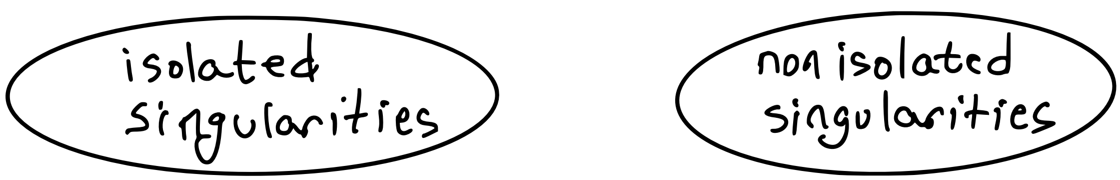
Lecture 37 — April 15, 2020

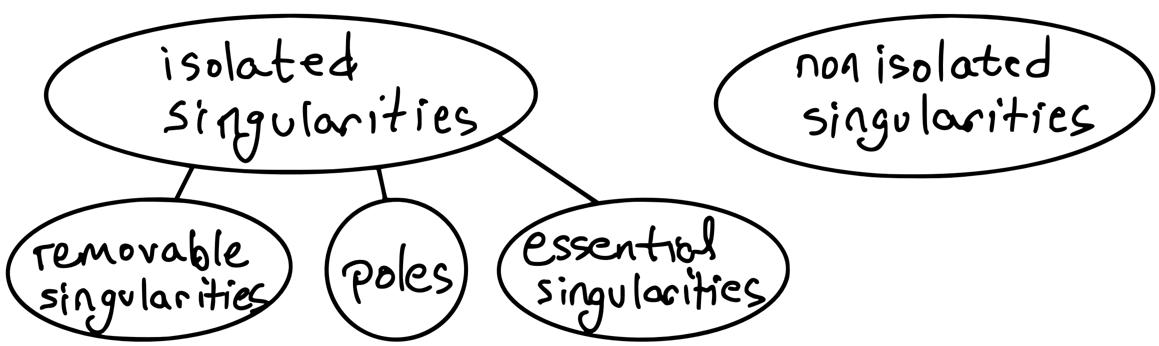
Covers sections 78, 79, 80, 81.

Last lecture we learned (some of) the power of residues. This lecture we get a nicer method to compute them. First, though, we need to talk a bit more about singularities.

We have divided singularities into two groups:



We will continue to ignore the nonisolated singularities, but will split the isolated singularities into three groups:



Definition. Suppose $f(z)$ has a singularity at z_0. Then we know that $f(z)$ has a Laurent series in some punctured disk $0 < |z - z_0| < \varepsilon$ centered at z_0.

z_0 is removable if this Laurent series has no negative powers, so for $0 < |z - z_0| < \varepsilon$,

$$f(z) = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots.$$

z_0 is a pole of order $m \geq 1$ if a_{-m} is the nonzero coefficient of lowest index, so for $0 < |z - z_0| < \varepsilon$,

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \cdots$$

z_0 is essential if there are infinitely many nonzero negative powers in this Laurent series.

Example. The function $\frac{\sin(z)}{z}$ has a removable singularity at $z_0 = 0$ because its Laurent series there is

$$\frac{\sin(z)}{z} = 1 - \frac{1}{6}z^2 + \frac{1}{120}z^4 - \frac{1}{5040}z^6 + \cdots.$$

Example. The function $\frac{\sin(z)}{z^7}$ has a pole of order 6 at $z_0 = 0$ because its Laurent series there is

$$\frac{\sin(z)}{z^7} = z^{-6} - \frac{1}{6}z^{-4} + \frac{1}{120}z^{-2} - \frac{1}{5040} + \cdots.$$

Example. The function $\sin(1/z)$ has an essential singularity at $z_0 = 0$ because its Laurent series there is

$$\sin\left(\frac{1}{z}\right) = z^{-1} - \frac{1}{6}z^{-3} + \frac{1}{120}z^{-5} - \frac{1}{5040}z^{-7} + \cdots.$$

Definition. A pole of order 1 is just called a **simple pole**.

If z_0 is a removable singularity of $f(z)$, then we can just... remove it.

More specifically, since we know that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $0 < |z-z_0| < \varepsilon$, we know that

$$\lim_{z \to z_0} f(z) = a_0.$$

If we now define

$$g(z) = \begin{cases} f(z) & \text{for } z \neq z_0, \\ \lim_{z \to z_0} f(z) & \text{for } z = z_0, \end{cases}$$

then we know that $g(z)$ is analytic at z_0 because for $|z| < \varepsilon$,

$$g(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

So removable singularities are boring.
What about poles?

Theorem. Suppose z_0 is an isolated singularity of $f(z)$. Then

z_0 is a pole of order $m \geq 1$

if and only if

$f(z) = \dfrac{g(z)}{(z-z_0)^m}$ for a function $g(z)$ that is analytic and nonzero at z_0.

Moreover, if these hold, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{g^{(m-1)}(z_0)}{(m-1)!}.$$

Recall. For $m=1$ we have $g^{(0)}(z_0) = g(z_0)$ and $0! = 1$, so in this case the residue of $f(z)$ at z_0 is simply $g(z_0)$.

