Lecture 30 - March 30, 2020

Last lecture we saw the

Cauchy-Goursat theorem

If C is a simple closed contour, and $f(z)$ is analytic on C and all points enclosed by C, then

$$\int_C f(z)\,dz = 0.$$

This lecture, we look at two surprising implications of this result.

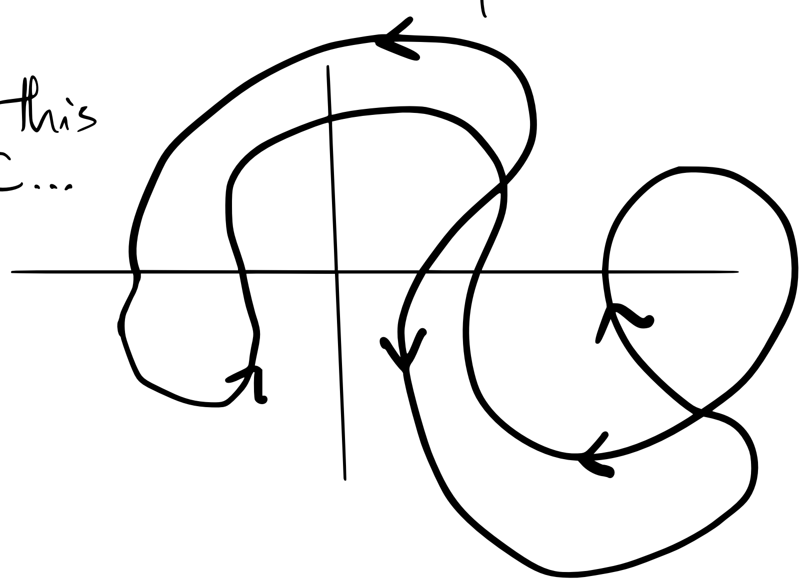
Our first implication lets us drop the "simple" requirement from C. Recall that a simply connected region is one without "holes". More formally, the region D is simply connected if every simple closed curve in D encloses only points of D.

Theorem. If $f(z)$ is analytic everywhere in a simply connected region* D, then for every closed contour C contained in D,

$$\int_C f(z)\,dz = 0.$$

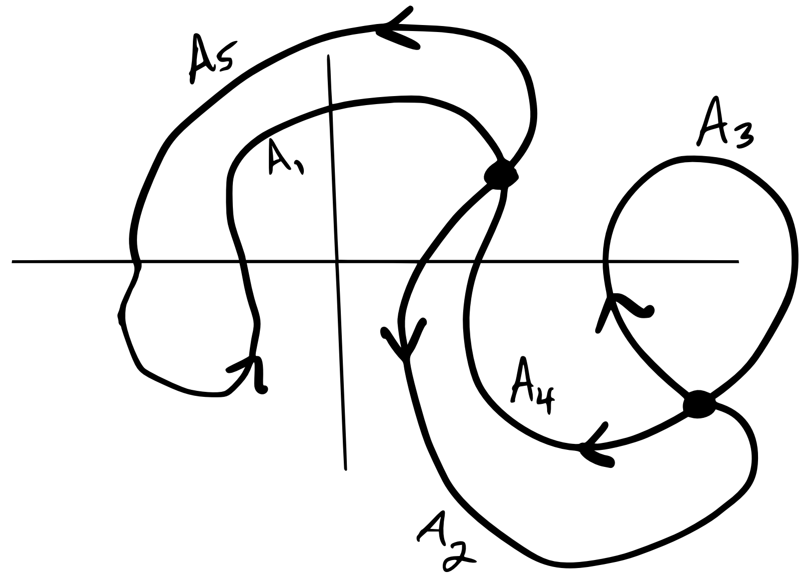
Proof. If C crosses itself only finitely many times, the theorem can essentially be proved with a picture...

Suppose this is C...



Then we can cut C every time it crosses itself...

* Recall that a region is a domain (nonempty, open, connected set) together with some or all of its boundary.



... and this process cuts C into a finite number of pieces, A_1, A_2, A_3, A_4, A_5 in this example, that can be recombined into a finite number of <u>simple</u> closed contours. In this example, those simple closed contours are

$$A_1 + A_5, \quad A_2 + A_4, \quad A_5.$$

Since Cauchy-Goursat says our integral along any of these is 0, our integral along C is also 0.

If C crosses itself infinitely many times, it is more complicated... $\blacksquare$

Recall, we proved that if $f(z)$ is continuous in a domain D and

$$\int_C f(z)\,dz = 0$$

for all closed contours C lying entirely in D, then $f(z)$ has an antiderivative on D!

So...

Corollary. Any function $f(z)$ that is analytic everywhere in a simply connected region D has an antiderivative on D.

In particular, all entire functions have antiderivatives.

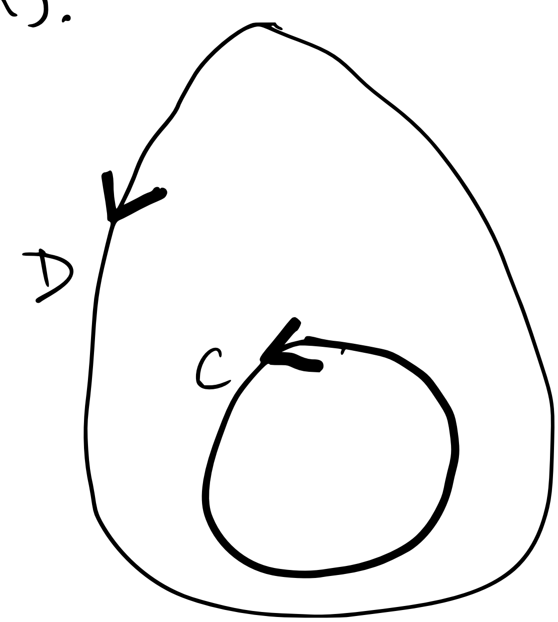
Our next implication of the Cauchy-Goursat theorem should be even more surprising.

