In-class review for Sections 7.2–7.5 — Solutions

1 Find the Laplace transform of the solution y to the initial value problem

$$y'' + 2y' + 2y = \begin{cases} 1 & \text{for } 0 \le t \le 7, \\ t & \text{for } t > 7. \end{cases} \quad ; \quad y(0) = 2, \quad y'(0) = 1.$$

Solution: Let $Y = \mathscr{L}\{y\}$. For the lefthand side, we have the following transforms:

$$\begin{aligned} \mathscr{L}\{y\} &= Y, \\ \mathscr{L}\{y'\} &= sY - y(0) &&= sY - 2, \\ \mathscr{L}\{y''\} &= s\mathscr{L}\{y'\} - y'(0) &&= s^2Y - 2s - 1. \end{aligned}$$

For the righthand side, we just have to do the integral:

$$\begin{aligned} \mathscr{L}\left\{\begin{cases} 1 & \text{for } 0 \le t \le 7, \\ t & \text{for } t > 7. \end{cases}\right\} &= \int_0^7 e^{-st}\,dt + \int_7^\infty te^{-st}\,dt, \\ &= \left[-\frac{1}{s}e^{-st}\right]_0^7 + \lim_{N\to\infty}\left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_7^N, \\ &= -\frac{e^{-7s}}{s} + \frac{1}{s} + \lim_{N\to\infty}\left(-\frac{Ne^{-Ns}}{s} - \frac{e^{-Ns}}{s^2} + \frac{7e^{-7s}}{s} + \frac{e^{-7s}}{s^2}\right), \\ &= \frac{6e^{-7s}}{s} + \frac{1}{s} + \frac{e^{-7s}}{s^2}. \end{aligned}$$

Now we need to put this all together. Taking the Laplace transform of both sides, we have

$$\mathscr{L}\left\{y'' + 2y' + 2y\right\} = \mathscr{L}\left\{\begin{cases} 1 & \text{for } 0 \le t \le 7, \\ t & \text{for } t > 7, \end{cases}\right\}$$

so

$$\left(s^2Y - 2s - 1\right) + 2\left(sY - 2\right) + 2Y = \frac{6e^{-7s}}{s} + \frac{1}{s} + \frac{e^{-7s}}{s^2}.$$

Solving for Y, we get

$$Y = \frac{\left(\frac{6e^{-7s}}{s} + \frac{1}{s} + \frac{e^{-7s}}{s^2}\right) + 2s + 5}{s^2 + 2s + 2}.$$

There is no need to simplify this any further.

$\boxed{2}$ Suppose that

$$\mathscr{L}\{y\} = \frac{2s-7}{(s^2-2s+5)(s-1)}.$$

What is y?

Solution: We first need to use partial fractions on the righthand side:

$$\frac{2s-7}{(s^2-2s+5)(s-1)} = \frac{As+B}{s^2-2s+5} + \frac{C}{s-1}.$$

Canceling denominators, we see that we need

$$(As+B)(s-1) + C(s^2-2s+5) = 2s-7.$$

We can now expand the lefthand side and match coefficients of s, or we can plug in three particular values for s to get three equations in A, B, and C. With either method, we get that

$$\mathscr{L}\{y\} = \frac{\frac{5}{4}s + \frac{3}{4}}{s^2-2s+5} + \frac{-\frac{5}{4}}{s-1}.$$

In order to undo the Laplace transform of the first fraction, we need to complete the square in the denominator. We do this by matching the first two terms, $s^2 - 2s$ to a square. In this case we get the first two terms from $(s-1)^2$, and then need to add 4 to get the right constant term; in other words,

$$s^2 - 2s + 5 = (s-1)^2 + 4.$$

Now we have

$$\mathscr{L}\{y\} = \frac{\frac{5}{4}s + \frac{3}{4}}{(s-1)^2+4} + \frac{-\frac{5}{4}}{s-1}.$$

