Lecture 27 - March 20, 2020

We finally look at **antiderivatives**. When a function has an antiderivative, we will see that contour integrals are much easier to compute.

Def. Suppose $f(z)$ is continuous on a domain D. If there is a function $F(z)$ that satisfies

$$F'(z) = f(z)$$

for all $z \in D$, then $F(z)$ is an **antiderivative** of $f(z)$.

Note 1. Antiderivatives are always analytic because... they have derivatives!

Note 2. If $F(z)$ and $G(z)$ are both antiderivatives of $f(z)$, then

$$F'(z) - G'(z) = 0.$$

We have a theorem (in Sec. 25) that says that if the derivative of an analytic function is 0, then that analytic function must be constant. Therefore,

$$F(z) - G(z)$$

must be constant. This means that **when a function has an antiderivative, that antiderivative is unique up to an additive constant.**

This is the "$+C$" you are used to from calculus.

Theorem. Suppose that $f(z)$ is continuous on a domain D. If any of the following is true, then so are the others:

(a) $f(z)$ has an antiderivative $F(z)$ on D;

(b) for any contour C lying entirely in D and going from z_1 to z_2,

$$\int_C f(z)\, dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1) \ldots$$

we denote this value by $\int_{z_1}^{z_2} f(z)\, dz$;

(c) the integral of $f(z)$ along any closed contour lying entirely in D is 0.

Note. The theorem doesn't guarantee that any of these hold!

Proof. Not this lecture... maybe the next one.

Example 1. Does $f(z) = \frac{1}{z^2}$ have an antiderivative?

Solution. First of all, the question doesn't make sense without specifying a domain. Since $1/z^2$ isn't defined at the origin, the largest possible domain would be

$$D = \mathbb{C} \setminus \{0\}.$$

On this domain, we do have an antiderivative,

$$F(z) = -\frac{1}{z}.$$

By the theorem, this means that

$$\int_C \frac{1}{z^2}\, dz = 0$$

for any contour that doesn't pass through the origin, including, for example, the unit circle (any orientation).

Example 2. Does $f(z) = \frac{1}{z}$ have an antiderivative on the domain $D = \mathbb{C} \setminus \{0\}$?

Solution. The very first contour integral we ever did (in our second lecture on contour integration) showed that

$$\int_C \frac{1}{z}\, dz = 2\pi i$$

when C is the unit circle with positive orientation. This shows that condition (c) of the theorem doesn't hold, so $f(z) = 1/z$ cannot have an antiderivative on D.

But what about $\log z$? Let $F(z)$ denote any branch of $\log(z)$. We have seen that

$$F'(z) = \frac{d}{dz} \log z = \frac{1}{z},$$

so isn't that an antiderivative???

No! The problem is the branch cut, which every branch of $\log(z)$ has. Along that ray, $\log(z)$ isn't defined, so it is **NOT** an antiderivative of $1/z$ there, and hence not an antiderivative of $1/z$ on the domain $D = \mathbb{C} \setminus \{0\}$.

Example 3. Fix an angle α. Does $f(z) = 1/z$ have an antiderivative on the domain $D = \mathbb{C} \setminus \{re^{i\alpha} : r > 0\}$?

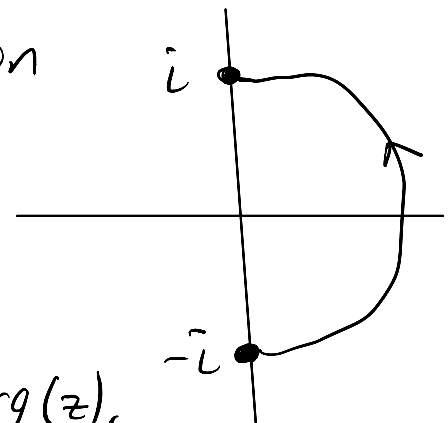
Solution. Yes! Once we exclude the ray $\{re^{i\alpha} : r > 0\}$, the antiderivative of $f(z) = 1/z$ is the branch of $\log(z)$ with that branch cut.

Because of this, we can still use antiderivatives to integrate $1/z$ along the unit circle...

Example 4. Let C_1 denote the right half of the unit circle, oriented from $-i$ to i. Compute

$$\int_{C_1} \frac{1}{z}\, dz.$$

Solution. We start by drawing a picture. Any branch of $\log(z)$ that is defined on all of C_1 will work, including the principal branch,



$$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z),$$

where

$$-\pi < \mathrm{Arg}(z) < \pi.$$

Since $\mathrm{Log}(z)$ is an antiderivative of $1/z$ on the domain

$$\mathbb{C} \setminus \{re^{i\pi} : r > 0\},$$

we can use part (b) of the theorem...

this part of the theorem shows that

$$\int_{C_1} \frac{1}{z}\,dz = \mathrm{Log}(z)\Big|_{-i}^{i}$$

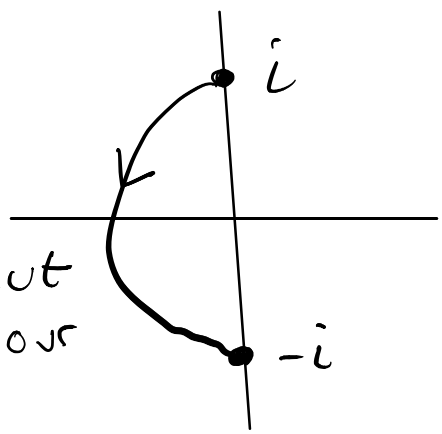
$$= \left(\ln(1) + i\frac{\pi}{2}\right) - \left(\ln(1) - i\frac{\pi}{2}\right)$$

$$= \pi i.$$

Example 4. Let C_2 denote the left half of the unit circle, oriented from i to $-i$. Compute

$$\int_{C_2} \frac{1}{z}\,dz.$$

Solution. The picture is here:



