Lecture 31 - April 1, 2020

Last lecture we extended the Cauchy-Goursat theorem to not-necessarily-simple closed contours, and we proved the principle of path deformation: where a function is analytic, we can change the path and it doesn't matter.

In this lecture we establish another surprising implication, the Cauchy integral FORMULA.

Cauchy integral formula. Suppose C is a positively oriented, simple, closed contour. If $f(z)$ is analytic on C and all points enclosed by C, then for every z_0 enclosed by C, we have

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz.$$

Note. You should think of this as a sort of "averaging" operation... $f(z_0)$ is determined by a "weighted average" of f along the contour C.

Note 2. That should be very surprising! An analytic function is determined at any point by a weighted average of its values along (almost) <u>any</u> contour encircling that point? Yes!

The proof isn't even all that difficult!

Proof. Let D denote the (closed) region enclosed by C.

Since $f(z)$ is analytic on D, $f'(z_0)$ exists. Therefore we can define a function $g(z)$ by

$$g(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} & \text{if } z \neq z_0, \\ f'(z_0) & \text{if } z = z_0. \end{cases}$$

Why might we care about $g(z)$? Well...

$$\int_C g(z)\,dz = \int_C \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

$$= \left(\int_C \frac{f(z)}{z - z_0}\,dz\right) - \left(f(z_0)\int_C \frac{1}{z - z_0}\,dz\right).$$

We saw in the last lecture that

$$\int_C \frac{1}{z-z_0}\,dz = 2\pi i$$

for <u>any</u> positively oriented, closed contour enclosing z_0. Thus we have

$$f(z_0)\int_C \frac{1}{z-z_0}\,dz = 2\pi i f(z_0).$$

Therefore,

$$\int_C g(z)\,dz = \left(\int_C \frac{f(z)}{z-z_0}\,dz\right) - 2\pi i f(z_0),$$

so

$$f(z_0) = \frac{1}{2\pi i}\left(\int_C \frac{f(z)}{z-z_0}\,dz - \int_C g(z)\,dz\right).$$

To complete the proof, we only need to show that

$$\int_C g(z)\,dz = 0.$$

Note that we can't use the Cauchy-Goursat theorem to get this, because we don't know that $g(z)$ is analytic at z_0.

Instead, we are going to show that

$$\int_C g(z)\,dz = 0$$

Using the tools we learned in our lecture on bounding contour integrals.

There we learned that if there is a nonnegative real number M such that

$$|g(z)| \leq M$$

for all points z on C, and if L is the length of C, then

$$\left| \int_C g(z)\,dz \right| \leq ML.$$

We are going to show that $g(z)$ must be bounded on D (which includes C), and then use the principle of path deformation to send L to 0. That will prove the result.

First, $g(z)$ is continuous...

We know $g(z)$ is continuous on $D\setminus\{z_0\}$ because there it is the product of $f(z)-f(z_0)$ and $\frac{1}{z-z_0}$, which are both continuous.

What about at $z=z_0$? There we have

$$\lim_{z\to z_0} g(z) = f'(z_0) = g(z_0),$$

so $g(z)$ is continuous there too. More importantly, the real-valued function $|g(z)|$ is continuous (we proved this a while ago).

From here, it's only a question of which theorem of real analysis you would like to appeal to. For example:

"D is closed and bounded, so it is compact, so the continuous function $|g(z)|$ attains its maximum somewhere on D."

However you convince yourself, $g(z)$ is bounded on D. So there is some nonnegative real number M such that

$$|g(z)| \le M$$

for all $z \in D$.

Now let C_r denote the circle of radius $r > 0$ centered at z_0.

By the principle of path deformation, since $g(z)$ is analytic everywhere except possibly at z_0, if r is small enough that C_r is enclosed by C, then

$$\int_C g(z)\,dz = \int_{C_r} g(z)\,dz.$$

Moreover, the length of C_r is $L = 2\pi r$, so by the above,

$$\left| \int_{C_r} g(z)\,dz \right| \le ML = 2\pi r M.$$

Since the quantity $2\pi r M$ tends to 0 as r tends to 0, we must conclude that

$$\left| \int_{C_r} g(z)\, dz \right| = 0,$$

which could only hold if

$$\int_{C_r} g(z)\, dz = 0,$$

which is what we wanted to prove. $\blacksquare$

Example. Compute $\displaystyle\int_C \frac{\cos z}{z}\, dz$, where C is the positively oriented unit circle.

Solution. This is an instance of the Cauchy integral formula with $z_0 = 0$ and $f(z) = \cos z$, so

$$\cos(0) = \frac{1}{2\pi i} \int_C \frac{\cos z}{z}\, dz,$$

which means $\displaystyle\int_C \frac{\cos z}{z}\, dz = 2\pi i.$

The Cauchy integral formula can be extended to give formulas for the derivatives of an analytic function too!

Cauchy integral formula for derivatives.

Suppose C is a positively oriented, simple, closed contour. If $f(z)$ is analytic on C and all points enclosed by C, then for every z_0 enclosed by C, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Proof. While the proof is not that difficult, we'll skip it.

Example. Compute

$$\int_C \frac{e^z}{(z^2+1)^3}\, dz,$$

where C is the unit circle centered at i.

Solution. We have

$$\frac{e^z}{(z^2+1)^3} = \frac{e^z}{(z-i)^3(z+i)^3}.$$

So this is an instance of the Cauchy integral formula with $n=2$, $z_0 = i$, and

$$f(z) = \frac{e^z}{(z+i)^3}.$$

The LHS of the formula is $f''(i)$. We compute

$$f'(z) = e^z(z+i)^{-3} - 3e^z(z+i)^{-4},$$

$$f''(z) = e^z(z+i)^{-3} - 6e^z(z+i)^{-4} + 12e^z(z+i)^{-5}.$$

So,

$$f''(i) = e^i\left((2i)^{-3} - 6(2i)^{-4} + 12(2i)^{-5}\right).$$

Putting everything together, we got

$$e^i\left((2i)^{-3}-6(2i)^{-4}+12(2i)^{-5}\right)=\frac{2}{2\pi i}\int_C \frac{e^z}{(z^2+1)^3}\,dz,$$

So

$$\int_C \frac{e^z}{(z^2+1)^3}\,dz = \pi i\, e^i\left((2i)^{-3}-6(2i)^{-4}+12(2i)^{-5}\right).$$

If you feel like it, you can simplify this to

$$\left(\frac{1}{4}-\frac{3i}{8}\right)\pi e^i.$$

Example. Let C be any simple closed contour. Show that

$$\int_C \frac{z^3+2z}{(z-z_0)^3}\,dz$$

is equal to 0 if z_0 lies outside C and is equal to $6\pi i z_0$ when z_0 lies inside C.

Solution. The integrand is analytic everywhere except at z_0, so if z_0 lies outside C, it follows from the Cauchy integral theorem that the integral is 0.

Now suppose z_0 lies inside C. Then we have an instance of the Cauchy integral formula with $n=2$, $z_0=z_0$, and $f(z) = z^3+2z$.

Since $f''(z) = 6z$, the LHS of this formula is

$$f''(z_0) = 6z_0,$$

so we get

$$6z_0 = \frac{2}{2\pi i}\int_C \frac{z^3+2z}{(z-z_0)^3}\,dz,$$

which simplifies to the desired result.