1 Limits

Suppose that \( c \) is a constant and the limits

\[
\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)
\]

exist. Then

\[
\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \tag{1}
\]

\[
\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \tag{2}
\]

\[
\lim_{x \to a} [cf(x)] = \lim_{x \to a} f(x) \tag{3}
\]

\[
\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \tag{4}
\]

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } g(x) \neq 0 \tag{5}
\]

\[
\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n \quad \text{where } n \text{ is a positive integer} \tag{6}
\]

\[
\lim_{x \to a} c = c \tag{7}
\]

\[
\lim_{x \to a} x = a \tag{8}
\]

\[
\lim_{x \to a} x^n = a^n \quad \text{where } n \text{ is a positive integer} \tag{9}
\]

\[
\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \quad \text{where } n \text{ is a positive integer} \tag{10}
\]

\[
\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \quad \text{where } n \text{ is a positive integer} \tag{11}
\]

(For (10) and (11) If \( n \) is even, we assume that \( a > 0 \).)

NOTE: If EITHER \( \lim_{x \to a} f(x) = \text{DNE} \) OR \( \lim_{x \to a} g(x) = \text{DNE} \), THEN WE CANT APPLY RULES (1) THROUGH (6) AND RULE (11) !!!

**Theorem 1.** \( \lim_{x \to a} f(x) = L \) if and only if \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \).

Meaning if \( \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x) \), then \( \lim_{x \to a} f(x) \) does not exist.

Essential Limits to know

\[
\lim_{u \to 0} \frac{\sin u}{u} = 1 \tag{12}
\]

\[
\lim_{u \to 0} \frac{u}{\sin u} = 1 \tag{13}
\]

\[
\lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \tag{14}
\]
Theorem 2. If $f$ is a polynomial or a rational function and $a$ is in the domain of $f$, then
\[ \lim_{x \to a} f(x) = f(a). \]

Theorem 3. The Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when $x$ is near $a$ (except for possibly $a$ and $b$), then
\[ \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \]
then
\[ \lim_{x \to a} g(x) = L. \]

2 Continuity

Definition 4. A function $f$ is \textit{continuous at a number} $a$ if
\[ \lim_{x \to a} f(x) = f(a). \]

Definition 5. A function $f$ is \textit{continuous from the right at a number} $a$ if
\[ \lim_{x \to a^+} f(x) = f(a) \]
and $f$ is \textit{continuous from the left at} $a$ if
\[ \lim_{x \to a^-} f(x) = f(a). \]

Definition 6. A function $f$ is \textit{continuous on an interval} if it is continuous at every number in the interval.
(If $f$ is defined only on one side of an endpoint of the interval, we understand \textit{continuous} at the endpoint to mean \textit{continuous from the right} or \textit{continuous from the left}.)

Theorem 7. If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:
1. $f + g$
2. $f - g$
3. $cf$
4. $fg$
5. $\frac{f}{g}$ if $g(a) \neq 0$.

Theorem 8. Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

Theorem 9. Any rational function is continuous whenever it is defined; that is, it is continuous on its domain.

Theorem 10. The following types of functions are continuous at every number \textit{in their domains}: polynomials, rational functions, root functions, trigonometric functions.

Theorem 11. If $f$ is continuous at $b$ and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,
\[ \lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)). \]

Theorem 12. If $f$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at $a$. Meaning, composition of continuous functions are continuous.

Theorem 13. Intermediate Value Theorem (IVT): Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$ (meaning either $f(a) \leq N \leq f(b)$ OR $f(b) \leq N \leq f(a)$), where $f(a) \neq f(b)$. Then there exist a number $c$ in $(a, b)$ such that $f(c) = N$. 
3 Limits Involving Infinity

**Definition 14.** The line $x = a$ is called a *vertical asymptote* (VA) of the curve $y = f(x)$ if at least one of the following statements is true:

\[
\begin{align*}
\lim_{x \to a^-} f(x) &= \infty \\
\lim_{x \to a^+} f(x) &= \infty \\
\lim_{x \to a^-} f(x) &= -\infty \\
\lim_{x \to a^+} f(x) &= -\infty
\end{align*}
\]

**Definition 15.** The line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

\[
\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.
\]

If $n$ is a positive integer, then

\[
\begin{align*}
\lim_{x \to \infty} \frac{1}{x^n} &= 0 \\
\lim_{x \to -\infty} \frac{1}{x^n} &= 0.
\end{align*}
\]

*Note: To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of $x$ that occurs in the denominator.*

4 Derivatives

**Definition 16.** The *tangent line* to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through $P$ with the slope $m$, where $m$ can be found in the following two ways:

\[
\begin{align*}
m &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
m &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\end{align*}
\]

provided that the limit exists.

