

Lecture 38 — April 17, 2020

Covers section 76,
with some examples
from section 81.

We have finally made it to Cauchy's residue theorem, which tells us how to compute the integral of an otherwise analytic function over a simple closed contour that encloses only finitely many isolated singularities.

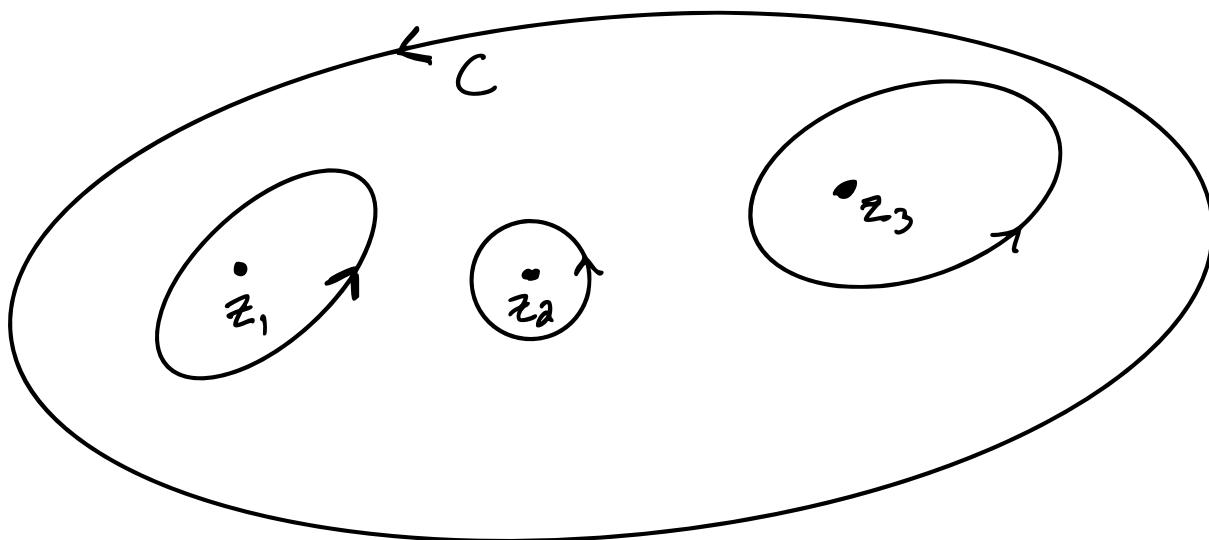
The residue theorem. Suppose that C is a positively oriented, simple closed contour, and that $f(z)$ is analytic inside and on C , except possibly at the finitely many points z_1, \dots, z_n , all of which lie inside (not on) C . Then,

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z).$$

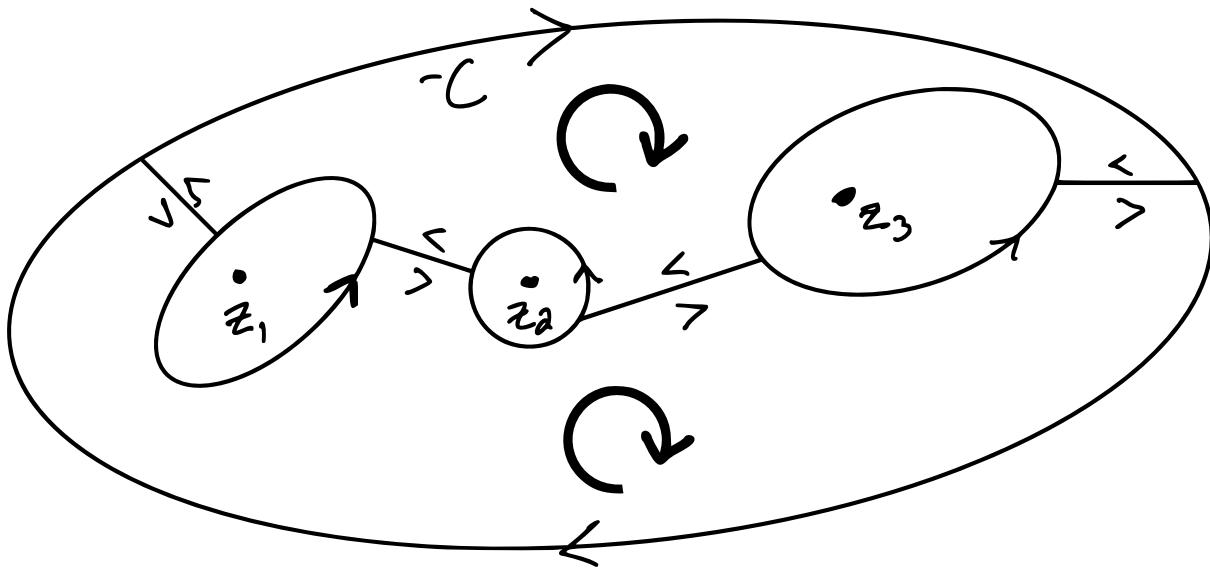
Note that the residue theorem generalizes the Cauchy-Goursat theorem (if there are no singularities, then there is nothing to sum, so the integral is 0). It also generalizes the Cauchy integral formula.

