$\boxed{1}$ Solve the following initial value problems.

a. $y'' - 4y' + 8y = 0$; $y(0) = 1$; $y'(0) = 0$.

Solution: This is a homogeneous equation with characteristic polynomial

$$r^2 - 4r + 8.$$

With the quadratic formula, we see that $r = 2 \pm 2i$, so the general solution is

$$y = e^{2t}(C \cos 2t + D \sin 2t).$$

Now we need to choose C and D to match the initial conditions. We see that

$$y(0) = e^0(C + 0) = C = 1,$$

so $C = 1$, and

$$y' = e^{2t}(-2C \sin 2t + 2D \cos 2t) + 2e^{2t}(C \cos 2t + D \sin 2t),$$

so plugging in $t = 0$,

$$y'(0) = e^0(0 + 2D) + 2e^0(C + 0) = 2D + 2C = 0.$$

Since $C = 1$, we see that $D = -1$. Putting it all together, we get

$$y(t) = e^{2t}(\cos 2t - \sin 2t)$$

as the final answer.

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b. $y'' + 2y' - 3y = 0$; $y(0) = 9$; $y'(0) = -3$.

Solution: The characteristic polynomial here is

$$r^2 + 2r - 3 = (r - 1)(r + 3),$$

so $r = 1, -3$, and our general solution is

$$y = Ce^t + De^{-3t}.$$

To find C and D we plug $t = 0$ into y and y':

$$\begin{aligned} y(0) &= C + D = 9, \\ y'(t) &= Ce^t - 3De^{-3t}, \\ y'(0) &= C - 3D = -3. \end{aligned}$$

From the first equation, we see that $D = 9 - C$. Plugging this into the last equation shows that $C - 3(9 - C) = -3$, so $4C = 24$. We conclude that $C = 6$ and $D = 3$, giving

$$y(t) = 6e^t + 3e^{-3t}$$

as the final solution.

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c. $y'' - 4y = 0$; $y(0) = 2$; $y'(0) = -98$.

Solution: The characteristic polynomial is

$$r^2 - 4 = (r - 2)(r + 2),$$

so the roots are $r = \pm 2$, and thus the solution is

$$y = Ce^{2t} + De^{-2t}.$$

To find C and D we plug $t = 0$ into y and y':

$$\begin{aligned} y(0) &= C + D = 2, \\ y'(t) &= 2Ce^{2t} - 2De^{-2t}, \\ y'(0) &= 2C - 2D = -98. \end{aligned}$$

From the first equation, $D = 2 - C$. Plugging this into the last equation gives us that $2C - 2(2 - C) = -98$, so $4C = -94$, $C = -47/2$, $D = 51/2$, and the final solution is

$$y(t) = -\frac{47}{2}e^{2t} + \frac{51}{2}e^{-2t}.$$

$\boxed{2}$ What form of particular solution y_p would you guess in order to solve the following differential equations using the method of undetermined coefficients? *Do not solve these problems.*

a. $y'' - 4y' - 21y = 2e^{7t}$.

Solution: First we have to solve the corresponding homogeneous equation,

$$y'' - 4y' - 21y = 0.$$

This equation has characteristic polynomial

$$r^2 - 4r - 21 = (r-7)(r+3),$$

so the roots are $r = 7, -3$ and the the homogeneous solution is

$$y_h = Ce^{7t} + De^{-3t}$$

Our first guess for y_p is to match the function on the righthand side and all its derivatives. In this case, this guess is

$$y_p = Ae^{7t}.$$

However, this guess overlaps with the homogeneous solution, so we need to multiply it by t:

$$y_p = Ate^{7t}.$$

This y_p does not overlap with the homogeneous solution, so it is the correct guess.

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b. $y'' - 2y' - 8y = 19t\cos(2t)$.

