1.

Evaluate $\int_0^1 x^5 e^{x^3}\,dx$

Integration by Parts + Usub Problem

<u>First</u> do u-sub $u = x^3$

$du = 3x^2\,dx$

$$\int_0^1 x^5 e^{x^3}\,dx = \int_0^1 x^3 \cdot \boxed{x^2}\, e^{x^3}\, \boxed{dx}$$

$$= \int_0^1 u e^u \left(\tfrac{1}{3}du\right)$$

$$= \frac{1}{3}\int_0^1 u e^u\,du$$

<u>Second</u>

IBP

Tabular Method

u	$+$	e^u
1	$-$	e^u
0		e^u

$$\int_0^1 u e^u\,du = u e^u - e^u$$

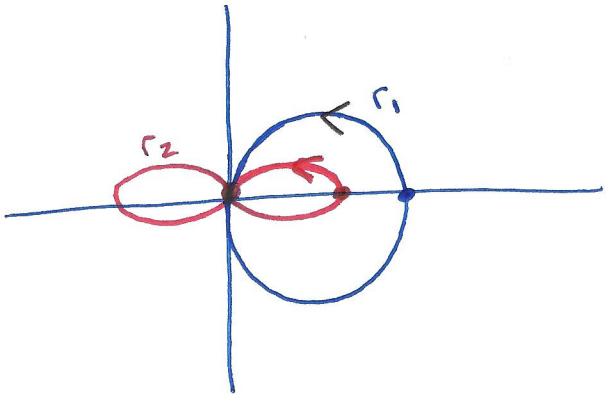
$$\frac{1}{3}\int_0^1 u e^u\,du = \frac{1}{3}\left[u e^u - e^u\right]\Big|_0^1 = \frac{1}{3}\left[e - e - (0-1)\right]$$

$$= \boxed{\frac{1}{3}}$$

EXTRA PROBLEM

2.

$r_1 = 2\cos(\theta)$ ← Circle w/ Radius 1 tangent to y-axis



$r_2^2 = 2\cos(2\theta)$ ← lemniscate (petal of length $\sqrt{2}$) on x-axis

Note r_1 hits the origin when $\theta = \frac{\pi}{2}$

and r_2 hits the origin when $\theta = \frac{\pi}{4}$ and $\frac{3\pi}{4}$

So if we were to try to solve these algebraically we would have mistakenly put the answer A. 0

Key: If they tell you to graph polar curves and count how many times the curves intersect each other then GRAPH It.

Answer B. 1 point (the origin)

EXTRA PROBLEM

3. $$\int_0^1 \frac{x}{x^2+2x+1}\,dx = \int_0^1 \frac{x}{(x+1)^2}\,dx$$

use cover up highest degree of linear factor

$$= \int_0^1 \frac{A}{(x+1)} + \frac{B}{(x+1)^2}\,dx$$

$$B = \frac{x}{1}\Big|_{x=-1} = -1$$

$$= \int_0^1 \frac{1}{x+1} + \frac{-1}{(x+1)^2}\,dx = \ln|x+1| + \frac{1}{x+1}\Big|_0^1$$

$$= \ln(2) + \frac{1}{2} - \ln(1) - \frac{1}{1}$$

$$= \boxed{\ln(2) - \frac{1}{2}}$$

A ← Compare Coeff. or Test points

$$x = A(x+1) + B$$

$$1x = Ax + \underbrace{A - 1}$$

$$A - 1 = 0 \Rightarrow A = 1$$

$$A = 1$$

$$\int_0^1 \frac{1}{x+1}\,dx = \ln|x+1|\Big|_0^1 = \ln(2) - \ln(1) = \boxed{\ln(2)}$$

$$\int_0^1 \frac{-1}{(x+1)^2}\,dx \overset{\substack{u=x+1 \\ du=dx}}{=} \int_{0+1=1}^{1+1=2} \frac{-1}{u^2}\,du = \int_1^2 -u^{-2}\,du = u^{-1}\Big|_1^2$$

$$= \frac{1}{2} - 1$$

$$(x+1)^{-1} = \frac{1}{x+1}\Big|_0^1 = \frac{1}{2} - 1 = \boxed{-\frac{1}{2}}$$

$$\boxed{\int_0^1 \frac{1}{x+1}\,dx + \int_0^1 \frac{-1}{(x+1)^2}\,dx = \ln 2 - \frac{1}{2}}$$

$$\boxed{\text{Answer: } \ln(2) - \frac{1}{2}}$$

Extra Problem 4

$x(t) = 2t - t^2$ $\qquad$ $x'(t) = 2 - 2t$

$y(t) = 3t - t^3$ $\qquad$ $y'(t) = 3 - 3t^2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{y'(t)}{x'(t)} = \frac{3-3t^2}{2-2t}$$

