Midterm 3 will cover sections 7.2–7.8.

$\boxed{1}$ Solve for $\mathscr{L}\{y\}$ given the following initial value problems.

a. $y'' - 4y' + 8y = e^{2t}\cos 3t$; $y(0) = 1$; $y'(0) = 3$.

Solution: Let $Y(s) = \mathscr{L}\{y\}$. We have

$$\begin{aligned} \mathscr{L}\{y\} &= Y, \\ \mathscr{L}\{y'\} &= sY - y(0) &&= sY - 1, \\ \mathscr{L}\{y''\} &= s(\mathscr{L}\{y\}) - y'(0) &&= s^2Y - s - 3, \end{aligned}$$

so

$$\mathscr{L}\{y'' - 4y' + 8y\} = (s^2Y - s - 3) - 4(sY - 1) + 8Y = (s^2 - 4s + 8)Y + (-s + 1).$$

For the righthand side, we first see that

$$\mathscr{L}\{\cos 3t\} = \frac{s}{s^2 + 9},$$

so using the rule $\mathscr{L}\{e^{at}f(t)\} = F(s - a)$, where $F(s) = \mathscr{L}\{f\}$, we see that

$$\mathscr{L}\{e^{2t}\cos 3t\} = \frac{s - 2}{(s - 2)^2 + 9}.$$

Putting these two together and solving for Y, we get

$$Y(s) = \frac{\frac{s-2}{(s-2)^2+9} + s - 1}{s^2 - 4s + 8}.$$

---

b. $y'' + 2y' - 3y = e^t + t + 1$; $y(0) = 9$; $y'(0) = -3$.

Solution: Letting $Y(s) = \mathscr{L}\{y\}$ we have

$$\begin{aligned} \mathscr{L}\{y\} &= Y, \\ \mathscr{L}\{y'\} &= sY - y(0) &&= sY - 9, \\ \mathscr{L}\{y''\} &= s(\mathscr{L}\{y\}) - y'(0) &&= s^2Y - 9s + 3, \end{aligned}$$

so

$$\mathscr{L}\left\{y'' + 2y' - 3y\right\} = (s^2Y - 9s + 3) + 2(sY - 9) - 3Y = (s^2 + 2s - 3)Y + (-9s - 15).$$

For the righthand side, we have

$$\mathscr{L}\left\{e^t + t + 1\right\} = \frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s}.$$

Solving for Y, we get

$$Y = \frac{\frac{1}{s-1} + \frac{1}{s^2} + \frac{1}{s} + 9s + 15}{s^2 + 2s - 3}.$$

---

c. $y'' - 4y = \begin{cases} \sin t & 0 < t < \pi, \\ -\sin t & t > \pi. \end{cases}$; $y(0) = y'(0) = 0.$

Solution: Letting $Y(s) = \mathscr{L}\{y\}$ we have

$$\begin{array}{lcllcl} \mathscr{L}\{y\} & = & Y, & & \\ \mathscr{L}\left\{y'\right\} & = & sY - y(0) & = & sY, \\ \mathscr{L}\left\{y''\right\} & = & s(\mathscr{L}\{y\}) - y'(0) & = & s^2Y, \end{array}$$

so the lefthand side is transformed to

$$\mathscr{L}\left\{y'' - 4y\right\} = s^2Y - 4Y = (s^2 - 4)Y.$$

In order to transform the righthand side, we first convert it into Heaviside functions:

$$\begin{cases} \sin t & 0 < t < \pi, \\ -\sin t & t > \pi. \end{cases} = \sin t - 2u(t - \pi)\sin t$$

Now we use the rule that $\mathscr{L}\{u(t-a)f(t)\} = e^{-as}\mathscr{L}\{f(t+a)\}$, so

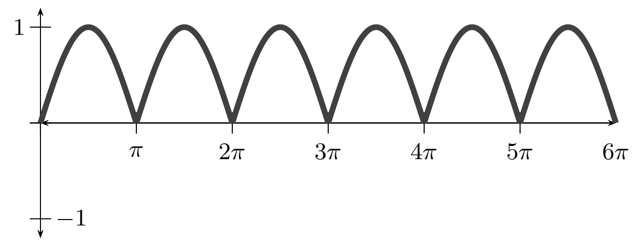
$$\begin{aligned} \mathscr{L}\{\sin t - 2u(t-\pi)\sin t\} &= \frac{1}{s^2+1} - 2e^{-\pi s}\mathscr{L}\{\sin(t+\pi)\}, \\ &= \frac{1}{s^2+1} - 2e^{-\pi s}\mathscr{L}\{-\sin t\}, \\ &= \frac{1}{s^2+1} + \frac{2e^{-\pi s}}{s^2+1}, \\ &= \frac{1 + 2e^{-\pi s}}{s^2+1}. \end{aligned}$$

Solving for Y, we get

$$Y = \frac{1+2e^{-\pi s}}{(s^2+1)(s^2-4)}.$$

---

d. $y'' + y' - 2y = f(t)$, where $f(t)$ is the fully-rectified sine wave below; $y(0) = y'(0) = 1$.