Solution: We first solve the homogeneous equation,

$$y'' - 2y' - 8y = 0.$$

The characteristic polynomial here is

$$r^2 - 2r - 8 = (r-4)(r+2),$$

so $r = 4, -2$, and the homogeneous solution is

$$y_h = Ce^{4t} + De^{-2t}.$$

Our first guess for y_p is an arbitrary constant multiple of the function on the righthand side, $At\cos(2t)$, together with arbitrary constant multiple of the functions which arise by differentiating this function, $Bt\sin(2t)$ and $E\cos(2t)$, together with arbitrary constant multiple of the functions which arise by differentiating *this* function, $F\sin(2t)$. Putting this all together, our initial guess is

$$y_p = At\cos(2t) + Bt\sin(2t) + E\cos(2t) + F\sin(2t).$$

Looking at the homogenous solution, we see that y_p and y_h do not overlap, so this is the correct guess for y_p.

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c. $y'' - 4y' + 5y = 7e^{2t}\sin t + t$.

Solution: The homogeneous equation,

$$y'' - 4y' + 5y = 0$$

has characteristic equation

$$r^2 - 4r + 5,$$

which has roots $r = 2 \pm i$ (by the quadratic formula). Therefore the homogeneous solution is

$$y_h = e^{2t}\left(C\cos t + D\sin t\right).$$

Our first guess for y_p is

$$y_p = Ae^{2t}\sin t + Be^{2t}\cos t + Et + F.$$

However, the $Ae^{2t}\sin t + Be^{2t}\cos t$ part duplicates the homogeneous solution, so we multiply it by t. This gives a guess of

$$y_p = Ate^{2t}\sin t + Bte^{2t}\cos t + Et + F.$$

$\boxed{3}$ Solve the following differential equations.

a. $y'' + 4y = \tan 2t$.

Solution: First we solve the homogeneous equation,

$$y'' + 4y = 0.$$

The characteristic polynomial,

$$r^2 + 4$$

has roots $r = \pm i$, so the homogeneous solution is

$$y_h = C \cos 2t + D \sin 2t.$$

The righthand side of the nonhomogeneous equation, $\tan 2t$, is not of a form we can apply the method of undetermined coefficients to, so we must use variation of parameters to find a solution of the form

$$y = C(t) \cos 2t + D(t) \sin 2t.$$

Let's start by figuring out the denominator. We have:

$$\begin{aligned} y_1 &= \cos 2t & \qquad y_2 &= \sin 2t \\ y_1' &= -2 \sin 2t & \qquad y_2' &= 2 \cos 2t, \end{aligned}$$

and a (the coefficient of y'' in the differential equation) is 1, so the denominator is

$$a(y_1 y_2' - y_2 y_1') = 2\cos^2 2t + 2 \sin^2 2t = 2.$$

Now it's time to use the formulas for $C(t)$ and $D(t)$. First,

$$\begin{aligned} C(t) &= \int \frac{-f y_2}{a(y_1 y_2' - y_2 y_1')}\, dt, \\ &= \int \frac{-\tan 2t \sin 2t}{2}\, dt, \\ &= -\frac{1}{2} \int \frac{\sin^2 2t}{\cos 2t}\, dt, \\ &= -\frac{1}{2} \int \frac{1 - \cos^2 2t}{\cos 2t}\, dt, \\ &= -\frac{1}{2} \int \sec 2t \, dt + \int \cos 2t \, dt, \\ &= -\frac{1}{4} \ln|\sec 2t + \tan 2t| + \sin t + E. \end{aligned}$$

Next,

$$\begin{aligned} D(t) &= \int \frac{f y_1}{a(y_1 y_2' - y_2 y_1')}\, dt, \\ &= \int \frac{\tan 2t \cos 2t}{2}\, dt, \\ &= \frac{1}{2}\int \sin 2t\, dt, \\ &= -\frac{1}{4}\cos 2t + F. \end{aligned}$$

Therefore the final solution is

$$y = C(t)y_1 + D(t)y_2 = \left(\frac{-\ln|\sec 2t + \tan 2t| + \sin 2t}{4} + E\right)\cos 2t + \left(\frac{-\cos 2t}{4} + F\right)\sin 2t.$$

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b. $y''' - 2y'' + 17y' = e^{3t}$.

