

Lecture 23 - March 11, 2020

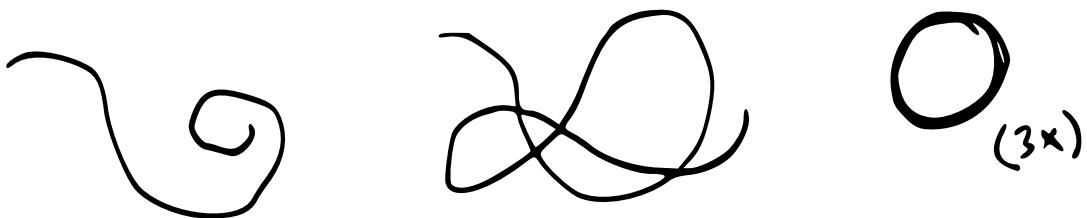
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First lecture on contour integrals.

- What is a contour?

It is a type of curve in the complex plane.

- What is a curve?



A continuous function/parameterization
(don't pick up pencil)

$$z = z(t) \text{ for } a \leq t \leq b.$$

So could also write

$$z: [a, b] \rightarrow \mathbb{C}.$$

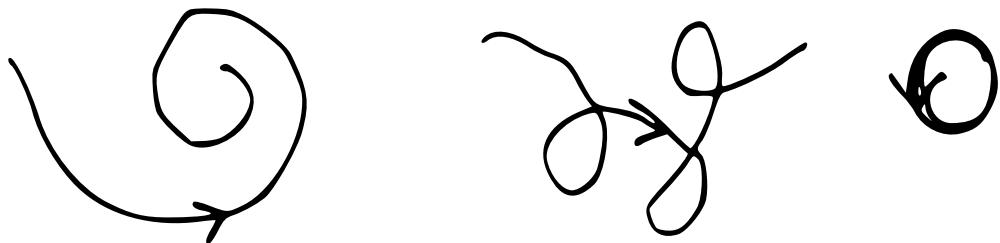
In rectangular coordinates,

$$z = z(t) = x(t) + iy(t)$$

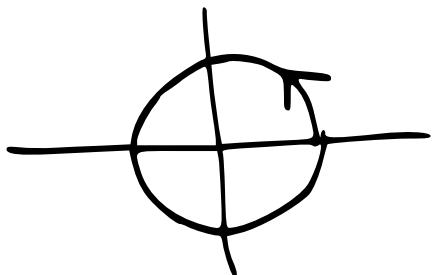
for $a \leq t \leq b$.

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Our definition means all curves are oriented (in the direction of increasing t).



Ex Unit circle



In MV calc:

$$\vec{r}(t) = (\cos t, \sin t) \text{ for } 0 \leq t \leq 2\pi$$

In complex analysis:

$$z(t) = \cos t + i \sin t \text{ for } 0 \leq t \leq 2\pi$$

Or even better:

$$z(t) = e^{it} \text{ for } 0 \leq t \leq 2\pi, \text{ or since angle,}$$

$$z(\theta) = e^{i\theta} \text{ for } 0 \leq t \leq 2\pi.$$

Opposite orientation:

$$z(\theta) = e^{-i\theta} \text{ for } 0 \leq t \leq 2\pi.$$

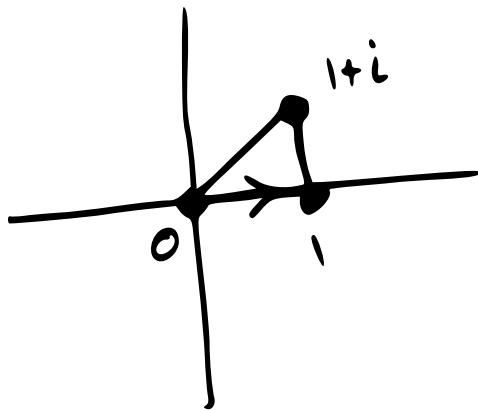
Trace out three times:

$$z(\theta) = e^{i\theta} \text{ for } 0 \leq t \leq 6\pi.$$

(is one way).

Ex This triangle

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Let's go from 0 to 1, then 1 to $1+i$,
then $1+i$ to 0.

$$z(t) = \begin{cases} t & 0 \leq t \leq 1, \\ 1 + i(t-1) & 1 \leq t \leq 2, \\ (3-t) + i(3-t) & 2 \leq t \leq 3. \end{cases}$$

Opposite orientation?

