2. A Remarkable Function

By Theorem 5.12 (p 311), we know that for each x in the interval $(0, \infty)$, the integral $\int_1^x \frac{1}{t}\,dt$ exists. For each $x > 0$, we could let $G(x)$ or $L(x)$ denote the value of this integral. The resulting function is a very important function having the interval $(0, \infty)$ as its domain. We should study this function and discover many of its properties. As we begin our study of this function, we would like to follow tradition and let $\ln x$ stand for the value of this integral [instead of using $G(x)$ or $L(x)$]. Thus in the following discussion, we shall let

$$\ln x = \int_1^x \frac{1}{t}\,dt \text{ for each } x > 0.$$

On pp. 338–339 in the text, the authors observe that the derivative of this function is positive on the whole domain which is an interval and that the range is the interval $(-\infty, \infty)$. Thus 1 is in the range of this function. This means that $1 = \ln x_0$ for some $x_0 > 0$. But since $\ln x$ is one-to-one, we know that x_0 is unique. Following tradition, we shall henceforth use e to denote this unique number x_0 such that $\ln x_0 = 1$.

Definition 2.1. The number e is that unique number such that $\ln e = 1$.

In Ex. 47, p 345, the authors indicate how we may show that

$$\ln b^r = r \ln b \text{ for each } b > 0 \text{ and for each rational number } r. \tag{1}$$

In particular, if we take b to be e and let r be an arbitrary rational number y, then we have

$$\ln e^y = y \ln e = y \text{ since } \ln e = 1.$$

Thus we have proved the following.

Theorem 2.2. If y is rational, then

$$\ln e^y = y. \tag{2}$$

3. Irrational Exponents on e

In a precalculus course, we gave the definition of a^y if $a > 0$ and y is any rational number. But we did not define a^y if y is irrational because this would have been an extremely difficult task without the use of calculus. We are now in a position where it is easy to define irrational exponents, and we must do so. [When a physicist or an engineering professor uses the function $g(x) = e^x$, it is essential that the domain of g include irrational numbers as well as rational numbers. We should remember there are far more irrational values of x than there are rational values, and we must not ignore them.]

We begin by defining irrational exponents on e. The question is: How should we define e^y when y is irrational so as to obtain a useful system?

Well— we choose some law which we already know holds for rational exponents and deliberately define e^y when y is irrational just so that this law will continue to hold. The law which we pick here is the one expressed in formula (2). We can restate that law in a more emphatic way as follows (since $\ln x$ is one-to-one).

(2*) If y is rational, then e^y is the unique number such that $\ln e^y = y$.

Definition 3.1. If y is irrational, then e^y is the unique number such that $\ln e^y = y$.

Combining Theorem 2.2 and Definition 3.1, we have the following result.

Theorem 3.2. If y is any real number, then

$$\ln e^y = y \quad \begin{cases} \text{by Theorem 2.2} & \text{if } y \text{ is rational} \\ \text{by Definition 3.1} & \text{if } y \text{ is irrational} \end{cases} \tag{3}$$

Examining (3), we see that (since $\ln x$ is one-to-one) this means:

$$\text{If } y = \ln x = f(x), \text{ then } x = e^y = f^{-1}(y). \tag{4}$$

Now if we properly interpret statement (4), we have the following result.

Theorem 3.3. The function e^x is the inverse function of $\ln x$. See Figure 1.

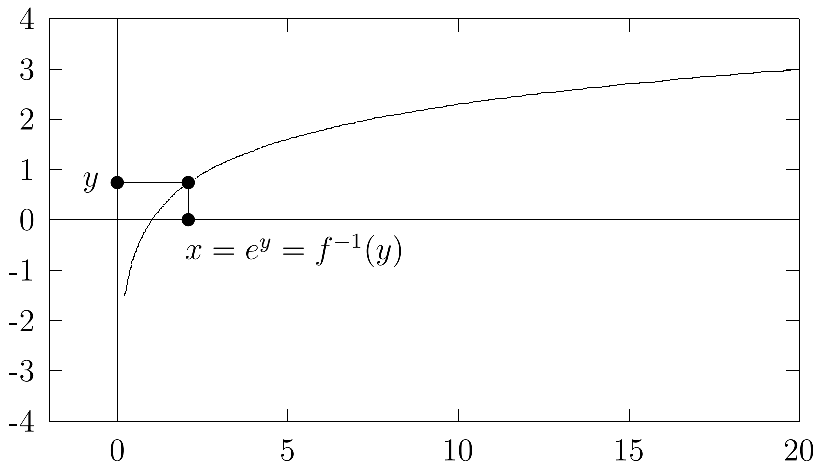

Figure 1. $y = \ln x = f(x)$

Now we use statement (4) to prove our next theorem.

Theorem 3.4. *The Surprise Theorem.* If $x > 0$, then $\ln x = \log_e x$.

Proof. Let $x > 0$ and let $y = \ln x$. Then

$$\begin{aligned} x &= e^y && \text{by (4)} \\ &= e^{\ln x} && \text{since } y = \ln x. \end{aligned} \tag{5}$$

Now statement (5) explicitly states that $\ln x$ is the exponent to which we raise e to obtain x. Hence $\ln x = \log_e x$ by Definition 1.10 in our first handout.

Corollary 3.5. For each $u > 0$, we have

$$u = e^{\ln u}. \tag{6}$$

4 . From Rational Exponents to Irrational Exponents

We are now in a position to define a^x where $a > 0$ and x is irrational. To do this, we choose some law which we already know holds if x is rational and deliberately define a^x when x is irrational just so that this law will continue to hold for irrational values of x. The law which we select is given in our next theorem.

Theorem 4.1. If $a > 0$ and x is rational, then

$$a^x = e^{x \ln a}. \tag{7}$$

Proof. Taking u to be a^x in (6), we have

$$\begin{aligned} a^x &= e^{\ln a^x} && \text{by (6)} \\ &= e^{x \ln a} && \text{by (1) (since } x \text{ is rational).} \end{aligned}$$

Definition 4.2. If $a > 0$ and x is irrational, then $a^x = e^{x \ln a}$

Combining Theorem 4.1 and Definition 4.2, we have the result: If x is any real number and $a > 0$, then

$$(7^*)\ a^x = e^{x \ln a} \quad \begin{cases} \text{by Theorem 4.1} & \text{if } x \text{ is rational} \\ \text{by Definition 4.2} & \text{if } x \text{ is irrational} \end{cases}$$

Now if we need to perform a calculus operation on a^u or on v^u, then we first rewrite the expression as

$$e^{u \ln a} \text{ or as } e^{u \ln v}$$

and then use the formulas which we know for e^u.

Calculus operations are differentiation, integration, and finding limits.

Likewise, if we need to perform a calculus operation on $\log_a u$, then we simply rewrite the expression as

$$\frac{\ln u}{\ln a}$$

and use formulas which we know for the natural logarithm.