WASHER Method

Rotated about $x$-axis OR any horizontal line ($y = \#$)

$$V = \int_a^b \pi [OR^2 - IR^2] \, dx$$

Shell Method

Rotated about $y$-axis or any other vertical line ($x = \#$)

$$V = \int_a^b 2\pi \left( \text{shell radius} \times \text{shell height} \right) \, dy$$

Rotated about $x$-axis ($y = \#$)

$$V = \int_a^b \pi [OR^2 - IR^2] \, dy$$

Shell radius $\perp$ to axis of rotation
Shell height $\parallel$ to axis of rotation
Region: \( y = x^2 + 1 \)
\[ y = 3 - x^2 \]
Rotation: \( x \)-axis

To find \( OR + IR \):
Graph functions to find top curve \& bottom curve

\( OR = \) Top curve \( = 3 - x^2 \)
\( IR = \) bottom curve \( = x^2 + 1 \)

\[ V = \pi \int_{a}^{b} (3-x^2)^2 - (x^2+1)^2 \, dx \]

Canceling choices C, D, \& E

Note: 
\( D \) would have been canceled also because the aren't squaring \( OR + IR \)

To find \( a + b \) set \( x^2 + 1 + 3 - x^2 \) equal to each other and solve for \( x \)

\[ x^2 + 1 = 3 - x^2 \]

\[ 2x^2 - 2 = 0 \Rightarrow 2(x^2 - 1) = 0 \]
\[ 2(x-1)(x+1) = 0 \]
\[ a = -1 \quad b = 1 \]

Answer:
A. \[ V = \pi \int_{-1}^{1} [(3-x^2)^2 - (x^2+1)^2] \, dx \]
\[ \int_0^1 \frac{5x + 8}{x^2 + 3x + 2} \, dx \]

\[ \frac{5x + 8}{x^2 + 3x + 2} = \frac{5x + 8}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \]

\[ \int_0^1 \frac{A}{x+1} + \frac{B}{x+2} \, dx = A \ln |x+1| + B \ln |x+2| \]

Use Cover Up method to find \( A + B \)

\[ A := \frac{5x + 8}{x + 2} \bigg|_{x=-1} = \frac{3}{1} = 3 \]

\[ B := \frac{5x + 8}{x + 1} \bigg|_{x=-2} = \frac{-10 + 8}{-1} = 2 \]

\[ \int_0^1 \frac{3}{x+1} + \frac{2}{x+2} \, dx = 3 \ln |x+1| + 2 \ln |x+2| \bigg|_{x=0} \]

\[ = 3 \ln 2 + 2 \ln 3 - \left[3 \ln 1 + 2 \ln 2\right]_{x=0} \]

\[ = 3 \ln 2 + 2 \ln 3 - 2 \ln 2 \]

\[ = \ln 2 + 2 \ln 3 \]

\[ = \ln (2^2) - \ln (3^2) = \ln (2) + \ln (9) \]

\[ = \ln (2) + \ln (3^2) = \ln (2 \cdot 9) = \ln (18) \]

Answer: \( \ln (18) \)
\[ \int_{-\infty}^{0} \frac{e^{2x}}{e^{x}+1} \, dx \]

**u-sub!**

1) Good first guess for u-sub is 

\[ u = \text{denominator} \quad (\text{might not always work but it's worth a try}) \]

\[ u = e^{x} + 1 \quad \Rightarrow \quad \text{NOTE: } u - 1 = e^{x} \]

\[ du = e^{x} \, dx \]

**Changing limits of integration:**

For \[ a = -\infty \]

\[ u(b) = u(0) = e^{0} + 1 = 1 + 1 = 2 \]

\[ u(a) = \lim_{x \to -\infty} u(x) = \lim_{x \to -\infty} e^{x} + 1 = 0 + 1 = 1 \]

\[ \int_{-\infty}^{2} \frac{(e^{2x})e^{x} \, dx}{e^{x}+1} = \int_{1}^{2} \frac{u - 1}{u} \, du = \int_{1}^{2} \left( 1 - \frac{1}{u} \right) \, du = u - \ln|u| \bigg|_{1}^{2} = 2 - \ln(2) - 1 + \ln(1) = 0 \]

Answer B
4. \[ \int_0^1 \frac{x}{x^2 + 2x + 2} \, dx = \int_0^1 \frac{x}{(x+1)^2 + 1} \, dx = \int_0^{u=1} \frac{u-1}{u^2 + 1} \, du \]

Let \( u = x + 1 \)
\( du = dx \)

And \( x = u - 1 \)

\[ = \int_1^2 \frac{u-1}{u^2 + 1} \, du \]

\[ = \int_1^2 \frac{u}{u^2 + 1} \, du - \int_1^2 \frac{1}{u^2 + 1} \, du \]

\[ = \frac{1}{2} \ln |u^2 + 1| - \arctan(u) \bigg|_1^2 \]

\[ = \frac{1}{2} \ln (4+1) - \arctan(2) - \frac{1}{2} \ln(2) \]

\[ = \frac{1}{2} \ln(5) - \arctan(2) - \frac{1}{2} \ln(2) + \frac{\pi}{4} \]

\[ = \frac{1}{2} \ln(\frac{5}{2}) - \arctan(2) + \frac{\pi}{4} \]
5) Compute \[\int_0^{0.5} \arcsin x \, dx.\]

\[\int_0^{0.5} \arcsin x \, dx = \sqrt{1-x^2} + x \arcsin x + C\]

\[\int_0^{0.5} \arcsin x \, dx = \sqrt{1-0.5^2} + 0.5 \arcsin(0.5) - 0\]

\[= \sqrt{1-\frac{1}{4}} + \frac{1}{2} \arcsin\left(\frac{1}{2}\right) - 1\]

\[= \sqrt{\frac{3}{4}} + \frac{1}{2} \left(\frac{\pi}{6}\right) - 1\]

\[= \frac{\sqrt{3}}{2} + \frac{\pi}{12} - 1\]

\[= \frac{2 - \sqrt{3}}{2} + \frac{\pi}{12}\]

