Find all singular points of the following differential equations.

a. \((x^2 - 1)y'' + xy' + 2y = 0\).

**Solution:** When put in standard form, this equation is

\[ y'' + \frac{x}{x^2 - 1} y' + \frac{2}{x^2 - 1} y = 0. \]

The functions \(x/(x^2 - 1)\) and \(2/(x^2 - 1)\) are analytic everywhere except \(x = \pm 1\), so those are the singular points.

b. \(x^3(x^2 + 1)y'' + xy' - y = 0\).

**Solution:** In standard form, this equation is

\[ y'' + \frac{1}{x^2(x^2 + 1)} y' - \frac{1}{x^3(x^2 + 1)} y = 0, \]

so the only singular point is \(x = 0\).

c. \((x^2 - 2)y'' + \sqrt{2}y' - (\sin x)y = 0\).

**Solution:** In standard form, this equation is

\[ y'' + \frac{\sqrt{2}}{x^2 - 2} y' \frac{\sin x}{x^2 - 2} y = 0, \]

so the singular points are \(x = \pm \sqrt{2}\).
d. \((\sin x)y'' + \pi y' - (\sin x)y = 0\).

**Solution:** In standard form, this equation is
\[ y'' + \frac{\pi}{\sin x} - y = 0, \]
so the singular points are \(x = \pi n\) for all integers \(n\).

e. \(xy'' + (\sin x)y = 0\).

**Solution:** In standard form, the equation is
\[ y'' + \frac{\sin x}{x}y = 0. \]
While it might look like 0 is a singular point, it is actually a removable discontinuity, and \((\sin x)/x\) is analytic near 0. Its power series is given by
\[
\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{2n}}{x}.
\]

Find the first four terms in a power series expansion at \(x = 0\) for a general solution to the given differential equation.

a. \(y' + (x + 2)y = 1\).

**Solution:** Let \(y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\).
Then \(y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots\),
so
\[
y' + (x + 2)y = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (x + 2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots),
\]
\[= (2a_0 + a_1) + (a_0 + 2a_1 + 2a_2)x + (a_1 + 2a_2 + 3a_3)x^2 + \cdots.\]
For this to equal 1 we see that
\[ a_1 = 1 - 2a_0, \]
\[ a_2 = \frac{-2a_1 + a_0}{2} = -1 + \frac{3}{2}a_0, \]
\[ a_3 = \frac{-2a_2 + a_1}{3} = \frac{1}{3} - \frac{a_0}{3}. \]

Therefore the first four terms of \( y \) are
\[ a_0 + (1 - 2a_0)x + \left(-1 + \frac{3}{2}a_0\right)x^2 + \left(\frac{1}{3} - \frac{a_0}{3}\right)x^3. \]

b. \( y' - (\sin x)y = 0. \)

Solution: Let
\[ y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots. \]

Then
\[ y' - (\sin x)y = a_1 + 2a_2 x + 3a_3 x^2 + \cdots - \left(x - \frac{x^3}{6} + \cdots\right) \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right), \]
\[ = a_1 + (2a_2 - a_0)x + (3a_3 - a_1)x^2 + \cdots \]

so we see that
\[ a_1 = 0, \]
\[ a_2 = \frac{a_0}{2}, \]
\[ a_3 = \frac{a_1}{3} = 0. \]

The first four terms of \( y \) are then
\[ a_0 + \frac{a_0}{2}x^2. \]

c. \( e^{2x}y' - y = e^x. \)
Solution: We know that
\[ e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots, \]
\[ e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3} x^3 + \cdots, \]
so letting \( y = a_0 + a_1 x + \cdots \), the lefthand side becomes
\[ e^{2x} y' - y = (1 + 2x + 2x^2 + \frac{4}{3} x^3 + \cdots)(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \]
\[ = (-a_0 + a_1) + (a_1 + 2a_2) x + (2a_1 + 3a_2 + 3a_3) x^2 + \cdots. \]

We want this to equal the first few terms of the expansion of \( e^x \), so we get
\[ a_1 = 1 + a_0, \]
\[ a_2 = \frac{1 - a_1}{2} = -\frac{a_0}{2}, \]
\[ a_3 = \frac{1}{3} - 2a_1 - 3a_2 = \frac{1}{2} - \frac{a_0}{6}. \]

So the first four terms of the expansion of \( y \) are
\[ a_0 + (1 + a_0)x - \frac{a_0}{2} x^2 + \left( \frac{1}{2} - \frac{a_0}{6} \right) x^3. \]

d. \( y'' - xy' + x^4 y = \sin x \).

