Lecture 35 – April 10, 2020

Last lecture we looked at Taylor series. We begin this lecture with a different sort of series.

Example. Find a series for $\frac{e^z}{z^3}$.

Solution. We have a series for e^z,

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

so why not just divide all the terms by z^3? That would give

$$\frac{e^z}{z^3} = z^{-3} + z^{-2} + \frac{1}{2}z^{-1} + \frac{1}{6} + \frac{1}{24}z + \cdots$$
$$= \sum_{n=-3}^{\infty} \frac{1}{(n+3)!} z^n.$$

In fact, since our series for e^z is valid everywhere (e^z is entire), this series for e^z/z^3 is valid everywhere $z \neq 0$, or equivalently, for all $|z| > 0$.

Example. Find a series for $\frac{1}{z - z^2}$ that is valid near the origin.

Solution. We have

$$\frac{1}{z - z^2} = \frac{1}{z} \cdot \frac{1}{1 - z}.$$

For $|z| < 1$, we know that

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n.$$

So for $0 < |z| < 1$, we have

$$\frac{1}{z - z^2} = \frac{1}{z} + 1 + z + \cdots = \sum_{n=-1}^{\infty} z^n.$$

These are both examples of **Laurent series**. A Laurent series is just a power series which is allowed to include **negative powers**.

If there are negative powers, then the series **won't** converge at its center (0 in both examples), but it might converge outside that.

Before we get into that, though, we ought to define what it even means for a doubly-infinite series to converge.

Definition. The doubly-infinite series $\sum_{n=-\infty}^{\infty} z_n$ **converges** if and only if the two series

$$\sum_{n=0}^{\infty} z_n \quad \text{and} \quad \sum_{n=1}^{\infty} z_{-n}$$

converge, and in that case,

$$\sum_{n=-\infty}^{\infty} z_n = \sum_{n=0}^{\infty} z_n + \sum_{n=1}^{\infty} z_{-n}.$$

Now consider a Laurent series centered at 0,

$$\sum_{n=-\infty}^{\infty} a_n z^n.$$

We've said this series converges if and only if

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

converge.

The first of these is a regular power series, so we expect it to converge inside some disk, say for

$$|z| < R_2.$$

For the second series, define $g(z)$ by

$$g(z) = \sum_{n=1}^{\infty} a_{-n} z^n.$$

We expect this series to also converge inside some disk, say for

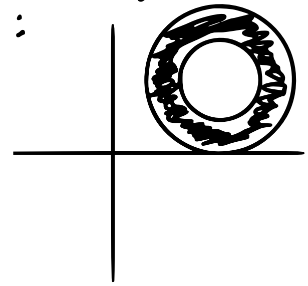
$$|z| < r.$$

Since our series is $g(1/z)$, it should therefore converge for

$$|1/z| < r,$$

or in other symbols, for $|z| > 1/r$. Setting $R_1 = 1/r$, we see that...

In general, Laurent series converge in an annulus (a 2-d donut):



In symbols, these annuli are defined as

$$\underbrace{R_1}_{\text{inner radius}} < |z - \underbrace{z_0}_{\text{center}}| < \underbrace{R_2}_{\text{outer radius}}.$$

As with Taylor series, we have formulae for the coefficients of Laurent series.

Laurent's theorem. Suppose $f(z)$ is analytic throughout the annulus $R_1 < |z-z_0| < R_2$, and let C denote any positively oriented simple closed contour enclosing z_0 and lying completely in this annulus. Then, for every point z in this annulus, we have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \quad \text{where for all } n \in \mathbb{Z},$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

Note. All contours satisfying the hypotheses give the same integrals by the principle of path deformation.

Note. If $f(z)$ is actually analytic in the whole disk $|z-z_0| < R_2$, then for all $n \leq -1$,

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{1}{2\pi i}\int_C f(z)\,(z-z_0)^{-n-1}\,dz.$$

Since $n \leq -1$, $-n-1 \geq 0$. That means that $f(z)\,(z-z_0)^{-n-1}$ is analytic in the disk $|z-z_0|<R_2$, so the integral above (and thus also a_n) is 0 by the Cauchy-Goursat theorem.

Moreover, the Cauchy integral formula tells us that for all $n \geq 0$,

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{f^{(n)}(z_0)}{n!}.$$

It follows that if $f(z)$ is actually analytic in the whole disk, then its Laurent series is the same as its Taylor series.

We won't prove Laurent's theorem, but we will sketch an argument that if a function has a Laurent series in some annulus, then the formula gives the correct coefficients (so the series is unique).

Suppose that we do have

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

for all points in the annulus $R_1 < |z-z_0| < R_2$.

Let k be an arbitrary integer and let C be any positively oriented simple closed contour enclosing z_0 and lying completely in this annulus.

We then have

$$\int_C \frac{f(w)}{(w-z_0)^{k+1}}\, dw = \int_C \sum_{n=-\infty}^{\infty} a_n (w-z_0)^{n-k-1}\, dw.$$

Like in our previous lecture, we can interchange the integral and the summation; unlike our previous lecture, we won't justify it this time.

Doing this interchange shows that

$$\int_C \frac{f(w)}{(w-z_0)^{k+1}}\, dw = \sum_{n=-\infty}^{\infty} \int_C a_n (w-z_0)^{n-k-1}\, dw.$$

Now we consider that integral. When $n=k$, this integral is

$$\int_C \frac{a_k}{(w-z_0)}\, dw = a_k \cdot 2\pi i$$

(as we have computed many times). Otherwise, when $n \neq k$, then the integrand $(w-z_0)^{n-k-1}$ has an antiderivative,

$$\frac{1}{n-k-1}(w-z_0)^{n-k}$$

in the domain $|w-z_0|>0$, so its integral along any closed contour in this domain is 0. (For $n>k$, the integrand is entire, actually.)

This shows that

$$\int_C \frac{f(w)}{(w-z_0)^{k+1}}\, dw = a_k \cdot 2\pi i,$$

so the coefficients are as the theorem says.

This uniqueness result shows that we don't have to compute Laurent series using the formula in Laurent's theorem (and we rarely do) — any way we can get a Laurent series is fine and will give us the same series.

Example. We already found a Laurent series for

$$f(z) = \frac{1}{z - z^2},$$

which was centered at $z_0 = 0$ and valid in the annulus $0 < |z| < 1$. But this function is <u>also</u> analytic in the "generalized annulus" $|z| > 1$. Find a Laurent series for $f(z)$ valid there.

Solution. Before we started with

$$f(z) = \frac{1}{z} \cdot \frac{1}{1 - z}.$$

This time, we need to use the geometric series

$$\frac{1}{1-(1/z)} = \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} z^n,$$

which is valid for $|1/z| < 1$, and thus for $|z| > 1$.

To use this, we manipulate f:

$$\begin{aligned}
f(z) &= \frac{1}{z}\cdot\frac{1}{1-z} \\
&= \frac{1}{z}\cdot\frac{1}{z}\cdot\frac{1}{(1/z)-1} \\
&= \frac{-1}{z^2}\cdot\frac{1}{1-(1/z)} \\
&= \frac{-1}{z^2}\sum_{n=-\infty}^{0} z^n \\
&= \sum_{n=-\infty}^{-2} -z^n,
\end{aligned}$$

which is valid for all $|z| > 1$.