

## Lecture 34 - April 8, 2020

We now switch from contour integrals to Taylor series (and later, Laurent series). As we'll see, these topics are not so far apart. First, though, we need to talk about sequences.

Definition. The sequence  $z_1, z_2, \dots \in \mathbb{C}$  converges to  $z \in \mathbb{C}$  if for every  $\epsilon > 0$ , there is some integer  $N \in \mathbb{N}$  such that

$$|z_n - z| < \epsilon \text{ whenever } n > N.$$

It is not hard to show that a sequence can converge to at most one point.

If  $z_1, z_2, \dots$  converges to  $z$ , we call  $z$  the limit of  $z_1, z_2, \dots$  and write

$$\lim_{n \rightarrow \infty} z_n = z.$$

If a sequence has no limit, then we say it diverges.

It is also not hard to show that if  $z_n = x_n + iy_n$  for all  $n$  and  $z = x + iy$ , then

$$\lim_{n \rightarrow \infty} z_n = z$$

if and only if

$$\lim_{n \rightarrow \infty} x_n = x \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = y.$$

This means that everything you know about limits of real sequences transfers to this context.

- Examples.
- ① the sequence  $z_n = i + \frac{(-1)^n}{n}$  converges to  $i$ .
  - ② the sequence  $z_n = i^n$  diverges.

We now move on to series. Recall that the series

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \dots$$

converges to the sum  $S$  if the sequence of partial sums

$$S_N = \sum_{n=1}^N z_n$$

Converges to  $S$ .

As with sequences, everything we know about real series translates to the complex context, because it is easy to show that

if  $z_n = x_n + iy_n$  for all  $n$  and  $S = X + iY$ , then

$$\sum_{n=1}^{\infty} z_n = S$$

if and only if

$$\sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

One of the things that transfers over is the "test for divergence":

If  $\sum_{n=1}^{\infty} z_n$  converges, then  $\lim_{n \rightarrow \infty} z_n = 0$ .

Note that this is a one-way-only implication: the converse does not hold.

Example. For any  $z \in \mathbb{C}$  with  $|z| < 1$ ,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

To see this, first look at the  $N^{\text{th}}$  partial sum,

$$S_N = 1 + z + z^2 + \cdots + z^N.$$

For every  $z \in \mathbb{C}$ , we have

$$\begin{aligned} S_N - zS_N &= (1 + z + \cdots + z^N) - (z + \cdots + z^{N+1}) \\ &= 1 - z^{N+1}. \end{aligned}$$

This shows that

$$(1-z) S_N = 1 - z^{N+1}$$

for all  $z \in \mathbb{C}$ . Thus for all  $z \neq 1$ , we have

$$S_N = \frac{1 - z^{N+1}}{1 - z}.$$

Now we see that if  $|z| < 1$ , then

$$\left| \frac{z^{N+1}}{1-z} \right| = \frac{|z|^{N+1}}{|1-z|}$$

converges to 0, so

$$\lim_{N \rightarrow \infty} S_N = \frac{1}{1-z}.$$

Finally we come to Taylor series. We'll start with the theorem, look at some examples, and then prove the theorem.

Taylor's theorem. If the function  $f(z)$  is analytic throughout the disk  $|z - z_0| < R$  of radius  $R$  centered at  $z_0$ , then for all  $z$  in this disc, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

Note. We define  $f^{(0)}(z_0) = f(z_0)$  and  $0! = 1$ .

Some examples. For the most part, our examples are the same examples you saw in calculus... (all centered at  $z_0=0$ )

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for } |z| < 1,$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for all } z \in \mathbb{C},$$

(because  $e^z$  is entire)

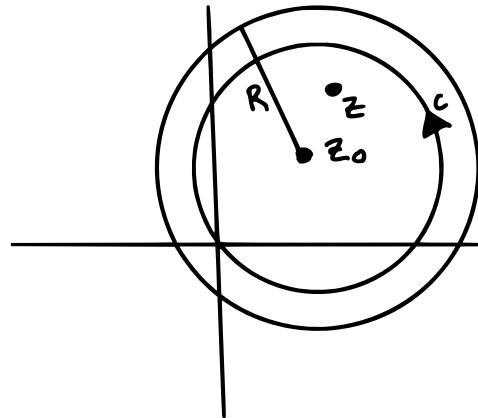
$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \text{for all } z \in \mathbb{C},$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad \text{for all } z \in \mathbb{C}.$$

## Proof of Taylor's theorem.

The idea of the proof is quite simple, but there is one step that is a bit of a pain.