Proof. Our proof uses cross-cuts like we used when we proved the principle of path deformation.

Let the points z_1, \dots, z_n be given. For each index k , let C_k be a (small) positively oriented contour enclosing z_k . Choose these small enough so that C_1, \dots, C_n do not intersect. We then have the picture below.



Next consider $-C$, and a cross cuts connecting $-C$ to C_1 , then C_1 to C_2 , etc. (We can change our indexing of the C_k if we need to.) We get the following.



We have split everything into a "top half" and a "bottom half." Each of these halves contains half of $-C$, C_1, \dots, C_k , and one orientation of each of our cross cuts. Our function is analytic on each half, so

$$\int_{\substack{\text{top} \\ \text{half}}} f(z) dz = \int_{\substack{\text{bottom} \\ \text{half}}} f(z) dz = 0,$$

by the Cauchy-Goursat theorem. Now if we add these two integrals, the integrals along the cross cuts cancel, and for the other contours we get the full integral.

This means that we get

$$\begin{aligned}0 &= \int_{\text{top half}} f(z) dz + \int_{\text{bottom half}} f(z) dz \\&= \int_{-C} f(z) dz + \sum_{k=1}^n \int_{C_k} f(z) dz.\end{aligned}$$

Therefore,

$$\int_C f(z) dz = \sum_{k=1}^n \int_{C_k} f(z) dz.$$

Finally, we know that

$$\int_{C_k} f(z) dz = 2\pi i \operatorname{Res}_{z=z_k} f(z),$$

so

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \operatorname{Res}_{z=z_k} f(z),$$

completing the proof. \square

Now for examples...

Example 1. Suppose C is a simple closed contour not passing through $\pm i$. What are the possible values of $\int_C \frac{z+4}{z^2+1} dz$?

Solution. Our integrand has singularities at $\pm i$. The contour could enclose zero, one, or both of these.

Since

$$\frac{z+4}{z^2+1} = \frac{\left(\frac{z+4}{z+i}\right)}{z-i},$$

where $\frac{z+4}{z+i}$ is analytic and nonzero at i , we see from the previous lecture that

$$\operatorname{Res}_{z=i} \frac{z+4}{z^2+1} = \frac{i+4}{2i} = \frac{1}{2} - 2i.$$

Similarly,

$$\operatorname{Res}_{z=-i} \frac{z+4}{z^2+1} = \frac{-i+4}{-2i} = \frac{1}{2} + 2i.$$

If C encloses neither i nor $-i$, then the integral is 0. (Cauchy-Goursat.)

If C encloses i but not $-i$, then the integral is

$$2\pi i \left(\frac{1}{2} - 2i \right) = \pi i + 4\pi.$$

(Note that we could have done this before...)

If C encloses $-i$ but not i , then the integral is

$$2\pi i \left(\frac{1}{2} + 2i \right) = \pi i - 4\pi.$$

If C encloses both i and $-i$, then the integral is

$$2\pi i \left(\frac{1}{2} - 2i + \frac{1}{2} + 2i \right) = 2\pi i.$$

Example 2. Find

$$\int_C \frac{dz}{z^3(z+4)},$$

where C is the positive orientation of the circle $|z|=2$.

Solution. Our integrand has singularities at 0 and -4. We only need to consider 0 because -4 lies outside C .

We can express $\frac{1}{z+4}$ as a geometric series to find this residue:

$$\frac{1}{z+4} = \frac{1}{4} \cdot \frac{1}{(\frac{z}{4})+1} = \frac{1}{4} \frac{1}{1 - (-\frac{z}{4})} = \sum_{n=0}^{\infty} \frac{1}{4}(-1)^n \left(\frac{z}{4}\right)^n,$$

so the residue at 0 is the coefficient of z^2 above, $(\frac{1}{4})^3 = \frac{1}{64}$.

It follows that the integral is

$$2\pi i \cdot \frac{1}{64} = \frac{\pi i}{32}.$$

We could also have used our theorem from last time.

Example 3. Find

$$\int_C \frac{dz}{z^3(z+4)},$$

where C is the positive orientation of the circle $|z+2|=4$.

Solution. Now we have to worry about that singularity at -4 . For this we'll use our theorem from the last lecture.

Since $\frac{1}{z^3(z+4)} = \frac{(1/z^3)}{z+4}$, where $1/z^3$ is analytic and nonzero at -4 , this is a simple pole and our residue there is

$$\frac{1}{(-4)^3} = -\frac{1}{64}.$$

It follows that the integral is

$$2\pi i \left(\frac{1}{64} + \frac{-1}{64} \right) = 0.$$