If we take our branch cut to be $\alpha = 0$, we get as our antiderivative

$$\log(z) = \ln|z| + i\arg(z),$$

where

$$0 < \arg(z) < 2\pi.$$

Now part (b) of the theorem states that

$$\int_{C_2} \frac{1}{z}\,dz = \log(z)\Big|_{i}^{-i}$$
$$= \left(\ln(1) + i\frac{3\pi}{2}\right) - \left(\ln(1) + i\frac{\pi}{2}\right)$$
$$= \pi i.$$

Example 5. Using the previous two examples, compute the integral of $1/z$ over the unit circle oriented positively (counterclockwise).

Solution. Let C denote this contour. At the end of our second lecture on contour integrals, we stated that

$$\int_{C+D} f(z)\,dz = \int_C f(z)\,dz + \int_D f(z)\,dz.$$

Since $C = C_1 + C_2$, we see that

$$\int_C \frac{1}{z}\,dz = \int_{C_1} \frac{1}{z}\,dz + \int_{C_2} \frac{1}{z}\,dz$$
$$= i\pi + i\pi = \boxed{2\pi i}$$

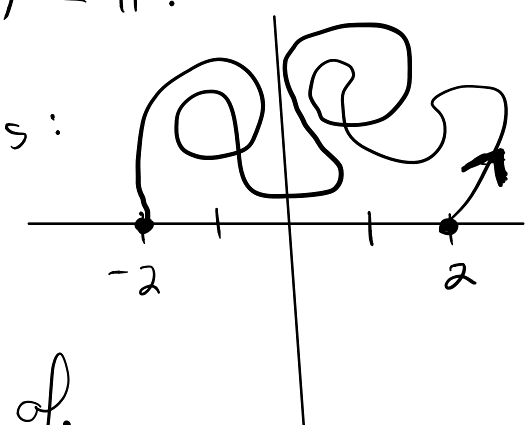
Example 6. Let C denote <u>any</u> contour from 2 to -2 that lies above the real axis, except at its endpoints. Evaluate

$$\int_C z^{1/2}\, dz,$$

where $z^{1/2}$ is the principal branch of $z^{1/2}$,

$$f(z) = \exp\left(\tfrac{1}{2} \operatorname{Log}(z)\right), \text{ where}$$
$$\operatorname{Log}(z) = \ln|z| + i \operatorname{Arg}(z), \text{ where}$$
$$-\pi < \operatorname{Arg}(z) < \pi.$$

Solution. A picture is:



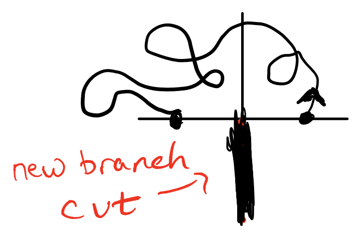
We need to worry about the fact that the principal branch of $z^{1/2}$ is not defined at $z = -2$.

But this is fine — we just need the function we are integrating to be piecewise continuous.

Still, the fact that the principal branch of $z^{1/2}$ isn't defined at $z = -2$ means we need to choose a different branch if we want to use our theorem.

So take our branch cut to be the negative complex axis... cut along $\{re^{-i\pi/2} : r > 0\}$, defining

$$g(z) = \exp\left(\tfrac{1}{2}\log(z)\right), \text{ where}$$
$$\log(z) = \ln|z| + i\arg(z), \text{ where}$$
$$-\tfrac{\pi}{2} < \arg(z) < \tfrac{3\pi}{2}.$$



Now, $g(z)$ is equal to the principal branch of $z^{1/2}$ everywhere except $z = -2$, where $g(z)$ is defined, but the principal branch of $z^{1/2}$ is not. As in our third lecture on contour integrals, this won't affect the answer.

Now, this branch of $z^{1/2}$ has an antiderivative on the domain

$$D = \mathbb{C} \setminus \{ re^{-i\pi/2} : r > 0 \}.$$

This antiderivative is

$$G(z) = \frac{2}{3} z^{3/2} = \frac{2}{3} \exp\left(\frac{3}{2} \log(z)\right), \text{ where}$$
$\log(z) = \ln|z| + i \arg(z)$, where
$-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$.

Since our contour C lies entirely within D, our theorem says that

$$\begin{aligned}
\int_C z^{1/2} dz &= \int_C g(z)\, dz \\
&= G(z) \Big|_{2}^{-2} \\
&= \frac{2}{3} \exp\left(\frac{3}{2}(\ln 2 + i\pi)\right) - \frac{2}{3} \exp\left(\frac{3}{2}(\ln 2 + 0)\right) \\
&= \frac{2}{3}\left(2^{3/2} \cdot e^{3i\pi/2} - 2^{3/2}\right) \\
&= \frac{2^{5/2}}{3}(-i-1).
\end{aligned}$$