In general, the opposite orientation of the curve

$$z = z(t), \quad a \leq t \leq b$$

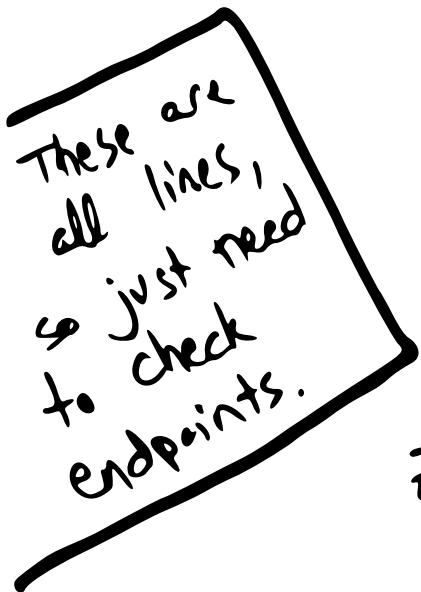
is the curve

$$z = z(-t) \quad -b \leq t \leq -a.$$

In the Δ example, this gives

$$z(t) = \begin{cases} -t & -1 \leq t \leq 0, \\ 1 + i(-t-1) & -2 \leq t \leq -1, \\ (3+t) + i(3+t) & -3 \leq t \leq -2. \end{cases}$$

$$z(t) = \begin{cases} (3+t) + i(3+t) & -3 \leq t \leq -2, \\ 1 + i(-t-1) & -2 \leq t \leq -1, \\ -t & -1 \leq t \leq 0. \end{cases}$$



- Trace out $3x$? Exercise for ^{viewer.} reader.

Def A simple curve is a curve 6/10

$$z: [a, b] \rightarrow \mathbb{C}$$

which does not cross itself.

This means that for $t_1 \neq t_2$, we have

$$z(t_1) = z(t_2)$$

only possibly at the endpoints:
 $t_1, t_2 \in \{a, b\}$.

one-to-one
except
possibly
at endpoints.

- Our $3x$ circle is NOT simple.
- Nor is .

Def If $z: [a, b] \rightarrow \mathbb{C}$ is a curve with $z(a) = z(b)$, then it is closed.

- Both of our examples were closed.

Arc length of a curve

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Suppose our curve is

$$z = z(t) = x(t) + iy(t) \text{ for } a \leq t \leq b.$$

If $x(t)$ and $y(t)$ are both differentiable (as real, single variable functions),

then we say z is differentiable.

From multivariable calc, we know that if z is a differentiable curve, then

$z'(t) = x'(t) + iy'(t)$ is velocity,

$|z'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$ is speed.

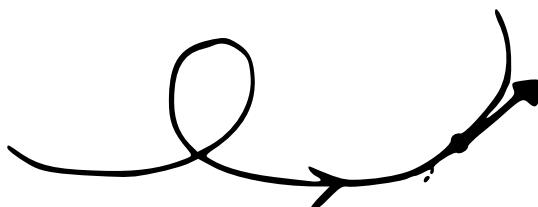
The arc length is

$$\int_a^b \text{speed} dt = \int_a^b |z'(t)| dt.$$

Tangent vectors

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From multivariable calc, we know that the derivative (velocity) vector $\mathbf{z}'(t)$ is tangent to the curve...



- If $\mathbf{z}'(t) \neq 0$.
- OTOH, if $\mathbf{z}'(t) = 0$ then we have all kinds of problems, so we just forbid this....

Def The curve

$$\mathbf{z} : [a, b] \rightarrow \mathbb{C}$$

is smooth if it is differentiable and $\mathbf{z}'(t) \neq 0$ for all $t \in [a, b]$.

Unit tangent vectors

If $z(t)$ is a smooth curve we can form the unit tangent vector

$$T(t) = \frac{z'(t)}{|z'(t)|}.$$

Note: We know the modulus of $T(t)$ is 1, so it is determined by its direction, which is the argument of $z'(t)$. So:

$$T(t) = e^{i \arg(z'(t))}.$$

Finally, what are contours?

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Def. A contour is a curve which is made up of a finite sequence of smooth curves whose endpoints joined end to end.

Def If only the first and last points of $z(t)$ are the same, then it is a simple closed contour.

The Jordan curve theorem ~~says~~ ^{implies} that every simple closed contour partitions the complex plane into two regions, the inside and the outside.

- It is surprisingly hard to prove.