Now we also need the numerator of the first fraction expressed in terms of $s-1$:

$$\frac{5}{4}s + \frac{3}{4} = \frac{5}{4}(s-1) + 2.$$

Putting this all together, we have

$$\mathscr{L}\{y\} = \frac{5}{4}\frac{s-1}{(s-1)^2+4} + \frac{2}{(s-1)^2+4} - \frac{5}{4}\frac{1}{s-1}.$$

We can now look these transforms up, and see that

$$y = \frac{5}{4}e^t\cos 2t + e^t \sin 2t - \frac{5}{4}e^t.$$

$\boxed{3}$ Find a first-order differential equation for the Laplace transform of the solution y to the initial value problem

$$y'' + ty' + 2y = e^{3t}; \quad y(0) = y'(0) = 0.$$

Hint: let $Y(s)$ denote the Laplace transform of y. Your answer will include $Y(s)$ and $Y'(s)$.

Solution: Let $Y = \mathscr{L}\{y\}$. For the lefthand side, let's start with the standard transforms:

$$\begin{array}{rcllcl} \mathscr{L}\{y\} & = & Y, & & & \\ \mathscr{L}\{y'\} & = & sY - y(0) & & = & sY, \\ \mathscr{L}\{y''\} & = & s\mathscr{L}\{y'\} - y'(0) & & = & s^2 Y. \end{array}$$

However, on the lefthand side of the differential equation we have ty', which we need to figure out how to transform. One of our rules states that

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\},$$

so,

$$\mathscr{L}\left\{ty'\right\} = -\frac{d}{ds}\mathscr{L}\left\{y'\right\} = -\frac{d}{ds}(sY) = -Y - sY'.$$

The righthand side is easy,

$$\mathscr{L}\left\{e^{3t}\right\} = \frac{1}{s-3},$$

and so we see that the Laplace transform of both sides,

$$\mathscr{L}\left\{y'' + ty' + 2y\right\} = \mathscr{L}\left\{e^{2t}\right\},$$

is given by

$$s^2 Y - Y - sY' + 2Y = \frac{1}{s-3}.$$

We can simplify this a bit into

$$Y - \frac{s}{s^2+1}Y' = \frac{1}{(s-3)(s^2+1)}.$$

Note that this is a linear first-order differential equation, so we *could* use an integrating factor to solve for Y. Theoretically, we could then undo the Laplace transform and find y. Best of luck to all who try that.

$\boxed{4}$ Suppose that

$$\mathscr{L}\{y\} = \frac{s^2 - s + 1}{s^4 - s^3 + s^2 - s}.$$

What is y?

Solution: The only real challenge here is factoring that quartic in the denominator of the fraction. We see immediately that s is a factor:

$$s^4 - s^3 + s^2 - s = s(s^3 - s^2 + s - 1).$$

Now we need to find one more factor. If we plug in $s = 1$ then $s^3 - s^2 + s - 1 = 0$, so $s - 1$ is a factor of this cubic. Dividing by $s - 1$ gives a quotient of $s^2 + 1$, so our factorization is

$$s^4 - s^3 + s^2 - s = s(s - 1)(s^2 + 1).$$

Next we compute the partial fraction decomposition:

$$\frac{s^2 - s + 1}{s^4 - s^3 + s^2 - s} = \frac{A}{s} + \frac{B}{s - 1} + \frac{Cs + D}{s^2 + 1}.$$

Canceling denominators shows that

$$s^2 - s + 1 = A(s - 1)(s^2 + 1) + Bs(s^2 + 1) + (Cs + D)s(s - 1).$$

Any way you solve this equation, you should get that $A = -1$ and $B = C = D = 1/2$. Therefore

$$\begin{aligned} \mathscr{L}\{y\} &= \frac{\frac{1}{2}s + \frac{1}{2}}{s^2 + 1} + \frac{\frac{1}{2}}{s - 1} - \frac{1}{s}, \\ &= \frac{1}{2}\frac{s}{s^2 + 1} + \frac{1}{2}\frac{1}{s^2 + 1} + \frac{1}{2}\frac{1}{s - 1} - \frac{1}{s}. \end{aligned}$$

These are all easy transforms to undo, and we see that

$$y = \frac{1}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}e^t - 1.$$