1 Find all singular points of the following differential equations.

a. $(x^2 - 1)y'' + xy' + 2y = 0.$

Solution: When put in standard form, this equation is

$$y'' + \frac{x}{x^2 - 1}y' + \frac{2}{x^2 - 1}y = 0.$$

The functions $x/(x^2 - 1)$ and $2/(x^2 - 1)$ are analytic everywhere except $x = \pm 1$, so those are the singular points.

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b. $x^3(x^2 + 1)y'' + xy' - y = 0.$

Solution: In standard form, this equation is

$$y'' + \frac{1}{x^2(x^2 + 1)}y' - \frac{1}{x^3(x^2 + 1)}y = 0,$$

so the only singular point is $x = 0$.

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c. $(x^2 - 2)y'' + \sqrt{2}y' - (\sin x)y = 0.$

Solution: In standard form, this equation is

$$y'' + \frac{\sqrt{2}}{x^2 - 2}y' \frac{\sin x}{x^2 - 2}y = 0,$$

so the singular points are $x = \pm\sqrt{2}$.

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d. $(\sin x)y'' + \pi y' - (\sin x)y = 0.$

Solution: In standard form, this equation is

$$y'' + \frac{\pi}{\sin x} - y = 0,$$

so the singular points are $x = \pi n$ for all integers n.

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e. $xy'' + (\sin x)y = 0.$

Solution: In standard form, the equation is

$$y'' + \frac{\sin x}{x} y = 0.$$

While it might look like 0 is a singular point, it is actually a removable discontinuity, and $(\sin x)/x$ is analytic near 0. Its power series is given by

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}.$$

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2 Find the first four terms in a power series expansion at $x = 0$ for a general solution to the given differential equation.

a. $y' + (x+2)y = 1.$

Solution: Let

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

Then

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots ,$$

so

$$\begin{aligned} y' + (x+2)y &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (x+2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots), \\ &= (2a_0 + a_1) + (a_0 + 2a_1 + 2a_2)x + (a_1 + 2a_2 + 3a_3)x^2 + \cdots . \end{aligned}$$

For this to equal 1 we see that

$$\begin{aligned} a_1 &= 1 - 2a_0, \\ a_2 &= -\frac{2a_1 + a_0}{2} = -1 + \frac{3}{2}a_0, \\ a_3 &= -\frac{2a_2 + a_1}{3} = \frac{1}{3} - \frac{a_0}{3}. \end{aligned}$$

Therefore the first four terms of y are

$$a_0 + (1 - 2a_0)x + \left(-1 + \frac{3}{2}a_0\right)x^2 + \left(\frac{1}{3} - \frac{a_0}{3}\right)x^3.$$

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b. $y' - (\sin x)y = 0$.

Solution: Let

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.$$

Then

$$\begin{aligned} y' - (\sin x)y &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots - \left(x - \frac{x^3}{6} + \cdots\right)\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right), \\ &= a_1 + (2a_2 - a_0)x + (3a_3 - a_1)x^2 + \cdots \end{aligned}$$

so we see that

$$\begin{aligned} a_1 &= 0, \\ a_2 &= \frac{a_0}{2}, \\ a_3 &= \frac{a_1}{3} = 0. \end{aligned}$$

The first four terms of y are then

$$a_0 + \frac{a_0}{2}x^2.$$

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c. $e^{2x}y' - y = e^x$.

Solution: We know that

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots , \\ e^{2x} &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots , \end{aligned}$$

so letting $y = a_0 + a_1 x + \cdots$, the lefthand of this equation side becomes

$$\begin{aligned} e^{2x}y' - y &= \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots\right)(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\ &= (-a_0 + a_1) + (a_1 + 2a_2)x + (2a_1 + 3a_2 + 3a_3)x^2 + \cdots . \end{aligned}$$

We want this to equal the first few terms of the expansion of e^x, so we get

$$\begin{aligned} a_1 &= 1 + a_0, \\ a_2 &= \frac{1 - a_1}{2} &= -\frac{a_0}{2}, \\ a_3 &= \frac{\frac{1}{2} - 2a_1 - 3a_2}{3} &= -\frac{1}{2} - \frac{a_0}{6}. \end{aligned}$$

So the first four terms of the expansion of y are

$$a_0 + (1 + a_0)x - \frac{a_0}{2}x^2 + \left(-\frac{1}{2} - \frac{a_0}{6}\right)x^3.$$

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d. $y'' - xy' + x^4 y = \sin x$.

