

Lecture 33 - April 6, 2020

We ended previous lecture with Cauchy's inequality: if $f(z)$ is analytic on and inside the circle of radius R centered at z_0 and $|f(z)| \leq M_R$ on this circle, then for all $n \in \mathbb{N}$,

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \cdot M_R.$$

We begin this lecture by seeing what this inequality means for a bounded entire function.

Suppose $f(z)$ is entire (analytic everywhere) and $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then applying Cauchy's inequality with $n=1$, we see that

$$|f'(z)| \leq \frac{M}{R}$$

for all $z \in \mathbb{C}$ and all radii $R > 0$.

But the only way this can hold for all R is if

$$f'(z) = 0$$

everywhere. A while ago, we proved a theorem stating that if $f'(z) = 0$ everywhere, then $f(z)$ must be constant. Therefore we have proved the following.

Liouville's theorem. If an entire function is bounded, then it is constant.

Next we use Liouville's theorem to prove the fundamental theorem of algebra.

There are a few ways to state the fundamental theorem of algebra, but the most convenient for us is the following.

The fundamental theorem of algebra.

Every nonconstant polynomial has a root (zero) in \mathbb{C} .

Our proof is by contradiction.

Suppose that $n \geq 1$ but that the polynomial

$$p(z) = a_n z^n + \dots + a_1 z + a_0$$

(also assume $a_n \neq 0$!)

does not have any roots. This implies that its reciprocal, $1/p(z)$, is entire. We'll complete the proof by showing that $1/p(z)$ is bounded, so it must be constant by Liouville's theorem, but since $n \geq 1$, it is not constant (and thus we will have contradicted our assumption that $p(z)$ has no roots).

First we find a lower bound on $|p(z)|$ for large z .

We have

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right).$$

If $|z|$ is large,

$$\begin{aligned} & \left| \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \\ &= \frac{1}{|z|} \cdot \left| a_{n-1} + \dots + \frac{a_1}{z^{n-2}} + \frac{a_0}{z^{n-1}} \right| \end{aligned}$$

will be small. In particular, for some choice of $R > 0$, if $|z| \geq R$, then

$$\left| \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \leq \frac{|a_n|}{2}.$$

(this is why we need $a_n \neq 0$)

Therefore, for $|z| \geq R$, we have

$$\begin{aligned} |p(z)| &= |z|^n \cdot \left| a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right| \\ &\geq |z|^n \cdot \left(|a_n| - \frac{|a_n|}{2} \right) \\ &= \frac{|z|^n |a_n|}{2} \\ &\geq \frac{R^n |a_n|}{2}. \end{aligned}$$

This shows that for $|z| \geq R$, we have

$$\left| \frac{1}{p(z)} \right| = \frac{1}{|p(z)|} \leq \frac{2}{R^n |a_n|},$$

so $1/p(z)$ is bounded outside the circle $|z| = R$.

Moreover, since $1/p(z)$ is entire, it is continuous on the disc $|z| \leq R$, so it is bounded there too.

It follows that $1/p(z)$ is bounded everywhere, so Liouville's theorem says that $1/p(z)$ must be constant, and that's the contradiction we sought, completing the proof.