NOTES ON SETS AND LOGIC
MHF 3202

ALEXANDRE TURULL

1. The Cantor-Bernstein-Schröder Theorem

A version of the book's proof of the theorem.

Theorem (Cantor-Bernstein-Schröder Theorem)**.** *Let A and B be sets. Suppose $|A| \leq |B|$ and $|B| \leq |A|$. Then $|A| = |B|$.*

Proof. By definition, there exist injective functions

$$f\colon A \to B$$

and

$$g\colon B \to A.$$

Let $h = g \circ f$ so that h is an injective function

$$h\colon A \to A.$$

As is customary, we set

$$h^0 = Id_A\colon A \to A$$

and, for every $n \in \mathbf{N}$,

$$h^n : A \to A$$

to be the result of composing $h^n = h \circ \cdots \circ h$ where there are n copies of h. For each $n \in \mathbf{Z}$ with $n \geq 0$, h^n is an injective function. We let

$$G = \bigcup_{i=0}^{\infty} h^i \left(A - g(B)\right).$$

Then

$$A - g(B) \subseteq G \subseteq A.$$

Furthermore,

$$h(G) \subseteq G.$$

We set

$$W = A - G.$$

Then

$$W \subseteq g(B) \subseteq A.$$

We notice that, for each $a \in W$, there exists exactly one $b \in B$ such that $g(b) = a$. We now define a map

$$\beta\colon A \to B$$

by, for all $a \in A$, setting

$$\beta(a) = \begin{cases} f(a), & \text{if } a \in G; \\ b, & \text{where } b \in B \text{ is such that } g(b) = a \text{ if } a \in W. \end{cases}$$

β is indeed a map from A to B.

Suppose that β is not injective. Then there exist $a_1, a_2 \in A$, such that $a_1 \neq a_2$ but $\beta(a_1) = \beta(a_2)$, and among all these we pick a_1 and a_2 such that $a_1 \in G$ and $a_2 \in W$ if possible. If $a_1, a_2 \in G$ then $\beta(a_1) = f(a_1) = \beta(a_2) = f(a_2)$, a contradiction since f is injective. If $a_1, a_2 \in W$ then $g(\beta(a_1)) = a_1 = a_2$, a contradiction. Hence $a_1 \in G$ and $a_2 \in W$. In this case, $\beta(a_1) = f(a_1)$ and $g(\beta(a_2)) = a_2$. Hence $g(f(a_1)) = a_2$. In other words $h(a_1) = a_2$. Since $a_1 \in G$ and $h(G) \subseteq G$, this implies $a_2 \in G$, a contradiction. Hence β is injective.

Suppose that β is not surjective. Then there exists some $b \in B$ such that $b \notin \beta(A)$. Suppose that $g(b) \in W$. Then $g(b) \in A$ and $\beta(g(b)) = b$, a contradiction. Hence $g(b) \in G$. Then, by the definition of G, there exists some $d \in (A - g(B))$ and some $k \in \mathbf{Z}$ such that $k \geq 0$ and $g(b) = h^k(d)$. If $k = 0$, then $g(b) = d \in (A - g(B))$, a contradiction, so $k \geq 1$. Hence, we get

$$g(b) = g\left(f\left(h^{k-1}(d)\right)\right).$$

Since g is injective, this implies

$$b = f\left(h^{k-1}(d)\right).$$

Since $d \in (A - g(B))$, we know that $h^{k-1}(d) \in G$, and it follows from the definition of β and the previous equation that

$$\beta\left(h^{k-1}(d)\right) = f\left(h^{k-1}(d)\right) = b.$$

This contradicts our choice of b. Hence, β is surjective.

It follows that β is a bijection. Therefore $|A| = |B|$. The proof is complete. □

References

[1] Paul R. Halmos, *Naive set theory*, van Nostrand, Princeton, Toronto, London, New York, 1960.

Department of Mathematics, University of Florida, Gainesville, FL 32611, USA

Email address: `turull@math.ufl.edu`