Solution: Making the usual substitution transforms the lefthand side into

$$y'' - xy' + x^4 y = 2a_2 + (6a_3 - a_1)x + \cdots .$$

For this to equal $\sin x = x - x^3/6 + \cdots$, we want

$$\begin{aligned} 2a_2 &= 0, \\ 6a_3 - a_1 &= 1. \end{aligned}$$

This shows that $a_2 = 0$ and $a_3 = (1 + a_1)/6$, giving the first four terms as

$$a_0 + a_1 x + \frac{1 + a_1}{6}x^3.$$

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3 Find the general solution for the following differential equations. (Your answer should contain a recurrence relation for the power series coefficients.)

a. $y'' + 4y = 0$.

Solution: We have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n, \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}. \end{aligned}$$

By reindexing, we can rewrite y'' as

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

We can now combine the two series:

$$y'' + 4y = \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} + 4a_n\right) x^n = 0.$$

This shows that for $n \geq 0$,

$$a_{n+2} = \frac{-4a_n}{(n+2)(n+1)}.$$

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b. $y' + (x-2)y = 0$.

Solution: Here we begin with

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n, \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1}. \end{aligned}$$

Now we need to expand $(x-2)y$:

$$(x-2)y = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} -2a_n x^n.$$

We want to express each series in terms of x^n. To do this for y' we just reindex, but the xy terms presents a different challenge. Expanding this out, we see

$$\sum_{n=0}^{\infty} a_n x^{n+1} = a_0 x + a_1 x^2 + a_2 x^3 + \cdots = \sum_{n=1}^{\infty} a_{n-1} x^n.$$

Now we need to make all of the summations start at the same value ($n = 1$). We do this by expressing their $n = 0$ terms separately (after we reindex the y' series):

$$\left(a_1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}x^n\right) + \left(\sum_{n=1}^{\infty} a_{n-1}x^n\right) + \left(-2a_0 + \sum_{n=1}^{\infty} -2a_n x^n\right) = 0.$$

Immediately we see from the constants that $a_1 = 2a_0$. Combining the other terms, we have

$$\sum_{n=1}^{\infty} \left((n+1)a_{n+1} + a_{n-1} - 2a_n\right) x^n = 0.$$

This shows that for $n \geq 1$,

$$a_{n+1} = \frac{-a_{n-1} + 2a_n}{n+1}.$$

---

c. $y'' + x^2 y = 0$.

Solution: We substitute

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n, \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}. \end{aligned}$$

This means that

$$x^2 y = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n,$$

while

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

In order to combine these two series, we must separate off the first two terms of the series for y'':

$$\left(2a_2 + 6a_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)a_{n+2}x^n\right) + \left(\sum_{n=2}^{\infty} a_{n-2}x^n\right) = 0.$$

This shows immediately that $a_2 = a_3 = 0$, and for $n \geq 2$,

$$a_{n+2} = \frac{a_{n-2}}{(n+2)(n+1)}.$$

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4 The function $f(t)$ is said to be *eventually bounded* if there is a constant M such that $|f(t)| < M$ for all sufficiently large t. Use the mass-spring analogy to determine whether all solutions to each of the following differential equations are eventually bounded.

a. $y'' + t^2 y = 0$.

Solution: Yes. We can think of this as modeling a spring with stiffness t^2. As $t \to \infty$, this stiffness increases, so the solutions are eventually bounded.

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b. $y'' - t^2 y = 0$.

Solution: No. As $t \to \infty$, the stiffness factor $-t^2$ decreases, thus there are solutions that are not eventually bounded.

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c. $y'' + y^6 = 0$.

Solution: No. The stiffness constant here is y^5. If y starts out negative, this will be a negative number, forcing the "mass" further to the negative side of equilibrium. Therefore there are solutions which are not eventually bounded.

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d. $y'' + (4 + 2\cos t)y = 0$ (Mathieu's equation).

Solution: Yes. The stiffness factor is between 2 and 6 for all t.

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5 The following differential equations represent the movement of a mass-spring system. For each, determine if it is underdamped, overdamped, or critically damped.

a. $y'' + 5y' + 6y = 0$.

Solution: The characteristic polynomial

$$r^2 + 5r + 6 = (r + 2)(r + 3)$$

has roots at $r = -2, -3$, so the differential equation is overdamped.

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b. $y'' + 12y' + 36y = 0$.

