Lecture 26 – March 18, 2020

Later, it will be important to bound the moduli of contour integrals, so we will learn how to do that now.

We want to establish results like:

$$\left| \int_C \frac{z-2}{z^4+1}\, dz \right| \leq \frac{4\pi}{15}, \quad \text{(Example 1)}$$

where C is the semicircular arc of the circle $|z|=2$ from $z=2$ to $z=2i$.

Our tool is the following theorem...

Theorem. Let C denote a contour of (arc) length L, and suppose that the function $f(z)$ is piecewise continuous on C. If there is a nonnegative real number M such that

$$|f(z)| \leq M$$

for all points z on C at which $f(z)$ is defined, then

$$\left| \int_C f(z)\, dz \right| \leq ML.$$

Proof. Parameterize C as $z(t)$ for $t \in [a,b]$, so we are interested in the modulus of

$$I = \int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

Write I in polar form as

$$I = |I| e^{i\alpha}. \quad (\text{If } I = 0, \text{ just choose } \alpha = 0.)$$

We now have, solving for $|I|$, that

$$|I| = e^{-i\alpha} \int_a^b f(z(t)) z'(t)\, dt$$
$$= \int_a^b e^{-i\alpha} f(z(t))\, z'(t)\, dt.$$

Since $|I|$ is real, it is equal to the real part of this integral:

$$|I| = \mathrm{Re}\left[\int_a^b e^{-i\alpha} f(z(t))\, z'(t)\, dt\right]$$
$$= \int_a^b \mathrm{Re}\left[e^{-i\alpha} f(z(t))\, z'(t)\right] dt.$$

We have

$$\mathrm{Re}\left[e^{-i\alpha} f(z(t))\, z'(t)\right]$$
$$\leq \left|e^{-i\alpha} f(z(t))\, z'(t)\right|$$
$$= \left|e^{-i\alpha}\right| \cdot \left|f(z(t))\right| \cdot \left|z'(t)\right|$$
$$\leq 1 \cdot M \cdot |z'(t)|.$$

This shows that

$$
\begin{aligned}
|I| &\leq \int_a^b M\, |z'(t)|\, dt \\
&= M \int_a^b |z'(t)|\, dt \\
&= M \cdot (\text{arc length of } C) \\
&= ML. \quad \blacksquare
\end{aligned}
$$

To apply this theorem, we need:

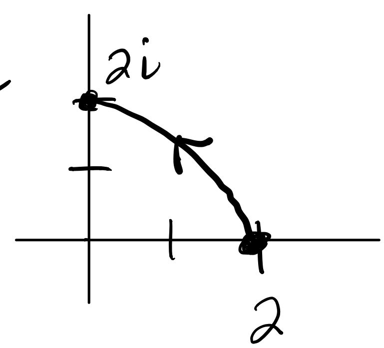
① A bound on $|f(z)|$ for all z along the contour.

② The length of the contour.

Example 1. Let C be the sector of the circle $|z|=2$ from $z=2$ to $z=2i$ in the first quadrant. Show (without evaluating the integral!) that

$$\left|\int_C \frac{z-2}{z^4+1}\,dz\right| \le \frac{4\pi}{15}.$$

Solution. The contour C is shown on the right. Clearly the length of C is



$$L = \frac{1}{4}\cdot \pi 2^2 = \pi.$$

Next we need to bound the modulus of the function

$$f(z) = \frac{z-2}{z^4+1}$$

for all $z \in C$.

We do this by writing

$$|f(z)| = \left| \frac{z-2}{z^4+1} \right| = \frac{|z-2|}{|z^4+1|}$$

and then bounding the numerator and denominator seperately.

