Lecture 24 — March 13, 2020

Contour integrals!

Recall. A contour is a ~~curve~~ finite sequence of $\boxed{\text{smooth}}$ curves joined end-to-end.

differentiable, derivative never 0.

Set up. We have a complex function

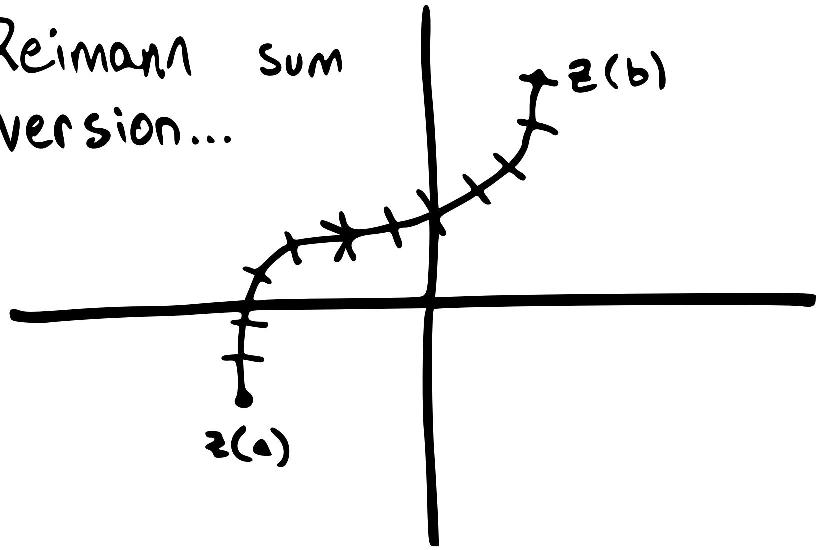
$$f : \mathbb{C} \to \mathbb{C}$$

and a contour

$$z : [a,b] \to \mathbb{C}.$$

We want to integrate f along z.

Reimann sum version...



Line integral (MV calc)

$$\int_C f(z(t))\, ds$$ ← wrt arc length

$$= \int_a^b f(z(t))\, \underbrace{|z'(t)|}_{\text{speed}}\, dt$$

Contour integral

$$\int_C f(z(t))\, dz$$ ← wrt change in z

$$= \int_a^b f(z(t))\, z'(t)\, dt$$

Note: contour integrals are NOT areas.

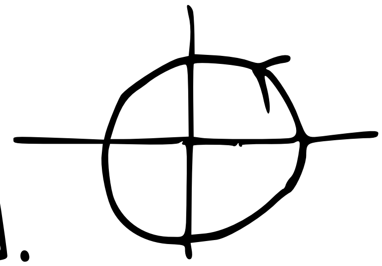
This integral will exist if...

$f(z(t))$ is piecewise continuous, because $z'(t)$ is also piecewise continuous (z is piecewise smooth...) so then $f(z(t))\,z'(t)$ is piecewise continuous.

(This is massive overkill...)

Let's skip the boring facts and do one!

Ex. Compute $\int_C \frac{dz}{z}$, where C is the unit circle with $\underbrace{\text{positive}}_{\text{counterclockwise}}$ orientation.



First. Parameterize C:

$$z(\theta) = e^{i\theta} \text{ for } \theta \in [0, 2\pi].$$

Second. Find derivative:

$$z'(\theta) = ie^{i\theta}.$$

Third. Evaluate:

$$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{z(t)} z'(t)\, dt$$

$$= \int_0^{2\pi} \frac{1}{e^{i\theta}} i e^{i\theta}\, dt$$

$$= \int_0^{2\pi} i\, dt$$

$$= \boxed{2\pi i.}$$

Variations

- Go around 3x.
- Go in clockwise orientation.
- Upper hemisphere from -1 to 1.
- Lower hemisphere from -1 to 1.

Note: We will NOT see path independence.

Ex. Compute $\int_C z\,dz$, where C is <u>any</u> smooth curve parameterized by

$$z : [a,b] \to \mathbb{C}.$$

Huh? It won't matter!

We have

$$\int_C z\,dz = \int_a^b z(t)\,z'(t)\,dt$$

Now observe:

$$\frac{d}{dt}\left(z^2(t)\right) \overset{\text{chain}}{\underset{\text{rule}}{=}} 2\,z(t)\,z'(t).$$

This means:

$$\int_a^b z(t)\,z'(t)\,dt = \frac{1}{2} z^2(t)\Big|_a^b$$

$$= \frac{1}{2} z^2(b) - \frac{1}{2} z^2(a).$$

Observations:

- path independence, unlike previous ex,
- suggests there are antiderivatives! (we'll get to those in a few days)

Boring properties

$$\int_C f(z) + g(z)\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz$$

$$\int_{C+D} f(z)\, dz = \int_C f(z)\, dz + \int_D f(z)\, dz$$

Slightly more interesting

Given a contour C, $-C$ denotes the contour of opposite orientation.

If $z(t)$ parameterizes C (for $a \le t \le b$), then $-C$ can be parameterized as $z(-t)$ for $-b \le t \le -a$. So

$$\int_{-C} f(z)\, dz = \int_{-b}^{-a} f(z(-t)) \frac{d}{dt}\left(z(-t)\right) dt$$

chain rule

$$= -\int_{-b}^{a} f(z(-t))\, z'(-t)\, dt$$

substitution

$$= \int_b^a f(z(t))\, z'(t)\, dt$$

$$= -\int_C f(z)\, dz$$