Solution: The characteristic polynomial

$$r^2 + 12r + 36 = (r + 6)^2$$

has a repeated root at $r = -6$, so the differential equation is critically damped.

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c. $y'' + \frac{9}{2}y' + 2y = 0$.

Solution: The characteristic polynomial

$$r^2 + \frac{9}{2}r + 2 = \frac{(r+4)(2r+1)}{2}$$

has roots at $r = -4, -1/2$, so the differential equation is overdamped.

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d. $y'' + 4y' + 13y = 0$.

Solution: The characteristic polynomial

$$r^2 + 4r + 13$$

has complex roots $r = -2 \pm 3i$, so the system is underdamped.

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6 A $1/8$ kg mass is attached to a spring with stiffness 16 N/m. The damping constant (friction coefficient) for the system is 2 N-sec/m. If the mass is moved $3/4$ m to the left of equilibrium and given an initial leftward (negative) velocity of 2 m/sec, determine the equation of motion of the mass and give its damping factor, quasiperiod, and quasifrequency.

Solution: The corresponding differential equation is

$$\frac{1}{8}y'' + 2y' + 16y = 0,$$

with initial conditions $y(0) = -3/4$, $y'(0) = -2$. The general solution to this equation is

$$y = Ce^{-8t}\cos 8t + De^{-8t}\sin 8t.$$

solving for the constants C and D gives the equation

$$y = -\frac{3}{4}e^{-8t}\cos 8t - e^{-8t}\sin 8t.$$

The damping factor is e^{-8t}, the amplitude is $\sqrt{13}/4$, the quasiperiod is $2\pi/\beta = \pi/4$, and the quasifrequency is $4/\pi$.

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7 At what time does the mass in the previous problem first return to equilibrium?

Solution: To answer the question we need to convert the answer in the previous problem to the form

$$y = Ae^{\alpha t}\sin(\beta t + \phi) = Ae^{\alpha t}\cos\beta t\sin\phi + Ae^{\alpha t}\sin\beta t\cos\phi.$$

The above equation shows that we have

$$A = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{13}}{4},$$

and

$$\tan\phi = \frac{3}{2}.$$

Because we need both $\sin\phi$ and $\cos\phi$ to be negative, ϕ is not $\arctan 3/2$, but actually

$$\phi = \pi + \arctan 3/2.$$

Thus $y(t) = 0$ when

$$8t + \pi + \arctan 3/2 = n\pi$$

for an integer n. The first time this happens is at

$$t = \frac{\pi - \arctan 3/2}{8}.$$

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8 A 10 kg weight is attached to a vertical spring with damping constant 2 kg-s/m. At rest, the spring is stretched 2 m. What is the spring constant of this spring? (Acceleration due to gravity near the surface of the Earth is approximately 9.8 m/s^2.)

Solution: A vertical spring system will hang at equilibrium mg/k. Therefore in this system, $98/k = 2$, so $k = 49$.

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9 Determine the equation of motion for an undamped system at resonance governed by

$$y'' + 16y = 2\cos 4t, \quad y(0) = 1, \quad y'(0) = 0.$$

Solution: This question is just asking us to solve an IVP. The homogeneous solution is

$$y_h = C\cos 4t + D\sin 4t.$$

For the particular solution, we guess $y_p = At\cos 4t + Bt\sin 4t$. We then compute:

$$\begin{aligned} y_p &= At\cos 4t + Bt\sin 4t, \\ y_p' &= (A + 4Bt)\cos 4t + (B - 4At)\sin 4t, \\ y_p'' &= (8B - 16At)\cos 4t + (-8A - 16Bt)\sin 4t. \end{aligned}$$

Substituting into the differential equation, we get

$$8B\cos 4t - 8A\sin 4t = 2\cos 4t,$$

so we want $A = 0$ and $B = 1/4$.

Our solution (so far) is

$$y = C\cos 4t + D\sin 4t + \frac{t}{4}\sin 4t,$$

and we just have to match the initial conditions. This requires $C = 1$ and $D = 0$, so the final solution is

$$y = \cos 4t + \frac{t}{4}\sin 4t.$$

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10 A 1 kg mass is attached to a horizontal spring with damping constant 2 kg-s/m and spring constant 1 N/m. Does this system have a resonance frequency?

Solution: The corresponding differential equation is

$$y'' + 2y' + y = 0.$$

The solution is therefore $y = Ce^{-t} + Dte^{-t}$. This is a critically damped system, and we know that only underdamped systems can have resonance frequencies, so the answer is no.