Example. The residue of $f(z) = \dfrac{e^{\sin(z)}}{(z-i)\cos(z)}$ at $z_0 = i$ is

$$\operatorname*{Res}_{z=i} f(z) = \frac{e^{\sin(i)}}{\cos(i)}.$$

Proof. Suppose z_0 is a pole of order m for $f(z)$, so in some punctured disk $0<|z-z_0|<\varepsilon$, we have

$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n,$$

where $a_{-m} \neq 0$. Define the function $g(z)$ by

$$g(z) = \begin{cases} (z-z_0)^m f(z) & \text{for } z \neq z_0, \\ a_{-m} & \text{for } z = z_0. \end{cases}$$

Then for all z in the disk $|z|<\varepsilon$, we have

$$g(z) = \sum_{n=0}^{\infty} a_{n-m} (z-z_0)^n.$$

Since $g(z)$ has a Taylor series in this disk, it is analytic at z_0, and since $a_{-m} \neq 0$, $g(z)$ is nonzero at z_0.

Now suppose that

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

for some function $g(z)$ that is analytic and nonzero at z_0.

Since $g(z)$ is analytic at z_0, there is some disk $|z| < \varepsilon$ in which $g(z)$ has a Taylor series,

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z-z_0)^n.$$

Therefore for $0 < |z-z_0| < \varepsilon$, we have

$$f(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!} (z-z_0)^{n-m}.$$

Since $g(z_0) \neq 0$, the Laurent series above shows that z_0 is a pole of order m for $f(z)$.

Finally, the coefficient of $(z-z_0)^{-1}$ above is

$$\frac{g^{(m-1)}(z_0)}{(m-1)!}. \quad \square$$

Before we do examples, we prove one more result, which explains the name "pole".

Proposition. If $f(z)$ has a pole at z_0, then

$$\lim_{z \to z_0} f(z) = \infty.$$

Recall. Satisfying $\lim\limits_{z \to z_0} f(z) = \infty$ is, by definition, the same as satisfying

$$\lim_{z \to z_0} \frac{1}{f(z)} = 0.$$

Proof. Suppose z_0 is a pole of order m. Then

$$f(z) = \frac{g(z)}{(z-z_0)^m},$$

where $g(z)$ is analytic and nonzero at z_0. Thus

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z-z_0)^m}{g(z)}$$

$$= \frac{\lim_{z \to z_0} (z-z_0)^m}{g(z_0)}$$

$$= 0. \ \square$$

For the rest of the lecture, we do examples.

Example. Compute the residue of

$$f(z) = \frac{1-\cos(z)}{z^3}$$

at the point $z_0 = 0$.

Solution. We can't take $m=3$ and $g(z) = 1-\cos(z)$ in the theorem, because we need $g(z_0) \neq 0$.

In this case it is probably easiest just to expand the series as we did in the previous lecture:

$$\frac{1-\cos(z)}{z^3} = \frac{1-\left(1+\frac{1}{2}z^2-\frac{1}{24}z^4+\cdots\right)}{z^3}$$

$$= \frac{1}{2}z^{-1} - \frac{1}{24}z + \cdots,$$

So the residue at 0 is $1/2$. Note that 0 is a simple pole of $f(z)$, so we would have needed to take $m=1$ to use the theorem.

Example. Compute

$$\int_C \frac{17z^2+1}{(z-1)(z^2+9)}\,dz,$$

where C is the positively oriented circle $|z-2|=2$.

Solution. Our integrand has isolated singularities at 1 and $\pm 3i$. Since only one of these is enclosed by C, the answer will be

$$2\pi i \cdot \operatorname*{Res}_{z=1} f(z),$$

where $f(z)$ is our integrand. We can apply the theorem above with $m=1$ and

$$g(z) = \frac{17z^2+1}{z^2+9}$$

(which is analytic and nonzero at 1) to see that the residue is $18/10 = 9/5$, so the value of the integral is $\boxed{\dfrac{18\pi i}{5}}$

Example. Compute the residue of

$$f(z) = \frac{e^{\pi z}}{z^2+1}$$

at $z_0 = i$.

Solution. We have

$$\frac{e^{\pi z}}{z^2+1} = \frac{e^{\pi z}}{(z-i)(z+i)},$$

so the theorem shows that

$$\operatorname*{Res}_{z=i} f(z) = \frac{e^{\pi i}}{2i} = -\frac{1}{2i}.$$

Example. Compute the residue of

$$f(z) = \frac{\mathrm{Log}(z)}{(z-3)^6}$$

at the point $z_0 = 3$.

Solution. Since $z_0 = 3$ is a 6^{th} order pole, we need to compute the 5^{th} derivative of $\mathrm{Log}(z)$. We get

$$\begin{aligned} f'(z) &= z^{-1}, \\ f''(z) &= -z^{-2}, \\ f'''(z) &= 2z^{-3}, \\ f^{(4)}(z) &= -6z^{-4}, \\ f^{(5)}(z) &= 24z^{-5}. \end{aligned}$$

So the theorem says that

$$\operatorname*{Res}_{z=3} f(z) = \frac{24 \cdot 3^{-5}}{5!} = \frac{1}{5 \cdot 3^5}.$$