

Lecture 29 - March 27, 2020

Today we prove an important result known as the Cauchy integral theorem and discuss (but don't prove) a strengthening known as the Cauchy-Goursat theorem.

Note: Soon we will learn the Cauchy integral formula — that is a different result!

We start by considering a generic contour integral,

$$\int_C f(z) dz.$$

Next we express $f(z)$ in rectangular coordinates, as

$$f(z) = u(x, y) + i v(x, y).$$

Because C is a contour, we may parameterize it by

$$z(t) = x(t) + iy(t) \text{ for } t \in [a, b],$$

where $z(t)$ is differentiable at all but finitely many points. It follows that $x(t)$ and $y(t)$ are also differentiable at all but finitely many points.

As usual, we have

$$dz = z'(t) dt.$$

But we can go further:

$$\begin{aligned} dz &= z'(t) dt \\ &= (x'(t) + iy'(t)) dt \\ &= x'(t) dt + iy'(t) dt \\ &= dx + idy. \end{aligned}$$

Therefore we can express our contour integral as

$$\begin{aligned}
 \int_C f(z) dz &= \int_a^b (u(x(t), y(t)) + i v(x(t), y(t))) (dx + i dy) \\
 &= \int_a^b u dx - v dy + i(u dy + v dx) \\
 &= \left(\int_a^b u dx - v dy \right) + i \left(\int_a^b u dy + v dx \right).
 \end{aligned}$$

Both of these integrals are now real line integrals! So we have shown

$$\int_C f(z) dz = \left(\int_C u dx - v dy \right) + i \left(\int_C u dy + v dx \right).$$

This form of the contour integral should remind you of...

Green's Theorem for Circulation and Curl. Let C be a piecewise smooth curve enclosing a simply connected region R in the xy -plane. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field, where P and Q have continuous first partial derivatives in an open region containing R . Then

$$\underbrace{\int_C \mathbf{F} \cdot \mathbf{T} ds}_{\text{counterclockwise circulation}} = \underbrace{\int_C P dx + Q dy}_{\text{}} = \iint_R \text{curl } \mathbf{F} dA = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA.$$

integral of curl

What do we need to apply Green's theorem to our situation? We'll go through the hypotheses and see how they translate:

- (a) "C is a piecewise smooth curve." We have this, because C is a contour.
- (b) "C encloses a simply connected region." Connected means that every two points in the region can be connected by a path in the region, while simply connected means the region doesn't have any "holes".

We will get this if C is a simple (doesn't cross itself) closed (comes back to where it started) contour.

④ "P and Q have continuous first partial derivatives in an open region containing R." We want to apply Green's theorem where P & Q are u & v , so...

- To have first partial derivatives in the first place, $f(z)$ must be analytic (differentiable).
- But we also need those derivatives to be continuous!
- So for this part, we need:
there is an open region containing R on which $f(z)$ is analytic and $f'(z)$ is continuous.

Assuming we have all that, what does Green's theorem say?

First consider the real part,

$$\int_C u \, dx - v \, dy = \iint_R -v_x - u_y \, dA,$$

$\begin{matrix} \uparrow & \uparrow \\ P = u & Q = -v \end{matrix}$

where R is the region enclosed by C .

Furthermore, since we are assuming that $f(z)$ is analytic, it satisfies the Cauchy-Riemann equations, one of which is

$$u_y = -v_x.$$

Substituting this above shows that the real part of our contour integral is $0!$

For the imaginary part, we use the other Cauchy-Riemann equation ($u_x = v_y$) to get

$$\int_C u \, dy + v \, dx = \iint_R u_x - v_y \, dA = 0.$$

We have therefore proved

The Cauchy integral theorem.

Suppose C is a simple closed contour, and that $f(z)$ is analytic on C and all points enclosed by C and that $f'(z)$ is continuous there. Then

$$\int_C f(z) dz = 0.$$

(Note: letting R denote C together with the region enclosed by C , being analytic on R is the same as being analytic on an open region containing R .)

Goursat showed that we could remove the continuity condition on f' ...

The Cauchy-Goursat theorem.

If C is a simple closed contour, and $f(z)$ is analytic on C and all points enclosed by C , then

$$\int_C f(z) dz = 0.$$

(We won't prove this, but the proof is in the book.)

NOTE: We already knew $\int_C f(z) dz = 0$ if $f(z)$ has an antiderivative!

Examples: For any simple closed contour C , the following integrals all equal 0.

$$\int_C \sin(z) dz,$$

$$\int_C e^{z^2} dz,$$

$$\int_C z^3 + 3z + 1 dz.$$

That's because these are all entire functions — they are analytic everywhere.

Example. If C is the unit circle centered at $1+i$ and oriented positively, then

$$\int_C \frac{1}{z^2} dz = 0.$$

This is because $1/z^2$ is analytic on $\mathbb{C} \setminus \{0\}$, which encloses C .

Example. If C is the unit circle centered at $1+i$ and oriented positively, then

$$\int_C z^{1/2} dz = 0,$$

where $z^{1/2}$ denotes the principal branch.

This is because the principal branch of $z^{1/2}$ is analytic on

$$\mathbb{C} \setminus \{r : r \leq 0\},$$

which encloses C .

Non-example. If C is the unit circle (centered at the origin) and oriented positively, then

$$\int_C \frac{1}{z} dz \neq 0.$$

In fact, we know the value of this integral is $2\pi i$.

The Cauchy integral theorem does not apply because $\frac{1}{z}$ is not analytic at 0 , which is inside C .