Lecture 25 - March 16, 2020

Today we're going to look at contour integrals involving branch cuts. These require practice!

Ex 1 (from Section 46 of Brown + Churchill)

Integrate $f(z) = z^{1/2}$ along the semicircular path

$$z(\theta) = 3e^{i\theta} \quad (\theta \in [0, \pi]).$$

Solution As written, it is not clear what this example is asking for, since it does not specify which branch of $z^{1/2}$ we should choose.

However, we find out in the solution that we are to use the branch

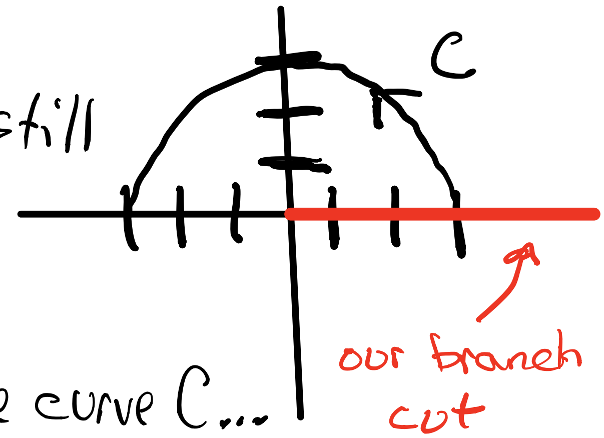
$$z^{1/2} = \exp\left(\tfrac{1}{2} \log z\right), \text{ where}$$

$$\log z = \ln|z| + i \arg z, \text{ where}$$

$$0 < \arg z < 2\pi$$

This means that our curve ($z = 3e^{i\theta}$) intersects our branch cut, which isn't great — our function isn't defined there!

Anyway, we can still make this work, as we will see.



Evaluating $z^{1/2}$ along the curve C...

$$f(z(\theta)) = \exp\left(\tfrac{1}{2}(\ln 3 + i\theta)\right)$$

$$= \sqrt{3}\, e^{i\theta/2}.$$

We also have

$$z'(\theta) = 3i\, e^{i\theta},$$

so our integrand is

$$f(z(\theta))\, z'(\theta) = 3\sqrt{3}\, i\, e^{3i\theta/2}.$$

The branch of $z^{1/2}$ we have chosen is not defined at $z = 3$ ($\theta = 0$ on our curve).

But, we can compute this as a limit.

Let

$$I = \int_C z^{1/2}\, dz.$$

We claim (and won't justify) that for the branch of $z^{1/2}$ we have chosen,

$$I = \lim_{\alpha \to 0} \int_\alpha^\pi 3\sqrt{3}\, i e^{3i\theta/2}\, d\theta.$$

Now we work the integral out:

$$\int_\alpha^\pi 3\sqrt{3}\, i e^{3i\theta/2}\, d\theta$$

$$= (3\sqrt{3}\, i)\left(\frac{2}{3i}\right) e^{3i\theta/2} \Big|_\alpha^\pi$$

$$= 2\sqrt{3}\left(\underbrace{e^{3i\pi/2}}_{-i} - e^{3i\alpha/2}\right)$$

$$= -2\sqrt{3}\, i - 2\sqrt{3}\, e^{3i\alpha/2}.$$

So we now have

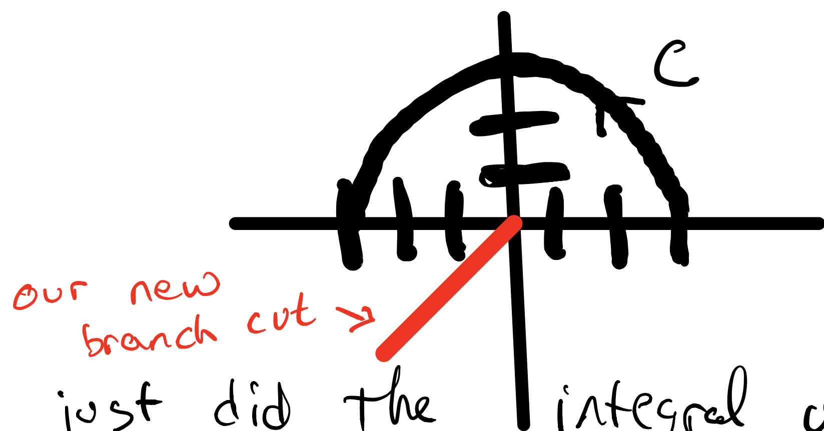
$$I = \lim_{\alpha \to 0} -2\sqrt{3}i - 2\sqrt{3}e^{3i\alpha/2}.$$

We have nothing to worry about with that limit, since e^z is continuous (entire, even), so

$$I = -2\sqrt{3}\,i - 2\sqrt{3}$$
$$= 2\sqrt{3}(-1-i).$$

Ex 1, revisited. What if we just chose a better branch cut for $z^{1/2}$? Would our lives have been easier? YES!

