$\boxed{1}$ Check that the function

$$y_h = C(1+t) + De^t$$

solves the homogeneous equation

$$ty'' - (1+t)y' + y = 0.$$

Then find a general solution to the non-homogeneous equation

$$ty'' - (1+t)y' + y = t^2e^{2t}.$$

Solution: To check the solution, we just need to compute derivatives and substitute them into the equation.

We then use variation of parameters to find the solution to the non-homogeneous equation. This solution will have the form

$$y = C(t)\underbrace{(1+t)}_{y_1} + D(t)\underbrace{e^t}_{y_2}$$

where

$$C(t) = \int \frac{-fy_2}{a(y_1y_2' - y_2y_1')}\,dt \quad \text{and} \quad D(t) = \int \frac{fy_1}{a(y_1y_2' - y_2y_1')}\,dt.$$

First we compute the denominator of these integrands:

$$a(y_1y_2' - y_2y_1') = t\begin{vmatrix} 1+t & e^t \\ 1 & e^t \end{vmatrix} = t(e^t + te^t - e^t) = t^2e^t.$$

Thus we see that

$$\begin{aligned} C(t) &= \int \frac{(-t^2e^{2t})e^t}{t^2e^t}\,dt \\ &= \int -e^{2t}\,dt \\ &= -\frac{e^{2t}}{2} + E. \end{aligned}$$

We also have

$$\begin{aligned} D(t) &= \int \frac{(t^2e^{2t})(1+t)}{t^2e^t}\,dt \\ &= \int e^t + te^t\,dt \\ &= te^t + F. \end{aligned}$$

Therefore the general solution to the non-homogeneous equation is

$$y = \left(-\frac{e^{2t}}{2} + E\right)(1+t) + \left(te^t + F\right)e^t.$$

$\boxed{2}$ In an initial value problem, you are given a differential equation, together with a value of y and a value of y'. In a *boundary value problem*, on the other hand, you are given a differential equation and two values of y (we think of these as the values of y "on the boundary"). The following questions concern the boundary value problem

$$y'' + \lambda^2 y = \sin t; \quad y(0) = 0; \quad y(\pi) = 1.$$

a. Find the general solution to the given differential equation for all $\lambda \neq \pm 1$ (ignoring the boundary conditions for now).

Solution: No matter what the value of λ is, the solution to the homogeneous equation is

$$y_h = C\cos\lambda t + D\sin\lambda t.$$

For $\lambda \neq \pm 1$, our guess for the particular solution is

$$y_p = A\cos t + B\sin t.$$

Computing derivatives shows

$$\begin{aligned} y_p' &= -A\sin t + B\cos t, \\ y_p'' &= -A\cos t - B\sin t. \end{aligned}$$

Substituting this into the differential equation gives

$$(-A\cos t - B\sin t) + \lambda^2(A\cos t + B\sin t) = \sin t,$$

and simplifying this shows that

$$(\lambda^2 A - A)\cos t + (\lambda^2 B - B)\sin t = \sin t.$$

So we want $A = 0$ and $\lambda^2 B - B = 1$, which means $B(\lambda^2 - 1) = 1$, so $B = 1/(\lambda^2 - 1)$. Therefore in this case the general solution is

$$y = C\cos\lambda t + D\sin\lambda t + \frac{\sin t}{\lambda^2 - 1}.$$

b. Find the general solution to the given differential equation when $\lambda = \pm 1$ (again ignoring the boundary conditions).

Solution: When $\lambda = \pm 1$, our initial guess for the particular solution overlaps with the homogeneous solution, so we have to multiply by t:

$$\begin{aligned} y_p &= At\cos t + Bt\sin t, \\ y_p' &= A\cos t - At\sin t + B\sin t + Bt\cos t, \\ y_p'' &= -A\sin t - A\sin t - At\cos t + B\cos t + B\cos t - Bt\sin t, \\ &= -2A\sin t - At\cos t + 2B\cos t - Bt\sin t. \end{aligned}$$

Next we substitute these values into the differential equation:

$$(-2A\sin t - At\cos t + 2B\cos t - Bt\sin t) + (At\cos t + Bt\sin t) = \sin t.$$

This simplifies to

$$-2A\sin t + 2B\cos t = \sin t,$$

so we want to set $A = -1/2$ and $B = 0$. This gives the general solution

$$y = C\cos t + D\sin t - \frac{t\sin t}{2}.$$

c. Show that the boundary value problem has a solution if and only if λ is not an integer.

Solution: In both cases, if λ is an integer then substituting $t = 0$ into the equation we have for y shows that we need

$$B\cos(\lambda 0) = 0,$$

so we must have $B = 0$. But then, again because λ is an integer, when we substitute $t = \pi$, we see that $y(\pi) = 0$, while we want it to equal 1.

On the other hand, when λ is not an integer, $\sin(\lambda\pi) \neq 0$, which allows us to choose D to satisfy the boundary value condition $y(\pi) = 1$.