For the numerator, we have

$$|z-2| = |z+(-2)| \leq \underbrace{|z|}_{|z|=2 \text{ because } z \in C} + |-2| = 2+2.$$

For the denominator, we want a lower bound, and we have

$$\begin{aligned} |z^4+1| &\geq \big| |z^4| - |1| \big| \\ &= |z|^4 - 1 \\ &= 2^4 - 1 \\ &= 15. \end{aligned}$$

We've shown that for all $z \in C$,

$$|f(z)| \leq \frac{4}{15},$$

so the theorem implies that

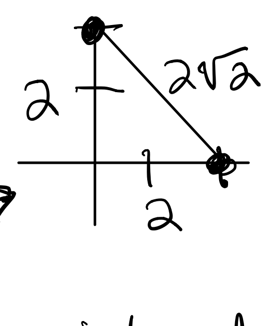
$$\left| \int_C \frac{z-2}{z^4+1}\, dz \right| \leq \frac{4}{15}\, \pi$$

bound on $|f(z)|$

length of C

Note. Both of the bounds we used can be improved. For example, $|z-2|$ is the distance between z and the point 2. The point of C that is farthest from 2 is $2i$, and we have

$$|2i-2| = 2\sqrt{2}.$$



This gives a bound of $\frac{2\sqrt{2}}{15}\pi$ on the modulus of the integral.

Our next example is much more typical of how we will use the theorem in the future.

Example 2. Let C_R denote the semicircle
$$z = Re^{i\theta} \text{ for } \theta \in [0, \pi],$$
oriented from R to $-R$. Show that
$$\lim_{R \to \infty} \int_{C_R} \frac{(z+1)}{(z^2+4)(z^2+9)}\, dz = 0.$$

Solution. For $z \in C_R$, we have
$$|z+1| \leq |z| + |1| = R+1,$$
$$|z^2+4| \geq \big||z^2| - |4|\big| = R^2 - 4,$$
$$|z^2+9| \geq \big||z^2| - |9|\big| = R^2 - 9.$$

The theorem now implies that

$$\left| \int_{C_R} \frac{(z+1)}{(z^2+4)(z^2+9)} \, dz \right| \leq \frac{R+1}{(R^2-4)(R^2-9)} \cdot \underbrace{\pi R}_{\text{length of } C} .$$

This quantity simplifies as

$$\frac{R^4}{R^4} \cdot \frac{\frac{\pi}{R^2} + \frac{\pi}{R^3}}{\left(1-\frac{4}{R^2}\right)\left(1-\frac{9}{R^2}\right)},$$

so the limit as $R \to \infty$ of our bounds on the moduli of these integrals is 0.

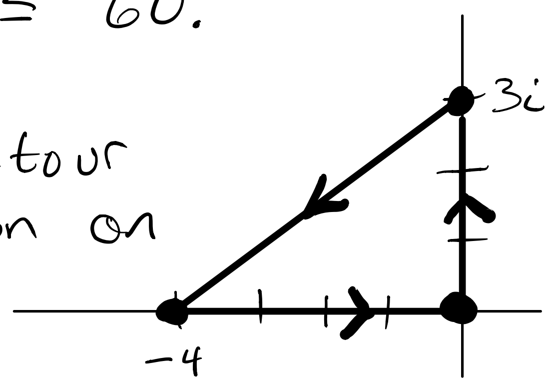
It follows that the limit as $R \to \infty$ of these integrals themselves must also be 0.

We'll do one more that looks trickier.

Exercise 3. Show that if C is the triangle with vertices 0, $3i$, and -4, oriented in that order, then

$$\left| \int_C (e^z - \bar{z})\, dz \right| \le 60.$$

Solution. The contour is shown on the right. Its length is



$$3 + \underbrace{5}_{\sqrt{3^2+4^2}} + 4 = 12.$$

Now we need to bound

$$|f(z)| = |e^z - \bar{z}|$$

on C.

We have

$$|f(z)| = |e^z - \bar{z}|$$
$$\leq |e^z| + |-\bar{z}|$$
$$= |e^z| + |z|.$$

Letting $z = x+iy$, we see that

$$|f(z)| \leq |e^{x+iy}| + |z|$$
$$= e^x + |z|.$$

We maximize these two terms seperately. On C, the maximum of e^x occurs when $x=0$, and it is 1. The maximum of $|z|$ occurs at $z=-4$, and it is 4.

This shows that $|f(z)| \leq 5$ on C, so

$$\left|\int_C (e^z - \bar{z})\, dz\right| \leq 5 \cdot 12 = 60.$$