frac{dy}{dx} > 0 \quad \text{at } t=3 \Rightarrow \text{increasing at } t=3
\]
\[
\frac{dy}{dx} < 0 \quad \text{at } t=3 \Rightarrow \text{decreasing at } t=3
\]

Evaluate \( \frac{dy}{dx} \) at \( t=3 \)

\[
\frac{3-3(3)^2}{2-2(3)} = \frac{3-27}{2-6} = \frac{-24}{-4} = 6 > 0
\]

\( y(t) \) is increasing at \( t=3 \) since \( \frac{dy}{dx} > 0 \) at \( t=3 \)

Concavity:

\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left( \frac{3-3t^2}{2-2t} \right)}{\frac{dx}{dt}}
\]

\[
\frac{d}{dt} \left( \frac{3-3t^2}{2-2t} \right) = \frac{(-6t)(2-2t) - (3-3t^2)(-2)}{(2-2t)^2} = \frac{12t^2-12t+6-6t^2}{(2-2t)^2}
\]

\[
\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{3-3t^2}{2-2t} \right) = \frac{12t^2-12t+6 - 6t^2}{(2-2t)^3} = \frac{6t^2-12t+6}{(2-2t)^3}
\]

Evaluate \( \frac{d^2y}{dx^2} \) at \( t=3 \):

\[
\frac{6 \cdot 9 - 12(3) + 6}{(2-6)^3} = \frac{54 - 36 + 6}{-4^3} = \frac{+}{-} < 0
\]

So \( y(t) \) is concave down at \( t=3 \).
5. \( C(t) = (2t-t^2, 3t-t^3) \)

\[
\frac{dy}{dt} = (3t-t^3)' = 3 - 3t^2 = 3(1-t^2) = 3(1+t)(1-t)
\]

\[
\frac{dy}{dt} = 0 \quad \text{at} \quad t = 1 \quad \text{and} \quad t = -1
\]

\[
\frac{dx}{dt} = (2t-t^2)' = 2 - 2t = 2(1-t)
\]

\[
\frac{dx}{dt} = 0 \quad \text{at} \quad t = 1
\]

Determining if \( t=1 \) has VTL, HTL or neither?

\[
\lim_{t \to 1} \frac{dy}{dx} = \lim_{t \to 1} \frac{3 - 3t^2}{2 - 2t} = \lim_{t \to 1} \frac{3(1-t^2)}{2(1-t)} = \lim_{t \to 1} \frac{3(1+t)(1-t)}{2(1-t)} = \frac{3(2)}{2} = 3
\]

Since \( \lim_{t \to 1} \frac{dy}{dx} \neq 0 \) then \( C(t) \) doesn't have HTL at \( t=1 \)

and since \( \lim_{t \to 1} \frac{dy}{dx} = 3 \neq \pm \infty \), \( C(t) \) doesn't have VTL at \( t=1 \)

So \( C(t) \) has a HTL at \( t=-1 \) since \( \frac{dy}{dx} = 0 \) at \( t=-1 \)

and \( C(t) \) has NO VTL.

Answer is A
6. Extra Problem CROSS-SECTION

Find the Volume $V$ of the described solid $S$.

The base of $S$ is triangular, with vertices $(0,0)$, $(0,6)$, and $(6,0)$.

Cross-sections are $\perp$ to $y$-axis are semicircles.

\[
\text{slicing it $\perp$ to $y$-axis}
\]

\[
\text{diameter is } x 
\Rightarrow \text{Radius is } \frac{x}{2}.
\]

\[
V = \int \text{Area (semicircles)} \, dy
\]

Area for circle : $\pi r^2$

Area for semicircle : $\frac{\pi r^2}{2}$ and Radius $r = \frac{x}{2}$

\[
V = \int \frac{\pi r^2}{2} \, dy \quad 0 \leq y \leq 6
\]

\[
= \int_0^6 \frac{\pi}{2} \left(\frac{x}{2}\right)^2 \, dy
\]

\[
= \int_0^6 \frac{\pi}{2} \cdot \frac{x^2}{4} \, dy
\]

\[
= \frac{\pi}{8} \int_0^6 x^2 \, dy
\]

Write $x$ into terms of $y$.