What about \[\int_{-1}^{1} \arcsin x \, dx = \text{DNE}\] (Well, it ends up being an imaginary number of some sort.)

\[\arcsin(x)\text{ isn't defined on } x > 1\]

or \(x \leq -1\)

\[\int_{-1}^{1} \frac{1}{x} \, dx = \]
\[ \int_0^{\frac{\pi}{2}} \cos^3(x) \sin^5(x) \, dx = \int_0^{\frac{\pi}{2}} \cos(x) \cos^2 x \sin^5 x \, dx \]

\[ = \int_0^{\frac{\pi}{2}} \cos(x) (1 - \sin^2 x) \sin^5 x \, dx \]

Let \( u = \sin x \)
\[ du = \cos x \, dx \]
\[ \int (1 - u^2)u^5 \, du \]

\[ = \int u^5 - u^7 \, du = \left[ \frac{1}{6} u^6 - \frac{1}{8} u^8 \right]_0 \]
\[ = \frac{1}{6} (1 - 0) - \frac{1}{8} (1 - 0) \]
\[ = \frac{1}{6} - \frac{1}{8} = \frac{4 - 3}{48} = \frac{1}{48} \]

\[ = \frac{1}{24} \]

**Answer A.**
7. \[ x = y^2 - 2y \]

Parabola, but because it's in terms of \( y \), it will look like this rather than \( y \)-axis.

Recall: \( f(x) = ax^2 + bx + c \)

So for \( x = y^2 - 2y \)

\[ \text{Vertex: } \left( f\left( \frac{-b}{2a} \right), \frac{f(-b)}{2a} \right) \]

Vertex: \( \left( f\left( \frac{-1}{2} \right), \frac{-1}{2} \right) \)

Swapping coordinates

Finding vertex

\[ \frac{-b}{2a} = -\frac{(-2)}{2 \cdot 1} = 1 \quad y = 1 \]

\[ x = \left( 1 \right)^2 - 2(1) = -1 \]

Vertex: \((-1, 1)\)

\[ y = 4 \]

[Diagram of a parabola with vertex at (-1, 1) and y-axis passing through it.]
Shell Method
For rotating about x-axis
or any horizontal line (y = k)
we integrate with respect to y
\[ 2\pi \int_a^b \text{shell radius} \cdot \text{shell height} \, dy \]
\[ a \leq y \leq b \]

Shell Radius: \( 4 - y \)
Shell Height: \( \text{"Top" Curve} - \text{"Bottom" Curve} \)
y-axis = Parabola
\[ 0 - (x) = 0 - (y^2 - 2y) = 2y - y^2 \]

Limits of integration: We're integrating with respect to y
So we look at y bounds on the region
\[ 0 \leq y \leq 2 \]

\[ V = 2\pi \int_0^2 (4-y)(2y-y^2) \, dy \]
Answer is D
"Cross-Sections ⊥ to y-axis are semi-circles"
Means we are slicing 3D figure horizontally and Thickness of Cross-Sections will be dy

\[ V = \int_a^b \text{area of cross-section} \cdot \text{thickness} \]
\[ V = \int_a^b \text{area of semi-circles} \cdot dy \]
\[ V = \int_a^b \frac{1}{2} \pi r^2 \ dy \]

But What's \( r \)? And how do we write in terms of \( y \)?

Diameter is length of line segment \( \overline{AB} \).
Note Points \( A \) and \( B \) lie on ellipse and the \( y \)-axis bisects line segment into two equal parts.

Radius: Distance between \( y \)-axis and Point \( B \)

Use \( x^2 + 9y^2 = 9 \) to write \( x \) in terms of \( y \).
\[ x^2 = 9 - 9y^2 \Rightarrow x = \sqrt{9 - 9y^2} \]
\[ r = \sqrt{9 - 9y^2} \]
8. Continued

\[
V = \int_{-a}^{b} \frac{1}{2} \pi \left( \sqrt{9 - 9y^2} \right)^2 dy
\]

Bounds \( a, b \) will correspond to \( y \) bounds on Ellipses.

\[
\int_{-1}^{1} \frac{1}{2} \pi (9 - 9y^2) dy = \int_{-1}^{1} \frac{1}{2} \pi 9(1 - y^2) dy = \frac{9\pi}{2} \int_{-1}^{1} (1 - y^2) dy
\]

\[
= \frac{9\pi}{2} \left[ y - \frac{1}{3}y^3 \right]_{y=1}^{y=-1}
\]

\[
= \frac{9\pi}{2} \left[ 1 - \frac{1}{3} - (1 - \frac{1}{3}) \right]
\]

\[
= \frac{9\pi}{2} \left[ 2 - \frac{2}{3} \right]
\]

\[
= \frac{3\pi}{2} \cdot 4 = \boxed{16\pi}
\]