In-class review for Section 7.6 — Solutions

1 Express the piecewise defined function

$$f(t) = \begin{cases} 2 & 0 < t < 3, \\ 0 & 3 < t < 7, \\ t^5 & 7 < t < 9, \\ \sin t & t > 9 \end{cases}$$

in a single line in terms of Heaviside (unit step) functions.

Solution: It is just

$$2 - 2u(t-3) + t^5 u(t-7) - t^5 u(t-9) + (\sin t)u(t-9).$$

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2 Find the Laplace transform of the solution y to the initial value problem

$$2y'' - 3y' + y = \begin{cases} t^2 & 0 < t < 5, \\ \sin t & t > 5 \end{cases} \qquad y(0) = 1; \quad y'(0) = 0.$$

Solution: To save some writing, let $Y(s) = \mathscr{L}\{y\}$. For the lefthand side, we have

$$\begin{aligned} \mathscr{L}\{y\} &= Y, \\ \mathscr{L}\{y'\} &= sY - y(0) &&= sY - 1, \\ \mathscr{L}\{y''\} &= s(\mathscr{L}\{y\}) - y'(0) &&= s^2 Y - s. \end{aligned}$$

So, we get

$$\mathscr{L}\left\{2y'' - 3y' + y\right\} = 2(s^2Y - s) - 3(sY - 1) + Y = (2s^2 - 3s + 1)Y + (-2s + 3).$$

To transform the righthand side, we first need to express it in terms of Heaviside functions. We have that

$$\begin{cases} t^2 & 0 < t < 5, \\ \sin t & t > 5 \end{cases} = t^2 - t^2 u(t-5) + (\sin t)u(t-5),$$

so using the rule that $\mathscr{L}\{f(t)u(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$, we get that

$$\mathscr{L}\left\{t^2 - t^2u(t-5) + (\sin t)u(t-5)\right\} = \frac{2}{s^3} - e^{-5s}\mathscr{L}\left\{(t+5)^2\right\} + e^{-5s}\mathscr{L}\left\{\sin(t+5)\right\}.$$

Now expand the $(t+5)^2$ to transform it:

$$\mathscr{L}\left\{(t+5)^2\right\} = \mathscr{L}\left\{t^2 + 10t + 25\right\} = \frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}.$$

For $\sin(t+5)$ we need the angle addition formula:

$$\begin{aligned}\mathscr{L}\{\sin(t+5)\} &= \mathscr{L}\{(\sin t)(\cos 5) + (\sin 5)(\cos 5)\} \\ &= \frac{\cos 5}{s^2+1} + \frac{s\sin 5}{s^2+1}.\end{aligned}$$

Putting this all together, we have

$$(2s^2-3s+1)Y + (-2s+3) = \frac{2}{s^3} - e^{-5s}\left(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}\right) + e^{-5s}\left(\frac{\cos 5}{s^2+1} + \frac{s\sin 5}{s^2+1}\right),$$

so

$$Y = \frac{\left(\frac{2}{s^3} - e^{-5s}\left(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}\right) + e^{-5s}\left(\frac{\cos 5}{s^2+1} + \frac{s\sin 5}{s^2+1}\right) + 2s - 3\right)}{2s^2 - 3s + 1}.$$

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3 Suppose that

$$\mathscr{L}\{y\} = \frac{e^{-2s} + se^{-3s}}{s^2+4}.$$

What is y?

Solution: We split the fraction into two parts,

$$\mathscr{L}\{y\} = e^{-2s}\frac{1}{s^2+4} + e^{-3s}\frac{s}{s^2+4},$$

and then solve each separately. First, the e^{-2s} term shows that the first fraction comes from a function of the form $f(t)u(t-5)$:

$$e^{-2s}\frac{1}{s^2+4} = f(t)u(t-5) = e^{-2s}\mathscr{L}\left\{f(t+2)\right\}.$$

Therefore,

$$\mathscr{L}\left\{f(t+2)\right\} = \frac{1}{s^2+4} = \frac{1}{2}\frac{2}{s^2+4} = \frac{1}{2}\sin 2t,$$

so $f(t) = 1/2\sin(2(t-2))$.

For the second fraction, we have

$$e^{-3s}\frac{s}{s^4+4} = f(t)u(t-3) = e^{-3s}\frac{s}{s^2+4} = e^{-3s}\mathscr{L}\left\{\cos 2t\right\},$$

from which is follows that the second fraction comes from $\cos(2(t-3))u(t-3)$. The final answer is

$$y(t) = \frac{1}{2}\sin(2(t-2))u(t-2) + \cos(2(t-3))u(t-3).$$

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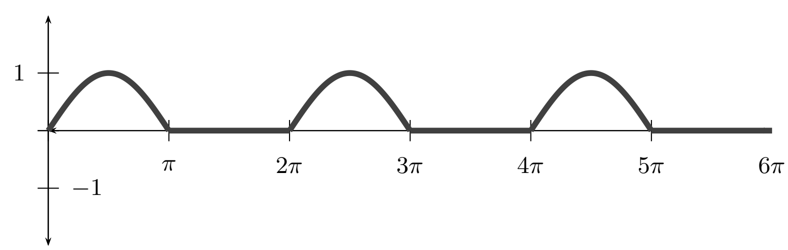
4 The *half-rectified sine wave* is equal to $\sin t$ when $\sin t$ is positive, and 0 otherwise.



Find the Laplace transform of the half-rectified sine wave.

Solution: The period of this function is $T = 2\pi$, and its fundamental window is

$$f_T(t) = \sin t - (\sin t)u(t-\pi).$$

We then have

$$\begin{aligned}\mathscr{L}\{f_T\} &= \frac{1}{s^2+1} - e^{-\pi s}\mathscr{L}\{\sin(t+\pi)\} \\ &= \frac{1}{s^2+1} - e^{-\pi s}\mathscr{L}\{-\sin t\} \\ &= \frac{1}{s^2+1} - e^{-\pi s}\frac{1}{s^2+1} \\ &= \frac{1-e^{-\pi s}}{s^2+1}.\end{aligned}$$

The Laplace transform of the half-rectified sine wave is therefore

$$\frac{\mathscr{L}\{f_T\}}{1-e^{-sT}} = \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})}.$$

Note that because $1-e^{-2\pi s} = (1-e^{-\pi s})(1+e^{-\pi s})$, this can be simplified to

$$\frac{1}{(s^2+1)(1+e^{-\pi s})},$$

but there is no need to do that.