In-class review for Sections 7.2–7.5 — Solutions

Find the Laplace transform of the solution \( y \) to the initial value problem

\[
\begin{aligned}
y'' + 2y' + 2y &= \begin{cases} 
1 & \text{for } 0 \leq t \leq 7, \\
t & \text{for } t > 7.
\end{cases} \\
; \quad y(0) = 2, \quad y'(0) = 1.
\end{aligned}
\]

**Solution:** Let \( Y = \mathcal{L}\{y\} \). For the lefthand side, we have the following transforms:

\[
\begin{aligned}
\mathcal{L}\{y\} &= Y, \\
\mathcal{L}\{y'\} &= sY - y(0) = sY - 2, \\
\mathcal{L}\{y''\} &= s\mathcal{L}\{y'\} - y'(0) = s^2Y - 2s - 1.
\end{aligned}
\]

For the righthand side, we just have to do the integral:

\[
\begin{aligned}
\mathcal{L}\left\{ \begin{cases} 1 & \text{for } 0 \leq t \leq 7, \\
t & \text{for } t > 7
\end{cases} \right\} &= \int_0^7 e^{-st} dt + \int_7^\infty te^{-st} dt, \\
&= \left[ \frac{1}{s}e^{-st} \right]_0^7 + \lim_{N \to \infty} \left[ \frac{4}{s}e^{-st} - \frac{1}{s^2}e^{-st} \right]_7^N \\
&= \frac{e^{-7s}}{s} + \frac{1}{s} + \lim_{N \to \infty} \left( \frac{N e^{-Ns}}{s} - \frac{e^{-Ns}}{s^2} + \frac{7e^{-7s}}{s} + \frac{e^{-7s}}{s^2} \right), \\
&= \frac{6e^{-7s}}{s} + \frac{1}{s} + \frac{e^{-7s}}{s^2}.
\end{aligned}
\]

Now we need to put this all together. Taking the Laplace transform of both sides, we have

\[
\mathcal{L}\left\{ \begin{cases} 1 & \text{for } 0 \leq t \leq 7, \\
t & \text{for } t > 7
\end{cases} \right\} = \mathcal{L}\left\{ \begin{cases} 1 & \text{for } 0 \leq t \leq 7, \\
t & \text{for } t > 7
\end{cases} \right\}
\]

so

\[
(s^2Y - 2s - 1) + 2(sY - 2) + 2Y = \frac{6e^{-7s}}{s} + \frac{1}{s} + \frac{e^{-7s}}{s^2}.
\]

Solving for \( Y \), we get

\[
Y = \frac{\left( \frac{6e^{-7s}}{s} + \frac{1}{s} + \frac{e^{-7s}}{s^2} \right) + 2s + 5}{s^2 + 2s + 2}.
\]

There is no need to simplify this any further.
Suppose that
\[ \mathcal{L}\{y\} = \frac{2s - 7}{(s^2 - 2s + 5)(s - 1)}. \]
What is \( y \)?

**Solution:** We first need to use partial fractions on the righthand side:
\[
\frac{2s - 7}{(s^2 - 2s + 5)(s - 1)} = \frac{As + B}{s^2 - 2s + 5} + \frac{C}{s - 1}.
\]
Canceling denominators, we see that we need
\[(As + B)(s - 1) + C(s^2 - 2s + 5) = 2s - 7.
\]
We can now expand the lefthand side and match coefficients of \( s \), or we can plug in three particular values for \( s \) to get three equations in \( A \), \( B \), and \( C \). With either method, we get that
\[ \mathcal{L}\{y\} = \frac{\frac{5}{4}s + \frac{3}{4}}{s^2 - 2s + 5} + \frac{-\frac{5}{4}}{s - 1}. \]
In order to undo the Laplace transform of the first fraction, we need to complete the square in the denominator. We do this by matching the first two terms, \( s^2 - 2s \) to a square. In this case we get the first two terms from \((s - 1)^2\), and then need to add 4 to get the right constant term; in other words,
\[ s^2 - 2s + 5 = (s - 1)^2 + 4. \]
Now we have
\[ \mathcal{L}\{y\} = \frac{\frac{5}{4}s + \frac{3}{4}}{(s - 1)^2 + 4} + \frac{-\frac{5}{4}}{s - 1}. \]
Now we also need the numerator of the first fraction expressed in terms of \( s - 1 \):
\[ \frac{5}{4}s + \frac{3}{4} = \frac{5}{4}(s - 1) + 2. \]
Putting this all together, we have
\[
\mathcal{L}\{y\} = \frac{5}{4} \frac{s - 1}{(s - 1)^2 + 4} + \frac{2}{(s - 1)^2 + 4} - \frac{\frac{5}{4}}{s - 1}.
\]
We can now look these transforms up, and see that
\[ y = \frac{5}{4} e^t \cos 2t + e^t \sin 2t - \frac{5}{4} e^t. \]
Find a first-order differential equation for the Laplace transform of the solution $y$ to the initial value problem