Solution: First we solve the homogeneous equation,

$$y''' - 2y'' + 17y' = 0.$$

The characteristic polynomial,

$$r^3 - 2r^2 + 17r = r(r^2 - 2r + 17)$$

has roots $r = 0, 1 \pm 4i$ (by the quadratic formula), so the homogeneous solution is

$$y_h = Ce^{0t} + e^t(D\cos 4t + E\sin 4t) = C + e^t(D\cos 4t + E\sin 4t).$$

The righthand side of the nonhomogeneous equation, e^{3t}, is of an appropriate form for the method of undetermined coefficients, so we do that. Our initial guess for the particular solution is

$$y_p = Ae^{3t}.$$

This guess does not overlap with the homogeneous equation, so we don't need to change it. Now we compute the first three derivatives of y_p:

$$\begin{aligned} y_p &= Ae^{3t}, \\ y_p' &= 3Ae^{3t}, \\ y_p'' &= 9Ae^{3t}, \\ y_p''' &= 27Ae^{3t}, \end{aligned}$$

and plug them in to the nonhomogeneous equation:

$$\left(27Ae^{3t}\right) - 2\left(9Ae^{3t}\right) + 17\left(3Ae^{3t}\right) = e^{3t}.$$

After collecting terms, we see that $60Ae^{3t} = e^{3t}$, so $A = {}^{1}\!/_{60}$, and our final solution is

$$y = y_h + y_p = C + e^t(D\cos 4t + E\sin 4t) + \frac{e^{3t}}{60}.$$

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c. $y'' - 4y' + 4y = te^{2t}$.

Solution: First we solve the homogeneous equation,

$$y'' - 4y' + 4y = 0.$$

The characteristic polynomial,

$$r^2 - 4r + 4 = (r-2)^2$$

has a repeated root at $r = 2$, so the homogeneous solution is

$$y_h = Ce^{2t} + Dte^{2t}.$$

The righthand side of the nonhomogeneous equation, te^{2t}, is of an appropriate form for the method of undetermined coefficients, so we do that. Our initial guess for the particular solution is

$$y_p = Ate^{2t} + Be^{2t}.$$

However, this guess overlaps with the homogeneous solution, so we multiply by t:

$$y_p = At^2e^{2t} + Bte^{2t}.$$

But we still overlap with the homogeneous solution, so we need yet another t[1]:

$$y_p = At^3e^{2t} + Bt^2e^{2t}.$$

[1]You might think that you just need to give the Bte^{2t} term another t, but you'd be wrong. If you only gave that term a t, then you'd get $y_p = At^2e^{2t} + Bt^2e^{2t} = (A+B)t^2e^{2t}$, which won't allow you to solve the equation.

Now we need the derivatives:

$$\begin{aligned}
y_p &= At^3e^{2t} + Bt^2e^{2t}, \\
y_p' &= 2At^3e^{2t} + 3At^2e^{2t} + 2Bt^2e^{2t} + 2Bte^{2t}, \\
&= 2At^3e^{2t} + (3A+2B)t^2e^{2t} + 2Bte^{2t}, \\
y_p'' &= 4At^3e^{2t} + 6At^2e^{2t} + 2(3A+2B)t^2e^{2t} + 2(3A+2B)te^{2t} + 4Bte^{2t} + 2Be^{2t}, \\
&= 4At^3e^{2t} + (12A+4B)t^2e^{2t} + (6A+8B)te^{2t} + 2Be^{2t},
\end{aligned}$$

and then plug them in to the nonhomogeneous equation:

$$\begin{aligned}
y'' - 4y' + 4y \;=\; & \left(4At^3e^{2t} + (12A+4B)t^2e^{2t} + (6A+8B)te^{2t} + 2Be^{2t}\right) \\
& -4\left(2At^3e^{2t} + (3A+2B)t^2e^{2t} + 2Bte^{2t}\right) \\
& +4\left(At^3e^{2t} + Bt^2e^{2t}\right) \qquad = \; te^{2t}.
\end{aligned}$$

