Multiple Choice Questions

1. Find the absolute maximum and minimum values of \( f(x) = \sin^2 x + \cos x \) on \([0, \pi]\)
   
   (a) \( \frac{1 + 2\sqrt{3}}{4} \) and \(-1\)
   
   (b) \( \frac{5}{4} \) and 0
   
   (c) 1 and \(-1\)
   
   (d) \( \frac{5}{4} \) and \(-1\)
   
   (e) \( \frac{1 + 2\sqrt{3}}{4} \) and 0

2. The inflection points of \( f(x) = \ln(x^2 + 4) \) occur at \( x = \ldots \)
   
   (a) 0 only
   
   (b) \(-2, 0, \) and 2
   
   (c) \(-1 \) and 1
   
   (d) \(-2 \) and 2 only
   
   (e) \( f(x) \) has no inflection points.

3. The local extrema of \( f(x) = 8x^2 - x^4 + 6 \) occur at which of the following \( x \)-values?
   
   (a) local maximum at \( x = -2, 0 \) local minimum at \( x = 2 \)
   
   (b) local maximum at \( x = 0 \) local minimum at \( x = -2, 2 \)
   
   (c) local maximum at \( x = -2 \) local minimum at \( x = 2 \)
   
   (d) local maximum at \( x = -2, 2 \) local minimum at \( x = 0 \)
   
   (e) local maximum at \( x = 2, 0 \) local minimum at \( x = -2 \)

4. Find each value of \( c \) that is guaranteed by the mean value theorem for \( f(x) = \sqrt{x - 6} \) on the interval \([7, 15]\)
   
   (a) \( c = 10 \)
   
   (b) \( c = \frac{29}{4} \)
   
   (c) \( c = 12 \)
   
   (d) \( c = 9 \)
   
   (e) There is no such value of \( c \) on the interval.
5. Consider the function \( f(x) \) with \( f'(x) = x^4 - x^3 \) and \( f''(x) = 4x^3 - 3x^2 \). Which of the following statements is/are true?

P. \( f(x) \) is concave up on the interval \( \left( \frac{3}{4}, \infty \right) \) only

Q. \( f(x) \) has inflection points at both \( x = 0 \) and \( x = \frac{3}{4} \)

R. \( f(x) \) is both concave up and increasing on \((1, \infty)\)

(a) Q only
(b) Q and R only
(c) P, Q and R
(d) P only
(e) P and R only

6. Which of the following statement is true regarding the function \( f \)?

(a) If \( f'(2) = 0 \), then \( f(x) \) must have a local maximum or minimum at \( x = 2 \).
(b) If \( f''(5) = 0 \), then \( x = 5 \) is an inflection point of \( f \)
(c) If \( f(x) \) has a local minimum at \( x = 3 \), then \( f'(3) = 0 \)
(d) If \( f \) is decreasing and concave up on \((0, 3)\) then \( f'(x) < 0 \) and \( f'(x) \) is decreasing on \((0, 3)\)
(e) None of the above

7. Use L'Hospital’s rule, if possible, to evaluate \( \lim_{x \to 1} \frac{x \ln x - x + 1}{e^x - ex} \)

(a) 0
(b) 1
(c) \( \frac{1}{e} \)
(d) \( \frac{2}{e} \)
(e) The limit does not exist.

8. Approximate the value of \( \int_0^2 \frac{6}{1 + 2x} \, dx \) with a Riemann Sum using four sub intervals of equal width and letting \( x_i^* \) be the left endpoint of the sub interval \([x_{i-1}, x_i]\).

(a) 77/20
(b) 25/2
(c) 13/4
(d) 25/4
(e) 77/10
9. Express the following limit of a Riemann sum as a definite integral:

\[
\lim_{n \to \infty} \sum_{i=1}^{\infty} \cos \left( 2 + \frac{4i}{n} \right) \frac{4}{n}
\]

If you find more than one integral that represents the sum, choose any one correct answer.

(a) \( \int_{0}^{4} \cos(2 + x) \, dx \)

(b) \( \int_{0}^{4} \cos(x) \, dx \)

(c) \( \int_{2}^{6} \cos(2 + x) \, dx \)

(d) \( \int_{2}^{6} \cos(x) \, dx \)

(e) \( \int_{0}^{4} \cos(2 + 4x) \, dx \)

10. Let \( f \) be a differentiable function such that \( f'(x) \geq 5 \) for all \( x \). If \( f(3) = 7 \), what is the largest possible value for \( f(-1) \)?

(a) 13

(b) −3

(c) −13

(d) 3

(e) 0
11. Suppose that the function $f(x)$ has horizontal tangent lines at $x = -2$, $x = 0$ and $x = 1$. If $f''(x) = 3x^2 + 2x - 2$, then which of the following statements is/are true?