The Principle of path deformation.

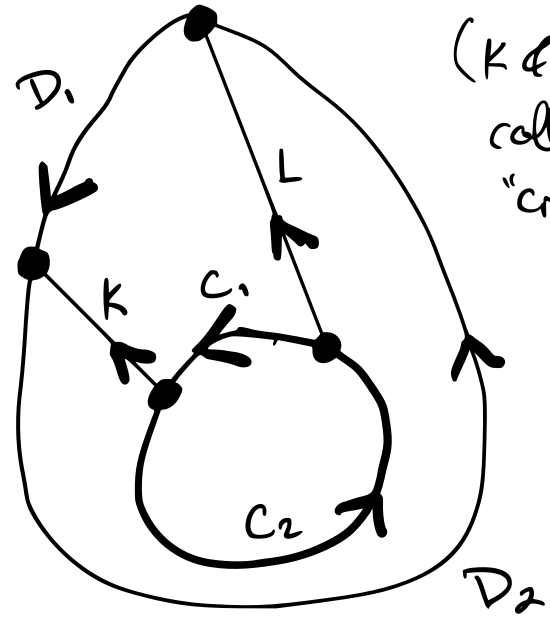
Suppose C and D are both closed contours oriented in the <u>same direction</u>, and that C is enclosed by D. If $f(z)$ is analytic on C, D, and all points in-between, then

$$\int_C f(z)\,dz = \int_D f(z)\,dz.$$

Proof. Again, we can almost prove this with just a picture. Suppose our contours look like the below (which is meant to look like an avocado, for what it's worth).



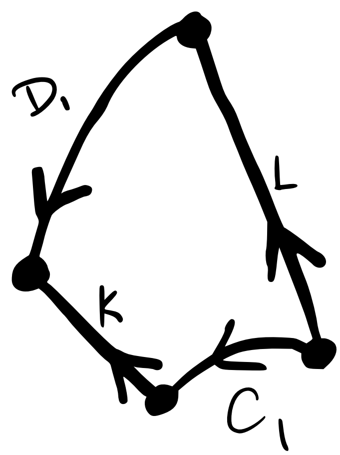
Now choose two points on each of C and D, and connect them with line segments that don't cross:



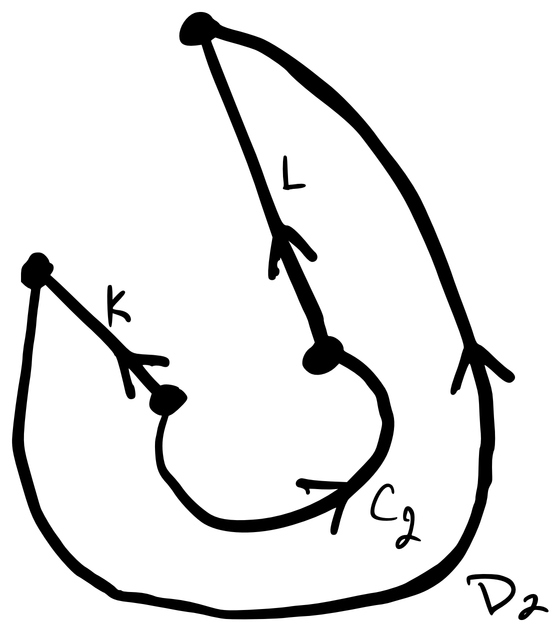
(K & L are called "cross cuts")

Label these two line segments K and L, and orient them from C to D. The points we chose on C and D split them both into two halves, say C_1, C_2 and D_1, D_2.

Starting at the top of the avocado, follow D_1 until K, follow K negatively, follow C_1 negatively, and then follow L. This gives one closed contour. The paths we didn't take give us another.



$D_1 - K - C_1 + L$



$-L - C_2 + K + D_2$

These two contours are plotted above. The first is equal to $D_1 - K + C_2 + L$.

Since this is a closed contour, and $f(z)$ is analytic on this contour and all points enclosed by it,

$$\int_{D_1 - K - C_1 + L} f(z)\, dz = 0.$$

But we can also break this up...

$$\int_{D_1 - K - C_1 + L} f(z)\, dz = \int_{D_1} f(z)\, dz - \int_{K} f(z)\, dz - \int_{C_1} f(z)\, dz + \int_{L} f(z)\, dz.$$

With the second contour, we can do the same thing:

$$0 = \int_{-L-C_2+K+D_2} f(z)\,dz$$

$$= -\int_L f(z)\,dz - \int_{C_2} f(z)\,dz + \int_K f(z)\,dz + \int_{D_2} f(z)\,dz.$$

If we add these together, we see that

$$0 = \int_{D_1} f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz + \int_{D_2} f(z)\,dz$$

$$= \int_D f(z)\,dz - \int_C f(z)\,dz,$$

and this proves the result. $\blacksquare$

Example. Let z_0 be any point in $\mathbb{C}$, and let C be any positively oriented closed contour that encloses but does not pass through z_0. Show that

$$\int_C \frac{1}{z-z_0}\,dz = 2\pi i.$$

Note. We've already seen this when $z=0$ and C is the unit circle.

Solution. Let D denote the positively oriented unit circle centered at z_0. By the principle of path deformation,

$$\int_C \frac{1}{z-z_0}\,dz = \int_D \frac{1}{z-z_0}\,dz$$

because the function $\frac{1}{z-z_0}$ is analytic everywhere except z_0. Now by making the substitution $w = z - z_0$, we see that this integral is equal to the integral of $1/z$ over the positively oriented unit circle centered at the origin.