Recall our situation...

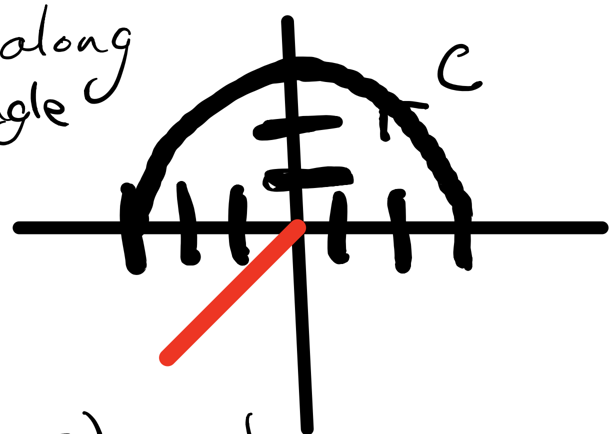


We just did the integral with the branch cut at the end of the curve C.

But why not take a branch cut that doesn't cut through C, like that shown above?

Before we do this, recall that taking a different branch of $z^{1/2}$ amounts to
taking a different branch of $\log z$
which amounts to
taking a different branch of $\arg z$.

We're going to cut along $\theta = \alpha$ for some angle $\alpha \in (-\pi, 0)$.



That means...

$$z^{1/2} = \exp\left(\tfrac{1}{2}\log z\right), \text{ where}$$
$$\log z = \ln|z| + i \arg z, \text{ where}$$

$$\boxed{\alpha < \arg z < \alpha + 2\pi}$$

(Note that along C, this is the principal value of the argument!)

The integrand stays the same, but now no limit! We just have...

$$I = \int_0^\pi 3\sqrt{3}\, i\, e^{3i\theta/2}\, d\theta$$
$$= (3\sqrt{3}\, i)\left(\frac{2}{3i}\right) e^{3i\theta/2}\Big|_0^\pi$$
$$= 2\sqrt{3}\left(\underbrace{e^{3i\pi/2}}_{-i} - 1\right)$$
$$= -2\sqrt{3}\, i - 2\sqrt{3}.$$

__Ex 2 (from Section 46 of Brown + Churchill).__

Using the principal branch, integrate

$$z^{-1+i}$$

over the unit circle, oriented positively (counterclockwise) and starting at $z=-1$.

__Solution.__ The principal branch of $\log z$ is

$$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z,$$

where

$$-\pi < \mathrm{Arg}\, z < \pi.$$

So the principal branch of z^{-1+i} is

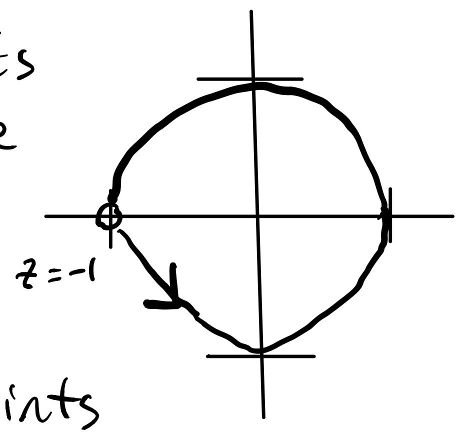
$$z^{-1+i} = \exp((1+i)\,\mathrm{Log}\, z).$$

Our contour is parameterized by

$$z(\theta) = e^{i\theta} \quad \text{for } \theta \in [-\pi, \pi].$$

EXCEPT at the endpoints of this curve, we have

$$\operatorname{Arg} z(\theta) = \theta.$$



HOWEVER, at the endpoints $\operatorname{Arg} z(\theta)$ is not defined.

BUT, we only need our integrand

$$f(z(\theta))\, z'(\theta)$$

to be piecewise continuous for $\theta \in (-\pi, \pi)$, so we're okay.

Repeating our calculations, we have

$$\operatorname{Log} z(\theta) = \theta, \quad \text{so}$$

$$\text{P.V. } z^{-1+i} = e^{(-1+i)i\theta} = e^{-(1+i)\theta}.$$

We also have

$$z'(\theta) = ie^{i\theta}, \quad \text{so}$$

$$\int_{-\pi}^{\pi} f(z(\theta))\, z'(\theta)\, d\theta = \int_{-\pi}^{\pi} i e^{-\theta}\, d\theta$$

$$= -ie^{-\theta}\Big|_{-\pi}^{\pi}$$

$$= -ie^{-\pi} + ie^{\pi}.$$

If we wanted to, we could simplify this to
$2i \sinh \pi$