Lecture 28 - March 23, 2020

Today we prove the theorem we discussed in the last lecture:

Theorem. Suppose that $f(z)$ is continuous on a domain D. If any of the following is true, then so are the others:

(a) $f(z)$ has an antiderivative $F(z)$ on D;

(b) for any contour C lying entirely in D and going from z_1 to z_2,

$$\int_C f(z)\,dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)\ldots$$

we denote this value by $\int_{z_1}^{z_2} f(z)\,dz$;

(c) the integral of $f(z)$ along any closed contour lying entirely in D is 0.

The standard way to prove such results is to prove

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a) \quad (\Rightarrow (b) \Rightarrow (c)).$$

Step 1 of proof: (a) ⇒ (b).

Suppose (a) holds, so there is a function $F(z)$ defined on all of D such that
$$F'(z) = f(z)$$
for all $z \in D$.

First we show that (b) holds for a single closed curve from z_1 to z_2 (for any choice of $z_1, z_2 \in D$). Suppose such a curve has parameterization

$$z(t) \quad \text{for} \quad t \in [a, b].$$

By the chain rule, we have
$$\begin{aligned} \frac{d}{dt}\Big(F(z(t))\Big) &= F'(z(t))\, z'(t) \\ &= f(z(t))\, z'(t). \end{aligned}$$

The definition of contour integrals says that

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt.$$

Moreover, we can break this integral into real and imaginary parts:

$$\int_a^b f(z(t))\,z'(t)\,dt$$

$$= Re\left[\int_a^b f(z(t))\,z'(t)dt\right] + i \cdot Im\left[\int_a^b f(z(t))\,z'(t)dt\right]$$

$$= \int_a^b Re\left[f(z(t))\,z'(t)\right] dt + i\int_a^b Im\left[f(z(t))\,z'(t)\right] dt.$$

Both of these are real integrals, so we can apply the fundamental theorem of calculus to conclude that

$$\int_a^b f(z(t))\,z'(t)\,dt$$

$$= Re\left[F(z(t))\right]\Big|_a^b + i \cdot Im\left[F(z(t))\right]\Big|_a^b$$

$$= F(z(t))\Big|_a^b.$$

Therefore, we have

$$\int_C f(z)\,dz = F(z(t))\Big|_a^b = F(z_2) - F(z_1).$$

The above holds for any smooth curve C from z_1 to z_2 (and lying entirely within D).

Recall that a contour is a finite sequence of smooth curves, joined end to end. Suppose that the contour C can be written as

$$C = C_1 + C_2 + \cdots + C_n,$$

where each C_k is a smooth curve, lying entirely within D, from z_k to z_{k+1}.

By the above, this means that

$$\int_{C_k} f(z)\,dz = F(z_{k+1}) - F(z_k).$$

Therefore we have

$$\int_C f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz + \cdots + \int_{C_k} f(z)\,dz$$

$$= F(z_2) - F(z_1) + \cdots + F(z_{k+1}) - F(z_k).$$

This sum telescopes, leaving only

$$\int_C f(z)\,dz = F(z_{k+1}) - F(z_1),$$

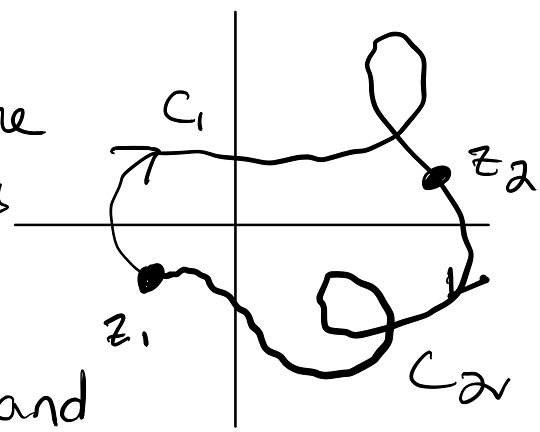
which is what we want, since C goes from z_1 to z_{k+1}.

Step 2 of proof: (b) ⇒ (c).

Assume not that (b) holds, meaning that the integral of $f(z)$ along any contour that lies entirely with D is independent of the path taken, meaning that it only depends on the endpoints.

Let C be any closed contour lying entirely in D, and take $z_1 \neq z_2$ to be any two distinct points along C.

As the picture on the right shows, C splits into two contours:



C_1 from z_1 to z_2 and
C_2 from z_2 to z_1.