Collecting terms, we have

$$(4A-8A+4A)t^3e^{2t} + (12A+4B-12A-8B+4B)t^2e^{2t} + (6A+8B-8B+4B)te^{2t} + (2B)e^{2t} = te^{2t}.$$

The first two of these cancel. Matching coefficients of equal terms (functions) in the remaining terms, we get the system of equations

$$\begin{cases} 6A + 4B = 1, \\ 2B = 0. \end{cases}$$

The second equation shows us that $B = 0$, so the first one says that $A = 1/6$. Our particular solution is therefore

$$y_p = \frac{t^3e^{2t}}{6}.$$

The final solution is therefore

$$y = y_h + y_p = Ce^{2t} + Dte^{2t} + \frac{t^3e^{2t}}{6}.$$

$\boxed{4}$ Solve the following initial value problems.

a. $y'' - 9y = 18t$; $y(0) = 1$; $y'(0) = 11$.

Solution: We'll divide this problem into two parts. First we need to solve the nonhomogeneous equation (which involves solving the homogeneous equation as well), and then we will worry about the initial conditions.

The homogeneous equation,

$$y'' - 9y = 0$$

has characteristic polynomial

$$r^2 - 9 = (r-3)(r+3),$$

which has roots at $r = \pm 3$. Therefore the homogeneous solution is

$$y_h = Ce^{3t} + De^{-3t}.$$

The righthand side of the nonhomogeneous equation, $18t$, is of an appropriate form for the method of undetermined coefficients, so we do that. Our initial guess for the particular solution is

$$y_p = At + B.$$

This guess for y_p does not overlap with the homogeneous solution, so we can proceed.

We need the derivatives of y_p:

$$\begin{aligned} y_p &= At + B, \\ y_p' &= A, \\ y_p'' &= 0. \end{aligned}$$

Now we substitute these into the nonhomogeneous equation and solve for A and B:

$$y'' - 9y = 0 - 9(At + B) = 18t.$$

We see from this that $A = -2$ and $B = 0$, so the particular solution is

$$y_p = -2t.$$

Putting these together, the general solution to this differential equation is

$$y = y_h + y_p = Ce^{3t} + De^{-3t} - 2t.$$

Now we need to use the initial conditions to find C and D. We see that

$$y(0) = C + D = 1$$

and

$$y'(0) = 3C - 3D - 2 = 11,$$

so our system of equations is

$$\begin{cases} C + D &= 1, \\ 3C - 3D - 2 &= 11. \end{cases}$$

From the first we see that $D = 1 - C$. Plugging this into the second we get $3C - 3(1-C) - 2 = 11$, so $6C = 16$, and thus $C = {}^{16}/_{6}$ and $D = -{}^{10}/_{6}$. Finally, the solution to the initial value problem is

$$y = \frac{16e^{3t}}{6} - \frac{10te^{3t}}{6} - 2t.$$

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b. $y'' - y' - 2y = 2e^{-t} + 4$; $y(0) = 10$; $y'(0) = -3$.

Solution: First we solve the homogeneous equation,

$$y'' - y' - 2y = 0.$$

This equation has the characteristic polynomial

$$r^2 - r - 2 = (r-2)(r+1),$$

so the homogeneous solution is

$$y_h = Ce^{2t} + De^{-t}.$$

Now we use the method of undetermined coefficients to solve the nonhomogeneous equation. Our first guess is

$$y_p = Ae^{-t} + B,$$

but this overlaps the homogeneous equation, so the e^{-t} term needs a t. The corrected guess is

$$y_p = Ate^{-t} + B.$$

To find A and B we plug y_p into the nonhomogeneous equation:

$$\begin{aligned} y_p &= Ate^{-t} + B, \\ y_p' &= -Ate^{-t} + Ae^{-t}, \\ y_p'' &= Ate^{-t} - Ae^{-t} - Ae^{-t}, \\ &= Ate^{-t} - 2Ae^{-t}. \end{aligned}$$