Solution: Letting $Y(s) = \mathscr{L}\{y\}$ we have

$$\begin{aligned}
\mathscr{L}\{y\} &= Y, \\
\mathscr{L}\{y'\} &= sY - y(0) &&= sY - 1, \\
\mathscr{L}\{y''\} &= s(\mathscr{L}\{y\}) - y'(0) &&= s^2Y - s - 1,
\end{aligned}$$

so

$$\mathscr{L}\left\{y'' + y' - 2y\right\} = (s^2Y - s - 1) + (sY - 1) - 2Y = (s^2 + s - 2)Y + (-s - 2).$$

The righthand side is a periodic function, so we use the rule that

$$\mathscr{L}\{f\} = \frac{\mathscr{L}\{f_T\}}{1 - e^{-sT}},$$

where T is the period of f (which in this case is π), and f_T is the function over one period, in this case,

$$f_T = \begin{cases} \sin t & 0 < t < \pi, \\ 0 & t > \pi. \end{cases} = \sin t - u(t-\pi)\sin t.$$

Thus

$$\mathscr{L}\{f_T\} = \mathscr{L}\{\sin t - u(t-\pi)\sin t\} = \frac{1}{s^2+1} - e^{-\pi s}\mathscr{L}\{\sin(t+\pi)\} = \frac{1+e^{-\pi s}}{s^2+1},$$

so

$$\mathscr{L}\{f\} = \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-\pi s})}.$$

Finally, we solve for Y:

$$Y = \frac{\frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-\pi s})} + s + 2}{s^2 + s - 2}.$$

---

e. $y'' - 4y' + ty = 0$; $y(0) = 1$; $y'(0) = 0$. (Find a differential equation satisfied by $\mathscr{L}\{y\}$.)

Solution:

$$\begin{aligned} \mathscr{L}\{y\} &= Y, \\ \mathscr{L}\{y'\} &= sY - y(0) &&= sY - 1, \\ \mathscr{L}\{y''\} &= s(\mathscr{L}\{y\}) - y'(0) &&= s^2Y - s. \end{aligned}$$

Now we need to use the rule $\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}$ to transform ty:

$$\mathscr{L}\{ty\} = -\frac{d}{ds}\mathscr{L}\{y\} = -Y'.$$

Therefore the lefthand side transforms into

$$(s^2Y - s) - 4(sY - 1) - Y' = (s^2Y - 4s)Y - Y' + (-s + 4).$$

Since the righthand side transforms to 0, we have

$$(s^2Y - 4s)Y - Y' = s - 4.$$

Not that the problem asked for it, but this differential equation is very difficult (impossible?) to solve explicitly.

---

f. $y'' + 4y = \delta(t - 2)$; $y(0) = y'(0) = 0$. (Here δ is the Dirac delta function.)

Solution: The lefthand side transforms into

$$\mathscr{L}\{y'' + 4y\} = s^2Y + 4Y = (s^2 + 4)Y.$$

The righthand side transforms into e^{-2s}, so we get

$$Y = \frac{e^{-2s}}{s^2 + 4}.$$

(This differential equation models a spring of unit mass, no damping, and spring constant $k = 4$ being hit by a hammer of unit force at time $t = 2$.)

g. $y'' + 5y' - y = e^{\sin t}\delta(t-3)$; $y(0) = 0$, $y'(0) = 3$.

Solution: Let lefthand side transforms into

$$(s^2 + 5s - 1)Y - 3,$$

where $Y(s)$ is the Laplace transform of $y(t)$. To transform the righthand side, note that $e^{\sin t}\delta(t-3)$ is nonzero only when $t = 3$, and at $t = 3$ it is $e^{\sin 3}$, so

$$\mathscr{L}\left\{e^{\sin t}\delta(t-3)\right\} = \mathscr{L}\left\{e^{\sin 3}\delta(t-e)\right\} = e^{\sin 3}e^{-3s}.$$

Therefore we have that

$$Y = \frac{e^{\sin 3}e^{-3s} + 3}{s^2 + 5s - 1}.$$

$\boxed{2}$ Compute the following inverse Laplace transforms.

a. $\mathscr{L}^{-1}\left\{\dfrac{2s^2 - 1}{s^3 + s^2 - 6s}\right\}$.