P. According to the second derivative test, $f(x)$ has a relative minimum at $x = -2$ and relative maxima at $x = 0$ and $x = 1$

Q. According to the second derivative test, $f(x)$ has a relative maximum at $x = 0$ and relative minima at $x = -2$ and $x = 1$

R. If $f'(3) = -1$, then the graph of $f(x)$ has the following shape around $x = 3$:

(a) P and Q only
(b) Q only
(c) P only
(d) Q and R only
(e) P and R only

12. A rocket rises from a launch pad that is four feet above the ground with acceleration after $t$ seconds given by $a(t) = 10 - 2t$ feet/sec$^2$. If the initial velocity of the rocket was 80 feet/sec, find the height $h(t)$ of the rocket above the ground 3 seconds later?  

$Hint$: what is $h(0)$?

(a) 276 feet
(b) 400 feet
(c) 242 feet
(d) 360 feet
(e) 280 feet

13. Find the $x$ coordinate of the point on the curve $y = \frac{x^2}{2}$ located at the shortest distance from the point $(1, 1)$.

(a) 1
(b) $\sqrt{2}$
(c) $\sqrt{2}$
(d) 2
(e) $\sqrt{2}$
14. The graph of \( y = f(x) \) is sketched below. Which of the following statements is/are true about an antiderivative \( F(x) \) of \( f \)?

P. \( F(x) \) has local extrema at \( x = 0 \) and \( x = 2 \).
Q. \( F(x) \) is concave down on \((1, \infty)\)
R. \( F(x) \) is decreasing on \((-2, 1) \) and \((3, \infty)\).

(a) P only  
(b) Q only  
(c) R only  
(d) P and Q only  
(e) None of the above

15. If \( f(x) = \frac{4x^2}{(x + 1)^2} \), then \( f'(x) = \frac{8x}{(x + 1)^3} \) and \( f''(x) = \frac{8(1 - 2x)}{(x + 1)^4} \).
Find each interval on which \( f(x) \) is both increasing and concave down.

(a) \((-\infty, -1) \cup (1/2, \infty)\)  
(b) \((-1, 0) \cup (1/2, \infty)\)  
(c) \((-\infty, -1)\) only  
(d) \((-\infty, -1) \cup (1/2, 0)\)  
(e) \((1/2, \infty)\) only
Free Response Questions

1. (a) State the Rolle’s Theorem.

(b) Can you apply the Rolle’s Theorem to the function \( f(x) = \frac{x - x^2}{6x - 2} \) on the interval \([0, 1]\)? If not, state a condition of the Theorem not satisfied by \( f(x) \).

(c) What is the value \( c \) guaranteed by the Rolle’s Theorem for \( f(x) = x^{4/3} - x^{1/3} \) on the interval \([0, 1]\)?
2. (a) Find the area of the largest right triangle that can be inscribed in the function \( f(x) = xe^{-0.5x} \) where the hypotenuse has one end at the origin, and the other at a point on the curve in the first quadrant. (See the picture below.)

Use the second derivative test to confirm your result.
(b) A homeowner wishes to enclose a rectangular garden plot. After research she has determined that she needs 2400 square feet of area to accommodate the plants she wishes to grow. She has decided to use fencing that costs 2$ per foot for three sides of the plot and a more decorative fencing on the side facing the road that costs 4$ per foot. What dimensions of the garden will minimize the cost of fencing?

Use the second derivative test to confirm your result.
3. If \( f(x) = x(x+5)^{3/2} \), then \( f'(x) = \frac{5x+15}{3(x+5)^{1/3}} \), and \( f''(x) = \frac{10(x+6)}{9(x+5)^{4/3}} \)

(a) Determine the following for \( f \) (if none, write none)

i. Domain of \( f \)

ii. Asymptotes of \( f \)

iii. \( x \)-intercept(s):......................... \( y \)-intercept:........................

iv. The critical numbers of \( f(x) \) occur at \( x = \) .................

v. Horizontal tangent lines at \( x = \) ......................

vertical tangent lines at \( x = \) .........................

vi. Number line indicating positive and negative value for \( f' \).

vii. Number line indicating positive and negative value for \( f'' \)

(b) Local maxima (as an ordered pair)

(c) Local minima (as an ordered pair)

(d) Inflection point(s) (as an ordered pair)
(e) Sketch the graph of \( y = x(x + 5)^{3/2} \), clearly labeling all the important features of the graph. Use the approximations \( f(-3) \approx -4.8 \).

4. Use L’Hospital’s rule to evaluate \( \lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{e^t - 1} \right) \).
5. Consider the area under the graph of $f(x) = 3x + x^2$ on $[0, 3]$.

(a) Find a Riemann sum which approximates the area using $n$ sub intervals of equal width letting $x_i^*$ be the right end point of the sub interval $[x_{i-1}, x_i]$.

(b) Find the area by considering the limit of the above as $n \to \infty$. You may use the facts that $\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$ and $\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$.
Multiple Choice Questions: 4 points each
1. D
2. D
3. D
4. A
5. E
6. E
7. C
8. D
9. A or C
10. C
11. D
12. E
13. E
14. C
15. E

For the Free Response Questions Refer to the worked out solutions in the link