Plugging this into the nonhomogeneous equation gives us that

$$y'' - y' - 2y = \left(Ate^{-t} - 2Ae^{-t}\right) - \left(-Ate^{-t} + Ae^{-t}\right) - 2\left(Ate^{-t} + B\right) = 2e^{-t} + 4,$$

so by collecting terms we see that

$$(A + A - 2A)te^{-t} + (-2A - A)e^{-t} - 2B = 2e^{-t} + 4.$$

The te^{-t} term cancels, the e^{-t} terms shows us that $-3A = 2$, so $A = -2/3$, and the constant term shows us that $-2B = 4$, so $B = -2$. The general solution to the nonhomogeneous equation is therefore

$$y = y_h + y_p = Ce^{2t} + De^{-t} - \frac{2te^{-t}}{3} - 2.$$

Finally, we use the initial conditions $y(0) = 10$ and $y'(0) = -3$ to find C and D:

$$y(0) = C + D - 2 = 10,$$

and

$$y'(t) = 2Ce^{2t} - De^{-t} + \frac{2te^{-t}}{3} - \frac{2e^{-t}}{3},$$

so

$$y'(0) = 2C - D - \frac{2}{3} = -3.$$

Our system of equations is therefore

$$\begin{cases} C + D - 2 & = & 10, \\ 2C - D - \frac{2}{3} & = & -3, \end{cases}$$

to which the solution is $C = 29/9$ and $D = 79/9$. The solution to the initial value problems is therefore

$$y = \frac{29e^{2t}}{9} + \frac{79e^{-t}}{9} - \frac{2te^{-t}}{3} - 2.$$

$\boxed{5}$ Suppose that you know that the general solution to the homogeneous differential equation

$$t^2y'' - 3ty' + 4y = 0$$

for $t > 0$ is

$$y_h = Ct^2 + Dt^2 \ln t.$$

Find the general solution to the differential equation

$$t^2y'' - 3ty' + 4y = t^2 \ln t.$$

Solution: We are going to use variation of parameters to find the solution to the non-homogeneous equation. This solution will have the form

$$y = C(t)\underbrace{t^2}_{y_1} + D(t)\underbrace{t^2 \ln t}_{y_2}$$

where

$$C(t) = \int \frac{-fy_2}{a(y_1y_2' - y_2y_1')}\,dt \quad \text{and} \quad D(t) = \int \frac{fy_1}{a(y_1y_2' - y_2y_1')}\,dt.$$

First we compute the denominator of these integrands:

$$a(y_1y_2' - y_2y_1') = t^2 \begin{vmatrix} t^2 & t^2 \ln t \\ 2t & 2t\ln t + t \end{vmatrix} = t^2(2t^3 \ln t + t^3 - 2t^3 \ln t) = t^5.$$

Thus we see that

$$\begin{aligned} C(t) &= \int \frac{-(t^2 \ln t)(t^2 \ln t)}{t^5}\,dt \\ &= \int \frac{-\ln^2 t}{t}\,dt \\ &= -\frac{\ln^3 t}{3} + E, \end{aligned}$$

(the integral is done by setting $u = \ln t$), while we also have

$$\begin{aligned} D(t) &= \int \frac{(t^2 \ln t)(t^2)}{t^5}\,dt \\ &= \int \frac{\ln t}{t}\,dt \\ &= \frac{\ln^2 t}{2} + F, \end{aligned}$$

(again found by setting $u = \ln t$). Therefore the general solution to the non-homogeneous equation is

$$y = \left(-\frac{\ln^3 t}{3} + E\right) t^2 + \left(\frac{\ln^2 t}{2} + F\right) t^2 \ln t.$$