Solution: We begin by using partial fractions on the righthand side:

$$\frac{2s^2 - 1}{s^3 + s^2 - 6s} = \frac{2s^2 - 1}{s(s^2 + s - 6)} = \frac{2s^2 - 1}{s(s-2)(s+3)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}.$$

Canceling denominators, we get

$$2s^2 - 1 = A(s-2)(s+3) + Bs(s+3) + Cs(s-2).$$

Plugging in $s = 0$, 2, and -3, we see:

$$\begin{aligned} -1 &= -6A, \\ 7 &= 10B, \\ 17 &= 15C, \end{aligned}$$

so
$$A=\frac{1}{6},\quad B=\frac{7}{10},\quad C=\frac{17}{15},$$
and thus
$$\frac{2s^2-1}{s^3+s^2-6s}=\frac{1}{6s}+\frac{7}{10(s-2)}+\frac{17}{15(s+3)}.$$
We now just need to invert this:
$$y(t)=\mathscr{L}^{-1}\left\{\frac{1}{6s}+\frac{7}{10(s-2)}+\frac{17}{15(s+3)}\right\}=\frac{1}{6}+\frac{7}{10}e^{2t}+\frac{17}{15}e^{-3s}.$$

---

b. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-8s+17}\right\}$.

Solution: Because the fraction doesn't factor (over the real numbers), we need to complete the square:
$$\frac{1}{s^2-8s+17}=\frac{1}{(s-4)+1}.$$
Therefore y is a shifted e^s times sin:
$$y(t)=e^{4s}\sin t.$$

---

c. $\mathscr{L}^{-1}\left\{\dfrac{9-s^2}{(s^2+9)^2}\right\}$.

Solution: This one takes a bit of thought. What rule would give us that denominator? Well, we know that
$$\mathscr{L}\left\{\sin 3t\right\}=\frac{3}{s^2+9},$$
so using the $\mathscr{L}\left\{tf(t)\right\}$ rule, we see that
$$\mathscr{L}\left\{t\sin 3t\right\}=-\frac{d}{ds}\frac{3}{s^2+9}=-\frac{6s}{(s^2+9)^2}.$$

This isn't quite what we were looking for, so let's try $t\cos 3t$ instead:

$$\mathscr{L}\{t\cos 3t\} = -\frac{d}{ds}\frac{s}{s^2+9} = -\frac{9-s^2}{(s^2+9)^2} = \frac{s^2-9}{(s^2+9)^2}.$$

The transform we were given is the negative us this, so

$$\mathscr{L}^{-1}\left\{\frac{9-s^2}{(s^2+9)^2}\right\} = -t\cos 3t.$$

---

d. $\mathscr{L}^{-1}\left\{\dfrac{3s}{s^2+4s+6}\right\}$.

Solution: The denominator doesn't factor, so we need to complete the square:

$$\frac{3s}{s^2+4s+6} = \frac{3s}{(s+2)^2+2}.$$

This shows that y will be a linear combination of $e^{-2t}\cos\sqrt{2}t$ and $e^{-2t}\sin\sqrt{2}t$. We just need to get it in the right form. Be careful here; for the term that corresponds to $e^{-2t}\cos\sqrt{2}t$ we need to get an $s+2$ in the numerator:

$$\frac{3s}{(s+2)^2+2} = 3\frac{s+2}{(s+2)^2+2} - 3\sqrt{2}\frac{\sqrt{2}}{(s+2)^2+2}$$

We then see that

$$y = 3e^{-2t}\cos\sqrt{2}t - 3\sqrt{2}e^{-2t}\sin\sqrt{2}t.$$

---

e. $\mathscr{L}^{-1}\left\{\dfrac{(1-e^{-s})^2}{s^3}\right\}$.

Solution: We are going to end up with Heaviside functions here because of the e^{-s} term in the numerator. To start, we want to expand that numerator:

$$\frac{(1-e^{-s})^2}{s^3} = \frac{1-2e^{-s}+e^{-2s}}{s^3} = \frac{1}{s^3} - 2\frac{e^{-s}}{s^3} + \frac{e^{-2s}}{s^3}.$$

The first fraction just gives us $t^2/2$. We have to work a bit harder for the second fraction:

$$e^{-s}\left(-\frac{2}{s^3}\right) = \mathscr{L}\left\{u(t-1)f(t)\right\} = e^{-s}\mathscr{L}\left\{f(t+1)\right\},$$

so $f(t+1) = -t^2$, and thus $f(t) = -(t-1)^2$, so

$$\mathscr{L}^{-1}\left\{e^{-s}\left(-\frac{2}{s^3}\right)\right\} = -(t-1)^2u(t-1).$$

The third fraction is similar:

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^3}\right\} = \frac{1}{2}(t-2)^2u(t-2).$$

Therefore,

$$y = \frac{t^2}{2} - (t-1)^2u(t-1) + \frac{1}{2}(t-2)^2u(t-2).$$

---

f. $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+2s+5}\right\}$.