Personal note: I believe that (18) is an easier method. For one thing I have seen many students use (17) but accidentally swap the $x$ and $a$. Usually for (17), it may involve some factor by grouping too. For students computing (18) for the first time, I suggest doing side work. Before trying to do $\frac{f(a + h) - f(a)}{h}$, on a separate line just compute $f(a + h)$ and then compute $f(a + h) - f(a)$. Try canceling out terms in $f(a + h) - f(a)$ first before trying to take the limit of $\frac{f(a + h) - f(a)}{h}$.

**Definition 17.** A function $f$ that measures the motion of an object is called the *position function* of the object.

**Definition 18.** In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. The *average velocity* over this time interval is

\[
\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a + h) - f(a)}{h}.
\]

If we let $h$ approach 0, we define the *velocity* (or *instantaneous velocity*) $v(a)$ at time $t = a$ to be the limit of these average velocities:

\[
v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]
Definition 19. The derivative of a function $f$ at a number $a$, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \quad \text{or} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(20)

(21)

if this limit exists.

Note: The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of $f$ at $a$.

Steps To finding the equation of the tangent line to $y = f(x)$ at $(a, f(a))$, we perform the following steps.

1. Find the derivative of $f'(x)$.
2. Use $f'(x)$ to find $f'(a)$, which will be the slope of our tangent line.
3. Use point-slope form (recall: given the slope $m$ of a line and the point $(x_1, y_1)$, the point slope form is $y - y_1 = m(x - x_1)$). So in this case we get $y - f(a) = f'(a)(x - a)$ or we can write $y = f'(a)(x-a)+f(a)$.

5 Practice Problems

1. Calculate the limit

$$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$$

2. Given that $\lim_{x \to a} h(x) = 8$ evaluate $\lim_{x \to a} \sqrt{h(x)}$.

3. Calculate the limit

$$\lim_{x \to -2} (3x^4 + 2x^2 - x + 1).$$

4. Calculate the limit

$$\lim_{\theta \to \frac{\pi}{2}} \theta \sin \theta.$$

5. Evaluate the limit if it exists

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}.$$

6. Evaluate the limit if it exists

$$\lim_{x \to 2} \frac{x^2 - x + 6}{x - 2}.$$

7. Evaluate the limit if it exists

$$\lim_{t \to -3} \frac{t^2 - 9}{2t^2 + 7t + 3}.$$

8. Evaluate the limit if it exists

$$\lim_{h \to 0} \frac{(4 + h)^2 - 16}{h}.$$

9. Evaluate the limit if it exists

$$\lim_{x \to -2} \frac{x + 2}{x^3 + 8}.$$

10. Evaluate the limit if it exists

$$\lim_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7}.$$
11. Evaluate the limit if it exists
\[ \lim_{{x \to -4}} \frac{\frac{1}{4} + \frac{1}{x}}{4 + x} \]

12. Evaluate the limit if it exists
\[ \lim_{{x \to 0}} \frac{x}{\sqrt{1 + 3x} - 1} \]

13. Use squeeze theorem to evaluate the limit
\[ \lim_{{x \to 0}} x^2 \cos(20\pi x) \]

14. Use Squeeze theorem to find the limit
\[ \lim_{{x \to 0}} \sin(\pi/x) \]

15. If \(4x - 9 \leq f(x) \leq x^2 - 4x + 7\) for \(x \geq 0\), find
\[ \lim_{{x \to 4}} f(x) \]

16. Evaluate
\[ \lim_{{x \to 0}} x^4 \cos(2/x) \]

17. Evaluate
\[ \lim_{{x \to 0^+}} \sqrt{7} |1 + \sin^2 (2\pi/x)| \]

18. Find the limit, if it exists.
\[ \lim_{{x \to 3}} (2x + |x - 3|) \]

19. Find the limit, if it exists.
\[ \lim_{{x \to -1}} \left( \frac{1}{x} - \frac{1}{|x|} \right) \]

20. Let \(g(x) = \begin{cases} -x & x \leq -1 \\ 1 - x^2 & -1 < x \leq 1 \\ x - 1 & x > 1 \end{cases} \)

Evaluate
\( \lim_{{x \to -1^+}} g(x) \)
\( \lim_{{x \to -1^-}} g(x) \)
\( \lim_{{x \to -1}} g(x) \)
\( \lim_{{x \to 1^+}} g(x) \)
\( \lim_{{x \to 1^-}} g(x) \)
\( \lim_{{x \to 1}} g(x) \)