$$y'' + ty' + 2y = e^{3t}; \quad y(0) = y'(0) = 0.$$ 

*Hint: let $Y(s)$ denote the Laplace transform of $y$. Your answer will include $Y(s)$ and $Y'(s)$. 

**Solution:** Let $Y = \mathcal{L}\{y\}$. For the left-hand side, let’s start with the standard transforms:

$$
\begin{align*}
\mathcal{L}\{y\} &= Y, \\
\mathcal{L}\{y'\} &= sY - y(0) = sY, \\
\mathcal{L}\{y''\} &= s\mathcal{L}\{y'\} - y'(0) = s^2Y.
\end{align*}
$$

However, on the left-hand side of the differential equation we have $ty'$, which we need to figure out how to transform. One of our rules states that

$$
\mathcal{L}\{tf(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\},
$$

so,

$$
\mathcal{L}\{ty'\} = -\frac{d}{ds}\mathcal{L}\{y'\} = -\frac{d}{ds}(sY) = -Y - sY'.
$$

The right-hand side is easy,

$$
\mathcal{L}\{e^{3t}\} = \frac{1}{s-3},
$$

and so we see that the Laplace transform of both sides,

$$
\mathcal{L}\{y'' + ty' + 2y\} = \mathcal{L}\{e^{3t}\},
$$

is given by

$$
s^2Y - Y - sY' + 2Y = \frac{1}{s-3}.
$$

We can simplify this a bit into

$$
Y - \frac{s}{s^2 + 1}Y' = \frac{1}{(s-3)(s^2 + 1)}.
$$

Note that this is a linear first-order differential equation, so we could use an integrating factor to solve for $Y$. Theoretically, we could then undo the Laplace transform and find $y$. Best of luck to all who try that.
Suppose that

\[ \mathcal{L}\{y\} = \frac{s^2 - s + 1}{s^4 - s^3 + s^2 - s}. \]

What is \( y \)?

**Solution:** The only real challenge here is factoring that quartic in the denominator of the fraction. We see immediately that \( s \) is a factor:

\[ s^4 - s^3 + s^2 - s = s(s^3 - s^2 + s - 1). \]

Now we need to find one more factor. If we plug in \( s = 1 \) then \( s^3 - s^2 + s - 1 = 0 \), so \( s - 1 \) is a factor of this cubic. Dividing by \( s - 1 \) gives a quotient of \( s^2 + 1 \), so our factorization is

\[ s^4 - s^3 + s^2 - s = s(s - 1)(s^2 + 1). \]

Next we compute the partial fraction decomposition:

\[ \frac{s^2 - s + 1}{s^4 - s^3 + s^2 - s} = \frac{A}{s} + \frac{B}{s - 1} + \frac{Cs + D}{s^2 + 1}. \]

Canceling denominators shows that

\[ s^2 - s + 1 = A(s - 1)(s^2 + 1) + Bs(s^2 + 1) + (Cs + D)s(s - 1). \]

Any way you solve this equation, you should get that \( A = -1 \) and \( B = C = D = 1/2 \). Therefore

\[ \mathcal{L}\{y\} = \frac{\frac{1}{2} s + \frac{1}{2}}{s^2 + 1} + \frac{\frac{1}{2}}{s - 1} - \frac{1}{s}, \]

\[ = \frac{1}{2} \frac{s}{s^2 + 1} + \frac{1}{2} \frac{1}{s^2 + 1} + \frac{1}{2} \frac{1}{s - 1} - \frac{1}{s}. \]

These are all easy transforms to undo, and we see that

\[ y = \frac{1}{2} \cos t + \frac{1}{2} \sin t + \frac{1}{2} e^t - 1. \]