$\frac{dy}{dx} > 0$ at $t \Rightarrow$ increasing at t

$\frac{dy}{dx} < 0$ at $t \Rightarrow$ decreasing at t

Evaluate $\frac{dy}{dx}$ at $t=3$

$$\frac{3-3(3)^2}{2-2(3)} = \frac{3-27}{2-6} = \frac{-24}{-4} = 6 > 0$$

$C(t)$ is <u>increasing</u> at $t=3$ since $\frac{dy}{dx} > 0$ at $t=3$

Concavity:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$\frac{d^2y}{dx^2} < 0$ at $t \Rightarrow$ concave down at t

$\frac{d^2y}{dx^2} > 0$ at $t \Rightarrow$ concave up at t

$$\frac{d}{dt}\left(\frac{3-3t^2}{2-2t}\right) = \frac{(-6t)(2-2t) - (3-3t^2)(-2)}{(2-2t)^2} = \frac{12t^2-12t+6-6t^2}{(2-2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3-3t^2}{2-2t}\right)}{\frac{dx}{dt}} = \frac{12t^2-12t+6-6t^2}{(2-2t)^3} = \frac{6t^2-12t+6}{(2-2t)^3}$$

Evaluate $\frac{d^2y}{dx^2}$ at $t=3$:

$$\frac{6\cdot 9 - 12(3) + 6}{(2-6)^3} = \frac{54-36+6}{-4^3} = \frac{+}{-} < 0$$

d^2y

So $C(t)$ is concave down at $t=3$!

EXTRA PROBLEMS

5. $C(t) = (2t - t^2,\ 3t - t^3)$

$$\frac{dy}{dt} = (3t - t^3)' = 3 - 3t^2 = 3(1-t^2) = 3(1+t)(1-t)$$

$\frac{dy}{dt} = 0$ at $t = 1$ and $\boxed{t = -1}$

$$\frac{dx}{dt} = (2t - t^2)' = 2 - 2t = 2(1-t)$$

$\frac{dx}{dt} = 0$ at $t = 1$

both $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} = 0$ at $t = 1$

Determining if $t=1$ has VTL, HTL or neither?

$$\lim_{t \to 1} \frac{dy}{dx} = \lim_{t \to 1} \frac{3 - 3t^2}{2 - 2t} = \lim_{t \to 1} \frac{3(1-t^2)}{2(1-t)} = \lim_{t \to 1} \frac{3(1+t)\cancel{(1-t)}}{2\cancel{(1-t)}}$$

$$= \frac{3(2)}{2} = 3$$

Since $\lim_{t \to 1} \frac{dy}{dx} \neq 0$ then $C(t)$ doesn't have HTL at $t=1$

and Since $\lim_{t \to 1} \frac{dy}{dx} = 3 \neq \pm\infty$, $C(t)$ doesn't have VTL at $t=1$

So $C(t)$ has a HTL at $t=-1$ since $\frac{dy}{dx} = 0$ at $t=-1$ and $C(t)$ has NO VTL!

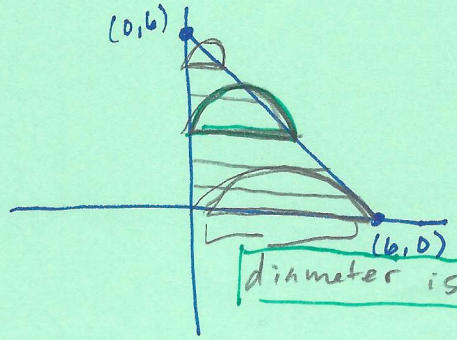
Answer is A

6. Extra Problem CROSS-SECTION

Find the Volume V of the described solid S

The base of S is triangular, with vertices (0,0), (0,6), and (6,0)

Cross-sections are $\perp$ to y-axis are semicircles



Slicing it $\perp$ to y-axis

$\boxed{\text{diameter is } x}$ $\Rightarrow$ Radius is $\frac{x}{2}$!

$$V = \int \text{Area(semicircles)}\, dy$$

Area for circle : πr^2

Area for SemiCircle : $\frac{\pi r^2}{2}$ and Radius $r = \frac{x}{2}$

$$V = \int \frac{\pi r^2}{2}\, dy \qquad 0 \le y \le 6$$

$$= \int_0^6 \frac{\pi}{2}\left(\frac{x}{2}\right)^2 dy \qquad \text{with } r = \frac{x}{2}$$

$$= \int_0^6 \frac{\pi}{2}\,\frac{x^2}{4}\, dy$$

$$= \frac{\pi}{8}\int_0^6 x^2\, dy \qquad \text{Write } x \text{ into terms of } y$$