21. Evaluate
\[ \lim_{{x \to 0}} \frac{\sin(3x)}{x} \]

22. Is there a number a such that
\[ \lim_{{x \to -2}} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} \]
exists.
23. Use the definition of continuity and the properties of limits to show that the function is continuous at 
\( x = -1 \)
\[ f(x) = (x + 2x^3)^4 \]

24. Explain why the function is discontinuous at \( x = 1 \)
\[ f(x) = \frac{1}{(x - 1)^2} \]

25. Explain why the function is discontinuous at \( x = 1 \)
\[ f(x) = \begin{cases} 
1 - x^2 & x < 1 \\
1/x & x \geq 1 
\end{cases} \]

26. Explain why the function is discontinuous at \( x = 1 \)
\[ f(x) = \begin{cases} 
x^2 - x & x \neq 1 \\
1 & x = 1 
\end{cases} \]

27. Find the constant \( c \) that makes \( g \) continuous on \(( -\infty, \infty)\)
\[ g(x) = \begin{cases} 
x^2 - x^2 & x < 4 \\
x^2 + 20 & x \geq 4 
\end{cases} \]

28. Use Intermediate value theorem to show that \( f(x) = x^2 + x - 3 \) has a root in \((1, 2)\).

29. Show that there is a root of the equation
\[ 4x^3 - 6x^2 + 3x - 2 = 0 \]
between 1 and 2.

30.
\[ \lim_{x \to \infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4} \]

31.
\[ \lim_{x \to \infty} \frac{x + 2}{\sqrt{9x^2 + 1}} \]

32.
\[ \lim_{x \to -\infty} \frac{x + 2}{\sqrt{9x^2 + 1}} \]

33.
\[ \lim_{x \to \infty} \frac{x^3 - 2x + 3}{5 - 2x^2} \]

34.
\[ \lim_{x \to \infty} \frac{x^3 + x^5}{1 - x^2 + x^4} \]

35. Find \( \lim_{x \to \infty} f(x) \) if, for all \( x > 5 \)
\[ \frac{4x - 1}{x} \leq f(x) \leq \frac{4x^2 + 3x}{x^2} \]

36. Find an equation of the tangent line to the curve at the given point
\[ y = f(x) = 2x^3 - 5x, \ (-1, 3) \]

37. Find an equation of the tangent line to the curve at the given point
\[ y = f(x) = \sqrt{x}, \ (1, 1) \]

38. Use the definition of derivative to find the derivative of \( f(x) = x^2 - 8x + 9 \)
6  Answer Key

1. 1/4
2. 2
3. 59
4. \(\frac{\pi}{2}\)
5. 5
6. DNE
7. 6/5
8. 8
9. 1/12
10. 1/6
11. -1/16
12. 2/3
13. 0
14. 0
15. 7
16. 0
17. 0
18. 0 [Note: use squeeze theorem]
19. 6
20. \(-\infty\)
21. (a) 0
   (b) 0
   (c) 0
   (d) 1
   (e) 0
   (f) DNE
22. 3
23. yes \(a = 15\)
24. Show that \(\lim_{x \to -1} f(x)\) exists and equals \(f(-1) = 81\).
25. \(f\) has an infinite discontinuity at \(x = 1\). More specifically \(\lim_{x \to 1} f(x) = -\infty\) and \(x = 1\) is a Vertical Asymptote.
26. \(\lim_{x \to 1} f(x) = dne\) since the left and right limits don’t agree. Here \(f\) has a jump discontinuity at \(x = 1\). We could also say that \(f\) is continuous from the right at \(x = 1\).
27. \( \lim_{x \to 1} f(x) = 1/2 \neq f(1) = 1 \). Here we say that \( f \) has a removable discontinuity.

28. \( c = -2 \)

29. \( f \) is continuous on \((1, 2)\), \( f(1) = -3 < 0 \), \( f(2) = 3 > 0 \).

30. Note that the function \( f(x) = 4x^3 - 6x^2 + 3x - 2 \) is a polynomial so \( f \) is continuous on \((-\infty, \infty)\).

31. \( 1/2 \)

32. \( 1/3 \)

33. \(-1/3\)

34. \(-\infty\)

35. \( \infty \)

36. \( 4 \)

37. \( f'(x) = 6x^2 - 5 \)
   \[ m = f'(-1) = 1 \]
   \[ y - 3 = 1(x - (-1)) \]
   \[ y = x + 4 \]

38. \( y = \frac{1}{2}x + \frac{1}{2} \)

39. \( f'(a) = 2a - 8 \)