Above we showed that $C = C_1 + C_2$. By our properties of contours, we have that

$$\int_{-C_2} f(z)\,dz = -\int_{C_2} f(z)\,dz,$$

where $-C_2$ is the oppositely oriented contour that runs from z_1 to z_2.

Since we have assumed that (b) holds, we have

$$\int_{C_1} f(z)\,dz = \int_{-C_2} f(z)\,dz.$$

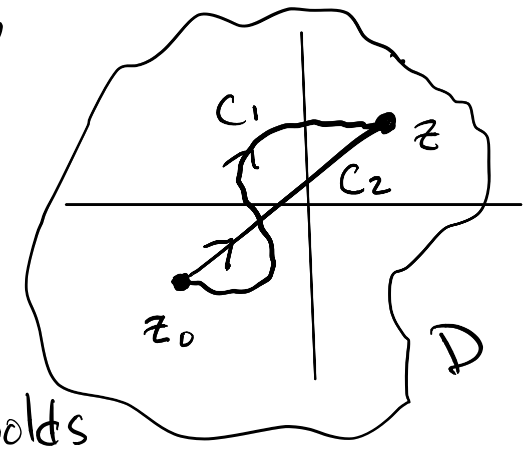
Therefore,

$$\begin{aligned}\int_C f(z)\,dz &= \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{-C_2} f(z)\,dz \\ &= 0\end{aligned}$$

by the above, establishing (c).

Step 3 of proof: c $\Rightarrow$ a.

Suppose that c holds, so the integral of $f(z)$ along any closed contour in D is 0. Let z_0 and z denote any two points in D and suppose C_1 and C_2 are both contours in D from z_0 to z, as on the right.



Let $C = C_1 + (-C_2)$, so C is a <u>closed</u> contour. By our hypothesis that c holds and the properties of contour integrals,

$$\begin{aligned} 0 &= \int_C f(z)\,dz \\ &= \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz \\ &= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz, \end{aligned}$$

so

$$\boxed{\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.}$$

The above shows that integrals of $f(z)$ are independent of path, so in fact we have established (b), but we wanted to establish (a).

Now fix some point $z_0 \in D$. By the above, we see that we may define a function

$$F(z) = \int_{z_0}^{z} f(w)\, dw,$$

where here we mean the integral of f on <u>any</u> contour in D from z_0 to z. (We saw above that all such integrals have the same value.)

We need to compute $F'(z)$. We have

$$\begin{aligned} F(z+\Delta z) &= \int_{z_0}^{z+\Delta z} f(w)\, dw \\ &= \int_{z_0}^{z} f(w)\, dw + \int_{z}^{z+\Delta z} f(w)\, dw, \end{aligned}$$

so long as $z + \Delta z \in D$.

Therefore, again as long as $z + \Delta z \in D$,

$$F(z+\Delta z) - F(z) = \int_z^{z+\Delta z} f(w)\, dw.$$

Since

$$\int_z^{z+\Delta z} 1 \, dw = \Delta z,$$

we can write

$$f(z) = \frac{f(z)}{\Delta z} \int_z^{z+\Delta z} 1 \, dw = \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)\, dw.$$

NOTE: constant in this integral

Therefore,

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right|$$

$$= \left| \frac{1}{\Delta z} \int_z^{z+\Delta z} f(w)\, dw - \frac{1}{\Delta z} \int_z^{z+\Delta z} f(z)\, dw \right|$$

$$= \frac{1}{|\Delta z|} \left| \int_z^{z+\Delta z} \left(f(w) - f(z) \right) dw \right|$$

$$\leq \frac{1}{|\Delta z|} \int_z^{z+\Delta z} |f(w) - f(z)|\, dw.$$

Finally, we must use the continuity of f. Since f is continuous at z, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(w) - f(z)| < \varepsilon \text{ whenever } |w - z| < \delta.$$

To bound

$$\int_z^{z+\Delta z} |f(w) - f(z)| \, dz,$$

we may assume the integral is along the line segment from z to $z + \Delta z$ (we have path independence).

Therefore if $|\Delta z| < \delta$, we'll have $|w - z| < \delta$, so

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right|$$

$$\leq \frac{1}{|\Delta z|} \cdot \varepsilon \cdot |\Delta z| = \varepsilon.$$

It follows that

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z),$$

completing our proof that (c) $\Rightarrow$ (a), and thus completing our proof of the theorem.