Note that the line connecting points (0,6) and (6,0) is $\boxed{y = -x + 6}$ use this to write x into terms of y

$x = 6 - y$

Extra Problem 6. Continued

$$V = \frac{\pi}{8}\int_0^6 x^2\,dy = \frac{\pi}{8}\int_0^6 (6-y)^2\,dy$$

$$\cancel{= \frac{\pi}{8}\int_0^6 36 - 2\cdot 6y + y^2}$$

$$= \frac{\pi}{8}\int_0^6 36 - 12y + y^2\,dy$$

$$= \frac{\pi}{8}\left[36y - 6y^2 + \frac{1}{3}y^3\right]\Bigg|_{y=0}^{y=6}$$

$$= \frac{\pi}{8}\left[36\cdot 6 - 6(36) + \frac{1}{3}\cdot 6^3 - 0\right]$$

$$= \frac{\pi}{8}\left[6^3 - 6^3 + \frac{1}{3}\cdot 6^3\right]$$

$$= \frac{\pi}{8}\cdot\frac{1}{3}\cdot 6^3$$

$$= \frac{\pi}{\cancel{24}\,4}\cdot \cancel{6}\cdot \cancel{36}^{\,9} = \boxed{9\pi}$$

7. $f(x) = \frac{1}{x^2}$

$f(3) = \frac{1}{9}$

$f'(x) = (x^{-2})' = -2x^{-3} = \frac{-2}{x^3}$

$f'(3) = \frac{-2}{27} = \frac{-2}{3^3}$

$f''(x) = (-2x^{-3})' = 6x^{-4}$

$f''(3) = 6(3)^{-4} = \frac{6}{3^4} = \frac{6}{81} = \frac{2}{27}$

$f(3) = \frac{1}{9}$ | $c_0 = \frac{1}{9}$

$f'(3) = -\frac{2}{27}$ | $c_1 = \frac{-2}{27}$

$f''(3) = \frac{2}{27}$ | $c_2 = \frac{1}{27}$

$$T_2(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2$$

$$T_2(x) = \frac{1}{9} + \left(-\frac{2}{27}\right)(x-3) + \frac{\frac{2}{27}}{2!}(x-3)^2$$

$$T_2(x) = \frac{1}{9} - \frac{2}{27}(x-3) + \frac{1}{27}(x-3)^2$$

$$T_2(3.1) \approx f(3.1) = \frac{1}{(3.1)^2}$$

$$T_2(3.1) = \frac{1}{9} - \frac{2}{27}(3.1-3) + \frac{1}{27}(3.1-3)^2$$

$$= \frac{1}{9} - \frac{2}{27}(0.1) + \frac{1}{27}(0.1)^2$$

$$= \frac{1}{9} - \frac{2}{270} + \frac{1}{2700}$$

$$\boxed{E. \frac{1}{(3.1)^2} \approx \frac{1}{9} - \frac{2}{270} + \frac{1}{2700}}$$

8. $$\int_1^\infty \frac{x^{1/2}}{\sqrt[3]{x^k+10x}}\,dx$$

Note $$\frac{x^{1/2}}{\sqrt[3]{x^k}} \geq \frac{x^{1/2}}{\sqrt[3]{x^k+10x}} \quad \text{for all } x \geq 1$$

$$\int_1^\infty \frac{x^{1/2}}{\sqrt[3]{x^k}}\,dx = \int_1^\infty \frac{x^{1/2}}{x^{k/3}}\,dx = \int_1^\infty \frac{1}{x^{\frac{k}{3}-\frac{1}{2}}}\,dx$$

By DCT for integrals

$$\int_1^\infty \frac{x^{1/2}}{\sqrt[3]{x^k+10x}}\,dx \quad \text{will converge}$$

if $$\int_1^\infty \frac{x^{1/2}}{\sqrt[3]{x^k}} \quad \text{converges.}$$

And by p-test for integrals $\int_1^\infty \frac{x^{1/2}}{x^{k/3}}dx = \int_1^\infty \frac{1}{x^{\frac{k}{3}-\frac{1}{2}}}dx$

converges when $\frac{k}{3}-\frac{1}{2} > 1$

$$\frac{k}{3} > 1+\frac{1}{2} \Rightarrow \frac{k}{3} > \frac{3}{2}$$

$$k > \frac{9}{2}$$

Therefore $\int_1^\infty \frac{x^{1/2}}{\sqrt[3]{x^k+10x}}dx$ converges when

$k > \frac{9}{2} \Rightarrow k$ is in $\left(\frac{9}{2}, \infty\right)$ Answer: E

~~8. Determine the values k for which the improper integral $\int_1^\infty \frac{x^{1/2}}{\sqrt[3]{x^k}}$~~