Solution: Making the usual substitution transforms the lefthand side into
\[ y'' - xy' + x^4 y = 2a_2 + (6a_3 - a_1) x + \cdots. \]
For this to equal \( \sin x = x - x^3/6 + \cdots \), we want
\[ 2a_2 = 0, \]
\[ 6a_3 - a_1 = 1. \]

This shows that \( a_2 = 0 \) and \( a_3 = (1 + a_1)/6 \), giving the first four terms as
\[ a_0 + a_1 x + \frac{1 + a_1}{6} x^3. \]
Find the general solution for the following differential equations. (Your answer should contain a recurrence relation for the power series coefficients.)

a. \( y'' + 4y = 0. \)

**Solution:** We have

\[
y = \sum_{n=0}^{\infty} a_n x^n,
\]

\[
y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.
\]

By reindexing, we can rewrite \( y'' \) as

\[
y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n.
\]

We can now combine the two series:

\[
y'' + 4y = \sum_{n=0}^{\infty} ((n+2)(n+1)a_{n+2} + 4a_n) x^n = 0.
\]

This shows that for \( n \geq 0, \)

\[
a_{n+2} = \frac{-4a_n}{(n+2)(n+1)}.
\]

b. \( y' + (x - 2)y = 0. \)

**Solution:** Here we begin with

\[
y = \sum_{n=0}^{\infty} a_n x^n,
\]

\[
y' = \sum_{n=1}^{\infty} na_n x^{n-1}.
\]
Now we need to expand \((x - 2)y\):

\[
(x - 2)y = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} -2a_n x^n.
\]

We want to express each series in terms of \(x^n\). To do this for \(y'\) we just reindex, but the \(xy\) terms presents a different challenge. Expanding this out, we see

\[
\sum_{n=0}^{\infty} a_n x^{n+1} = a_0 x + a_1 x^2 + a_2 x^3 + \cdots = \sum_{n=1}^{\infty} a_{n-1} x^n.
\]

Now we need to make all of the summations start at the same value \((n = 1)\). We do this by expressing their \(n = 0\) terms separately (after we reindex the \(y'\) series):

\[
\left( a_1 + \sum_{n=1}^{\infty} (n+1)a_{n+1} x^n \right) + \left( \sum_{n=1}^{\infty} a_{n-1} x^n \right) + \left( -2a_0 + \sum_{n=1}^{\infty} -2a_n x^n \right) = 0.
\]

Immediately we see from the constants that \(a_1 = 2a_0\). Combining the other terms, we have

\[
\sum_{n=1}^{\infty} ((n+1)a_{n+1} + a_{n-1} - 2a_n) x^n = 0.
\]

This shows that for \(n \geq 1\),

\[
a_{n+1} = \frac{-a_{n-1} + 2a_n}{n + 1}.
\]

c. \(y'' + x^2y = 0\).

**Solution:** We substitute

\[
y = \sum_{n=0}^{\infty} a_n x^n,
\]

\[
y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.
\]

This means that

\[
x^2y = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n,
\]
while
\[ y'' = \sum_{n=0}^{\infty} (n + 2)(n + 1)a_{n+2}x^n. \]

In order to combine these two series, we must separate off the first two terms of the series for \( y'' \):
\[
\left( 2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n + 2)(n + 1)a_{n+2}x^n \right) + \left( \sum_{n=2}^{\infty} a_{n-2}x^n \right) = 0.
\]

This shows immediately that \( a_2 = a_3 = 0 \), and for \( n \geq 2 \),
\[
a_{n+2} = \frac{a_{n-2}}{(n + 2)(n + 1)}.
\]

4 The function \( f(t) \) is said to be \textit{eventually bounded} if there is a constant \( M \) such that \(|f(t)| < M\) for all sufficiently large \( t \). Use the mass-spring analogy to determine whether all solutions to each of the following differential equations are eventually bounded.

a. \( y'' + t^2y = 0 \).

\textbf{Solution:} Yes. We can think of this as modeling a spring with stiffness \( t^2 \). As \( t \to \infty \), this stiffness increases, so the solutions are eventually bounded.

b. \( y'' - t^2y = 0 \).

\textbf{Solution:} No. As \( t \to \infty \), the stiffness factor \(-t^2\) decreases, thus there are solutions that are not eventually bounded.

c. \( y'' + y^6 = 0 \).
**Solution:** No. The stiffness constant here is \( y^5 \). If \( y \) starts out negative, this will be a negative number, forcing the “mass” further to the negative side of equilibrium. Therefore there are solutions which are not eventually bounded.

d. \( y'' + (4 + 2 \cos t)y = 0 \) (Mathieu’s equation).

**Solution:** Yes. The stiffness factor is between 2 and 6 for all \( t \).

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5. The following differential equations represent the movement of a mass-spring system. For each, determine if it is underdamped, overdamped, or critically damped.

a. \( y'' + 5y' + 6y = 0 \).

**Solution:** The characteristic polynomial

\[
r^2 + 5r + 6 = (r + 2)(r + 3)
\]

has roots at \( r = -2, -3 \), so the differential equation is overdamped.

b. \( y'' + 12y' + 36y = 0 \).