Solution: Again with the Heaviside functions! We will need to complete the square at some point, so let's do it first:

$$\frac{e^{-\pi s}}{s^2+2s+5} = \frac{e^{-\pi s}}{(s+1)^2+2^2}.$$

Now we know we'll have a $u(t-\pi)$ because of the $e^{-\pi s}$, so

$$\frac{e^{-\pi s}}{(s+1)^2+2^2} = \mathscr{L}\left\{u(t-\pi)f(t)\right\} = e^{-\pi s}\mathscr{L}\left\{f(t+\pi)\right\}.$$

We see therefore that

$$f(t+\pi) = \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\},$$

and thus $f(t+\pi) = \frac{1}{2}e^{-t}\sin 2t$, so

$$f(t) = \frac{1}{2}e^{-t-\pi}\sin(2(t-\pi)).$$

We could at this point notice that $\sin(2t - 2\pi) = \sin 2t$ (sin is periodic of period 2π), which would show us that

$$y = \frac{1}{2}u(t-\pi)e^{-t-\pi}\sin 2t$$

---

g. $\mathscr{L}^{-1}\{7\}$.

Solution: The Laplace transform of the Dirac delta function $\delta(t-a)$ is e^{-as}, so $\mathscr{L}^{-1}\{7\} = 7\delta(t)$.

---

h. $\mathscr{L}^{-1}\left\{\dfrac{s^2+2s}{s^2+4}\right\}$.

Solution: First we want to reduce the degree of the numerator:

$$\frac{s^2+2s}{s^2+4} = \frac{(s^2+4)+(2s-4)}{s^2+4} = 1 + \frac{2s-4}{s^2+4}.$$

Now, since the denominator is already in the form of a complete square, we see that

$$y = \mathscr{L}^{-1}\{1\} + 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 2\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \delta(t) + 2\cos 2t - 2\sin 2t.$$

---

i. $\mathscr{L}^{-1}\left\{\dfrac{s\mathscr{L}\{g\}}{s^2+4}\right\}$.

Solution: We can write the Laplace transformation as a product:

$$\frac{s\mathscr{L}\{g\}}{s^2+4} = \left(\frac{s}{s^2+4}\right)\mathscr{L}\{g\}.$$

We see that $s/(s^2+4)$ is the Laplace transform of $\cos 2t$, so by the Convolution Theorem,

$$\mathscr{L}^{-1}\left\{\frac{s\mathscr{L}\{g\}}{s^2+4}\right\} = (\cos 2t) * g = \int_0^t (\cos 2(t-v))g(v)\,dv.$$

---

i. $\mathscr{L}^{-1}\left\{\dfrac{\mathscr{L}\{g\}+s}{s}\right\}$.

Solution: We first simplify the Laplace transform:

$$\frac{\mathscr{L}\{g\}+s}{s} = \frac{\mathscr{L}\{g\}}{s} + 1 = \left(\frac{1}{s}\right)\mathscr{L}\{s\} + 1.$$

The 1 on the right is the Laplace transform of $\delta(t)$, while the other term follows from the Convolution Theorem:

$$\mathscr{L}^{-1}\left\{\left(\frac{1}{s}\right)\mathscr{L}\{g\}+1\right\} = 1 * g + \delta(t) = \left(\int_0^t g(v)\,dv\right) + \delta(t).$$

Thus we could also write this answer as

$$G(t) + \delta(t),$$

where $G(t)$ is the antiderivative of $g(t)$.

$\boxed{3}$ Solve the integro-differentential equation

$$y(t) + \int_0^t e^{t-v} y(v)\,dv = \sin t.$$

Solution: Let $Y(s) = \mathscr{L}\{y\}$. The integral on the lefthand side is the convolution $e^t * y$, so its Laplace transform is given by

$$\mathscr{L}\left\{\int_0^t e^{t-v} y(v)\,dv\right\} = \mathscr{L}\left\{e^t\right\}\mathscr{L}\{y\} = \frac{Y}{s-1}.$$

The Laplace transform of $\sin t$ is then $1/(s^2+1)$. Putting these together, the Laplace transform of the entire equation is given by

$$Y + \frac{Y}{s-1} = \frac{1}{s^2+1}.$$

We then see that

$$\frac{sY}{s-1} = \frac{1}{s^2+1},$$

so

$$Y = \frac{s-1}{s(s^2+1)} = \frac{2}{s+1} - \frac{1}{s},$$

where the last equality is obtained via partial fractions. It follows that

$$y(t) = 2e^{-t} - 1.$$