Lecture 32 — April 3, 2020

We ended the previous lecture with the most general form of the Cauchy integral formula:

Cauchy integral formula.

Suppose C is a positively oriented, simple, closed contour. If C is analytic on C and all points enclosed by C, then for every z_0 enclosed by C, we have

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz.$$

This lecture, we study three implications of the CIF.

For our first implication, suppose that the function $f(z)$ is analytic at the point z_0. By the definition of analytic functions, this means that $f(z)$ has a derivative in some neighborhood of z_0.

Consequently, $f(z)$ is analytic in some neighborhood of z_0. Let N denote this neighborhood, and let $C \subseteq N$ be a positively oriented, simple, closed contour enclosing z_0. By the CIF,

$$f''(z_0) = \frac{2}{2\pi i} \int_C \frac{f'(z)}{(z - z_0)^3} \, dz,$$

so $f''(z_0)$ exists. But more than this, for any other point z enclosed by C,

$$f''(z) = \frac{2}{2\pi i} \int_C \frac{f'(w)}{(w - z)^3} \, dw.$$

Thus f'' exists in a neighborhood of z_0, which means f' is analytic at z_0. Thus we have shown:

If f is analytic at z_0, then f' is also analytic at z_0.

But now we can apply this to f', then to f'', f''', etc. Thus we obtain the following.

Theorem. If the function f is analytic at the point z_0, then it is infinitely differentiable at z_0, and all of those derivatives are also analytic at z_0.

This result allows us to establish a converse to the Cauchy-Goursat theorem.

Suppose the function $f(z)$ is continuous on the domain D. Assume that for every closed contour $C \subseteq D$, we have

$$\int_C f(z)\,dz = 0.$$

(This is the conclusion of Cauchy-Goursat. Now we take it as our hypothesis.)

By our theorem on antiderivatives, this is enough to guarantee that $f(z)$ has an antiderivative throughout D, meaning that there is an analytic function $F(z)$ so that

$$F'(z) = f(z)$$

for all $z \in D$.

By our theorem above, $F' = f$ and all other derivatives of F are analytic at every point in D, so:

<u>Morera's theorem.</u> Suppose $f(z)$ is continuous on the domain D and that

$$\int_C f(z)\, dz = 0$$

for every closed contour $C \subseteq D$. Then f is analytic on D.

For our final implication of the CIF, we apply our tools for bounding contour integrals to the contour integral in CIF.

Let C_R denote the positively oriented circle of radius R centered at z_0. If $f(z)$ is analytic on C_R and all points enclosed by it, then the CIF says

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Now suppose that for some $M_R \geq 0$ we have

$$|f(z)| \leq M_R$$

for all $z \in C_R$. Then for all $z \in C_R$,

$$\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{|(z-z_0)^{n+1}|} \leq \frac{M_R}{R^{n+1}}.$$

It follows that

$$\int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \leq \frac{M_R}{R^{n+1}} \cdot 2\pi R,$$

and thus we get the following result.

Cauchy's inequality. Suppose $f(z)$ is analytic on and inside the circle of radius R centered at z_0. If $|f(z)| \leq M_R$ on C_R, then for all $n \in \mathbb{N}$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n} \cdot M_R.$$