Note that the line connecting points $(0,6)$ and $(6,0)$ is $y = -x + 6$ use this to write $x$ into terms of $y$.

\[
x = 6 - y
\]
Extra Problem 6. Continued

\[ V = \frac{\pi}{8} \int_0^6 x^2 \, dy = \frac{\pi}{8} \int_0^6 (6-y)^2 \, dy \]

\[ = \frac{\pi}{8} \int_0^6 (36 - 12y + y^2) \, dy \]

\[ = \frac{\pi}{8} \left[ 36y - 6y^2 + \frac{1}{3}y^3 \right]_0^6 \]

\[ = \frac{\pi}{8} \left[ 36 \cdot 6 - 6 \cdot 6^2 + \frac{1}{3} \cdot 6^3 \right] \]

\[ = \frac{\pi}{8} \left[ 6^3 - 6^3 + \frac{1}{3} \cdot 6^3 \right] \]

\[ = \frac{\pi}{8} \cdot \frac{1}{3} \cdot 6^3 \]

\[ = \frac{\pi}{24} \cdot 16 \cdot 36^9 = \boxed{9\pi} \]
7. \( f(x) = \frac{1}{x^2} \)

\[
f(3) = \frac{1}{9} 
\]

\[
f'(x) = \frac{d}{dx} (x^{-2}) = -2x^{-3} = \frac{-2}{x^3} 
\]

\[
f'(3) = \frac{-2}{27} = \frac{-2}{3^3} 
\]

\[
f''(x) = (-2x^{-3})' = 6x^{-4} 
\]

\[
f''(3) = 6(3)^{-4} = \frac{6}{3^4} = \frac{6}{81} = \frac{2}{27} 
\]

\[
T_2(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 
\]

\[
T_2(x) = \frac{1}{9} + \left( \frac{-2}{27} \right)(x-3) + \frac{\frac{2}{27}}{2!} (x-3)^2 
\]

\[
T_2(x) = \frac{1}{9} - \frac{2}{27}(x-3) + \frac{1}{27}(x-3)^2 
\]

\[
T_2(3.1) \approx f(3.1) = \frac{1}{(3.1)^2} 
\]

\[
T_2(3.1) = \frac{1}{9} - \frac{2}{27} (3.1-3) + \frac{1}{27} (3.1-3)^2 
\]

\[
= \frac{1}{9} - \frac{2}{27} (0.1) + \frac{1}{27} (0.1)^2 
\]

\[
= \frac{1}{9} - \frac{2}{270} + \frac{1}{2700} 
\]

\[
E. \frac{1}{(3.1)^2} \approx \frac{1}{9} - \frac{2}{270} + \frac{1}{2700} 
\]
8. \[ \int_1^\infty \frac{x^{1/2}}{3\sqrt[3]{x^k+10x}} \, dx \]

Note \[ \frac{x^{1/2}}{3\sqrt[3]{x^k}} \geq \frac{x^{1/2}}{3\sqrt[3]{x^k+10x}} \text{ for all } x \geq 1 \]

\[ \int_1^\infty \frac{x^{1/2}}{3\sqrt[3]{x^k}} \, dx = \int_1^\infty \frac{x^{1/2}}{x^{k/3}} \, dx = \int_1^\infty \frac{1}{x^{k/3 - 1/2}} \, dx \]

By DCT for integrals

\[ \int_1^\infty \frac{x^{1/2}}{3\sqrt[3]{x^k+10x}} \, dx \] will converge

if \[ \int_1^\infty \frac{x^{1/2}}{3\sqrt[3]{x^k}} \, dx \] converges.

And by p-test for integrals \[ \int_1^\infty \frac{x^{1/2}}{x^{k/3}} \, dx = \int_1^\infty \frac{1}{x^{k/3 - 1/2}} \, dx \]

converges when \[ \frac{k}{3} - \frac{1}{2} > 1 \]

\[ \frac{k}{3} > 1 + \frac{1}{2} \Rightarrow \frac{k}{3} > \frac{3}{2} \]

\[ k > \frac{9}{2} \]

There fore \[ \int_1^\infty \frac{x^{1/2}}{3\sqrt[3]{x^k+10x}} \, dx \] converges when \[ k > \frac{9}{2} \]

\[ k > \frac{9}{2} \Rightarrow k \text{ is in } (\frac{9}{2}, \infty) \] Answer: E