**Solution:** The characteristic polynomial

\[
r^2 + 12r + 36 = (r + 6)^2
\]

has a repeated root at \( r = -6 \), so the differential equation is critically damped.

c. \( y'' + \frac{9}{2}y' + 2y = 0 \). 
Solution: The characteristic polynomial
\[ r^2 + \frac{9}{2}r + 2 = \frac{(r + 4)(2r + 1)}{2} \]
has roots at \( r = -4, -\frac{1}{2} \), so the differential equation is overdamped.

d. \( y'' + 4y' + 13y = 0 \).

Solution: The characteristic polynomial
\[ r^2 + 4r + 13 \]
has complex roots \( r = -2 \pm 3i \), so the system is underdamped.

6 A \( \frac{1}{8} \) kg mass is attached to a spring with stiffness 16 N/m. The damping constant (friction coefficient) for the system is 2 N-sec/m. If the mass is moved \( \frac{3}{4} \) m to the left of equilibrium and given an initial leftward (negative) velocity of 2 m/sec, determine the equation of motion of the mass and give its damping factor, quasiperiod, and quasifrequency.

Solution: The corresponding differential equation is
\[ \frac{1}{8}y'' + 2y' + 16y = 0, \]
with initial conditions \( y(0) = -\frac{3}{4}, y'(0) = -2 \). The general solution to this equation is
\[ y = Ce^{-8t} \cos 8t + De^{-8t} \sin 8t. \]
solving for the constants \( C \) and \( D \) gives the equation
\[ y = -\frac{3}{4}e^{-8t} \cos 8t - e^{-8t} \sin 8t. \]

The damping factor is \( e^{-8t} \), the amplitude is \( \sqrt{13}/4 \), the quasiperiod is \( \frac{2\pi}{\beta} = \frac{\pi}{4} \), and the quasifrequency is \( 4/\pi \).
At what time does the mass in the previous problem first return to equilibrium?

**Solution:** To answer the question we need to convert the answer in the previous problem to the form

\[ y = Ae^{\alpha t} \sin(\beta t + \phi) = Ae^{\alpha t} \cos \beta t \sin \phi + Ae^{\alpha t} \sin \beta t \cos \phi. \]

The above equation shows that we have

\[ A = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{13}}{4}, \]

and

\[ \tan \phi = \frac{3}{2}. \]

Because we need both \( \sin \phi \) and \( \cos \phi \) to be negative, \( \phi \) is not \( \arctan \frac{3}{2} \), but actually

\[ \phi = \pi + \arctan \frac{3}{2}. \]

Thus \( y(t) = 0 \) when

\[ 8t + \pi + \arctan \frac{3}{2} = n\pi \]

for an integer \( n \). The first time this happens is at

\[ t = \frac{\pi - \arctan \frac{3}{2}}{8}. \]

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A 10 kg weight is attached to a vertical spring with damping constant 2 kg-s/m. At rest, the spring is stretched 2 m. What is the spring constant of this spring? (Acceleration due to gravity near the surface of the Earth is approximately 9.8 m/s\(^2\).)

**Solution:** A vertical spring system will hang at equilibrium \( mg/k \). Therefore in this system, \( 98/k = 2 \), so \( k = 49 \).
9. Determine the equation of motion for an undamped system at resonance governed by
\[ y'' + 16y = 2 \cos 4t, \quad y(0) = 1, \quad y'(0) = 0. \]

**Solution:** This question is just asking us to solve an IVP. The homogeneous solution is
\[ y_h = C \cos 4t + D \sin 4t. \]
For the particular solution, we guess \( y_p = At \cos 4t + Bt \sin 4t \). We then compute:
\[
\begin{align*}
y_p &= At \cos 4t + Bt \sin 4t, \\
y_p' &= (A + 4Bt) \cos 4t + (B - 4At) \sin 4t, \\
y_p'' &= (8B - 16At) \cos 4t + (-8A - 16Bt) \sin 4t.
\end{align*}
\]
Substituting into the differential equation, we get
\[ 8B \cos 4t - 8A \sin 4t = 2 \cos 4t, \]
so we want \( A = 0 \) and \( B = 1/4 \).
Our solution (so far) is
\[ y = C \cos 4t + D \sin 4t + \frac{t}{4} \sin 4t, \]
and we just have to match the initial conditions. This requires \( C = 1 \) and \( D = 0 \), so the final solution is
\[ y = \cos 4t + \frac{t}{4} \sin 4t. \]

10. A 1 kg mass is attached to a horizontal spring with damping constant 2 kg-s/m and spring constant 1 N/m. Does this system have a resonance frequency?

**Solution:** The corresponding differential equation is
\[ y'' + 2y' + y = 0. \]
The solution is therefore \( y = Ce^{-t} + Dte^{-t} \). This is a critically damped system, and we know that only underdamped systems can have resonance frequencies, so the answer is no.