Suppose that  $f(z)$  is analytic throughout the disk  $|z - z_0| < R$ . Fix a point  $z$  in this disk and let  $C$  denote any positively oriented circle centered at  $z_0$  with radius strictly between  $|z - z_0|$  and  $R$ , as shown above.



By the Cauchy integral formula, we know that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw.$$

All  $w$  in this integral lie on  $C$ , which has radius greater than  $|z - z_0|$ ,

so for all such  $w$  we have

$$|w - z_0| > |z - z_0|,$$

or,

$$\left| \frac{z - z_0}{w - z_0} \right| < 1.$$

Thus by our previous example, we can write

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} \\ &= \frac{1}{w-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}} \\ &= \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left( \frac{z-z_0}{w-z_0} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}}. \end{aligned}$$

Therefore, we have

$$f(z) = \frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} \left( \frac{f(w)}{(w-z_0)^{n+1}}, (z-z_0)^n \right) dw.$$

If we can interchange the integral and the sum (we'll justify that below), then

$$f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}} dw \right) (z-z_0)^n,$$

and then by the Cauchy integral formula for derivatives, this shows us that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n.$$

This completes the proof, pending the justification for why we can interchange the integral and the sum.

Why we can interchange the integral and the sum...

Again by our previous example, for every  $N \geq 1$  and  $z$  with  $|z| < 1$ ,

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z}.$$

Ignoring the constant out front,  
we are interested in

$$\int_C \sum_{n=0}^{\infty} \frac{f(w)}{w-z_0} \left( \frac{z-z_0}{w-z_0} \right)^n dw.$$

Since  $\left| \frac{z-z_0}{w-z_0} \right| < 1$ , for every  $N \geq 1$ , this is equal to

$$\begin{aligned} & \int_C \sum_{n=0}^{N-1} \frac{f(w)}{w-z_0} \left( \frac{z-z_0}{w-z_0} \right)^n dw \\ & + \boxed{\int_C \frac{f(w)}{w-z_0} \frac{\left( \frac{z-z_0}{w-z_0} \right)^N}{1 - \frac{z-z_0}{w-z_0}} dw}. \end{aligned}$$

↑  
let this be  
 $R_N(z)$ .

The first term above is the integral of a finite sum, so we can interchange the integral and summation there. Thus we're done if we can show that

$$\lim_{N \rightarrow \infty} R_N(z) = 0.$$

To do this, first we simplify,

$$R_N(z) = \int_C \frac{f(w)}{w-z} \cdot \left( \frac{z-z_0}{w-z_0} \right)^N dw.$$

Let

$$r = |z - z_0| \text{ and}$$

$s = \text{radius of } C,$

so by our construction of  $C$ ,  $r < s < R$ .

Also, for  $w \in C$ , we have

$$\begin{aligned} |w-z| &= |(w-z_0)-(z-z_0)| \\ &\geq |w-z_0| - |z-z_0| \\ &= s-r, \end{aligned}$$

which is positive.

Finally, let  $M$  denote any upper bound for  $|f(w)|$  for  $w \in C$ . Then for all  $w \in C$ , we have

$$\left| \frac{f(w)}{w-z} \cdot \left( \frac{z-z_0}{w-z_0} \right)^N \right| \leq \frac{M}{s-r} \left( \frac{r}{s} \right)^N.$$

Thus we get

$$|R_N(z)| = \left| \int_C \frac{f(w)}{w-z} \cdot \left( \frac{z-z_0}{w-z_0} \right)^N dw \right|$$

$$\leq (\text{length of } C) \cdot (\underset{\text{of integrand}}{\underset{\text{max modulus}}{\cdot}})$$
$$\leq 2\pi s \cdot \frac{M}{s-r} \left(\frac{r}{s}\right)^N.$$

Everything in this bound is constant except  $(r/s)^N$ , and we know  $r/s < 1$ . Therefore

$$\lim_{N \rightarrow \infty} R_N(z) = 0,$$

as desired.  $\square$