Contents

1. Review of Sets and Functions

It is assumed that the reader is familiar with the most basic set constructions and functions and knows the natural numbers, integers and rational numbers

$$\begin{aligned}\mathbb{N} =&\{0, 1, 2, \dots\}\\ \mathbb{Z} =&\{0, \pm1, \pm2, \dots\}\\ \mathbb{Q} =&\{m/n : m \in \mathbb{Z}, \quad n \in \mathbb{N}^+\}\end{aligned}$$

respectively as well as the real numbers $\mathbb{R}$, though we will carefully review the least upper bound property of $\mathbb{R}$ in Section 2. Let $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ denote the positive integers.

Familiarity with matrices $M_n(\mathbb{R})$ and $M_{m,n}(\mathbb{R})$ is also assumed. The complex numbers $\mathbb{C}$ will also appear in these course notes.

1.1. **Unions, Intersections, Complements, and Products.**

Definition 1.1. Given sets $X, Y \subseteq S$, the *union* and *intersection* of X and Y are

$$\begin{aligned}X \cup Y =&\{z \in S : z \in X \text{ or } z \in Y\} \subseteq S\\ X \cap Y =&\{z \in S : z \in X \text{ and } z \in Y\} \subseteq S,\end{aligned}$$

respectively.

The *complement* of X, denoted $\tilde{X}$, is the set

$$\tilde{X} = \{x \in S : x \notin X\}.$$

The *relative complement* of X in Y is

$$Y \setminus X = Y \cap \tilde{X} = \{z \in S : z \in Y \text{ and } z \notin X\}.$$

In particular, $\tilde{X} = S \setminus X$. ◁

Definition 1.2. Let X and Y be sets. The *Cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) : x \in X, \quad y \in Y\}.$$

◁

Example 1.3. The set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is known as the Cartesian plane.

The set $\mathbb{R}^3$ is the 3-dimensional Euclidean space of third semester Calculus. △

Definition 1.4. Given a set S, let $P(S)$ denote the *power set* of S, the set of all subsets of S. ◁

Example 1.5. Let $S = \{0, 1\}$. Then,

$$P(S) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.$$

As we shall see later, $P(\mathbb{N})$ is a large set. △

Definition 1.6. Let S be a given set. The *union* and *intersection* of the collection $\mathcal{F} \subseteq P(S)$ are

$$\begin{aligned}\bigcup_{A\in\mathcal{F}} A =& \{x \in S : \text{ there is a } A \in \mathcal{F} \text{ such that } x \in A\}\\ \bigcap_{A\in\mathcal{F}} A =& \{x \in S : x \in A \text{ for every } A \in \mathcal{F}\},\end{aligned}$$

respectively. ◁

For example, let $\mathcal{F} = \{A \subseteq \mathbb{N} : 0 \in A\}$. In this case,

$$\bigcap_{A\in\mathcal{F}} = \{0\}$$

and

$$\bigcup_{A\in\mathcal{F}} = \mathbb{N}.$$

Do Problem 1.1.

1.2. **Functions.**

Definition 1.7. A *function* f is a triple (f, A, B) where A and B are sets and f is a rule which assigns to each $a \in A$ a unique $b = f(a)$ in B. We write

$$f : A \to B.$$

(i) The set A is the *domain* of f.
(ii) The set B is the *codomain* of f
(iii) The *range* of f, sometimes denoted $\operatorname{rg}(f)$, is the set $\{f(a) : a \in A\}$.
(iv) The function $f : A \to B$ is *one-one* if $x, y \in A$ and $x \neq y$ implies $f(x) \neq f(y)$.
(v) The function $f : A \to B$ is *onto* if for each $b \in B$ there exists an $a \in A$ such that $b = f(a)$; i.e., if $\operatorname{rg}(f) = B$.
(vi) A function which is both one-one and onto is called a *bijection*.
(vii) The *graph* of f is the set

$$\operatorname{graph}(f) = \{(a, f(a)) : a \in A\} \subseteq A \times B.$$

(viii) If $f : A \to B$ and $Y \subseteq B$, the *inverse image* of Y under f is the set

$$f^{-1}(Y) = \{x \in A : f(x) \in Y\}.$$

(ix) If $f : A \to B$ and $C \subseteq A$, the set

$$\begin{aligned}f(C) =& \{f(c) : c \in C\}\\ =& \{b \in B : \text{ there is an } c \in C \text{ such that } b = f(c)\}\end{aligned}$$

is the *image of C under f*. Note, $\operatorname{rg}(f) = f(A)$.
(x) The *identity function* on a set A is the function $id_A : A \to A$ with rule $id_A(x) = x$.

◁

Example 1.8. Often one sees functions specified by giving the rule only, leaving the domain implicitly understood (and the codomain unspecified), a practice to be avoided. For example, given $f(x) = x^2$ it is left to the reader to guess that the domain is the set of real numbers. But it could also be $\mathbb{C}$ or even $M_n(\mathbb{C})$, the $n \times n$ matrices with entries from $\mathbb{C}$. If the domain is taken to be $\mathbb{R}$, then $\mathbb{R}$ is a reasonable choice of codomain. However, the range of f is $[0, \infty)$ (a fact which will be carefully proved later) and so the codomain could be any set containing $[0, \infty)$. The moral is that it is important to specify both the domain and codomain as well as the rule when defining a function. △

Example 1.9. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Note that f is neither one-one nor onto.

As an illustration of the notion of inverse image, $f^{-1}((4, \infty)) = (-\infty, 2) \cup (2, \infty)$ and $f^{-1}((-2, -1)) = \emptyset$. △

Example 1.10. The function $g : \mathbb{R} \to [0, \infty)$ defined by $g(x) = x^2$ is not one-one, but it is, as we'll see in Subsection 2.2, onto.

The function $h : [0, \infty) \to [0, \infty)$ defined by $h(x) = x^2$ is both one-one and onto. Note $h^{-1}((4, \infty)) = (2, \infty)$. △

Do Exercise 1.1 and Problem 1.2.

Remark 1.11. Given a set S, an (index) set I and a function $\alpha : I \to P(S)$, let $A_i = \alpha(i)$. Let $\mathcal{F}$ denote the range of α. In this context, the union and intersection over $\mathcal{F}$ are written,

$$\bigcup_{i \in I} A_i = \bigcup_{A \in \mathcal{F}} A$$

and

$$\bigcap_{i \in I} A_i = \bigcap_{A \in \mathcal{F}} A$$

respectively. See Exercise 1.3.

Finally, in the case that the index set is $\mathbb{N}$ or $\mathbb{Z}$, it is customary to write,

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n=0}^{\infty} A_n$$

and

$$\bigcup_{n \in \mathbb{Z}} A_n = \bigcup_{n=-\infty}^{\infty} A_n.$$

◇

Definition 1.12. Given sets A, B and X, Y and functions $f : A \to X$ and $g : B \to Y$, define $f \times g : A \times B \to X \times Y$ by $f \times g(a, b) = (f(a), g(b))$. ◁

Example 1.13. If $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(n) = 2n$ and $g : \mathbb{Z} \to \mathbb{N}$ is defined by $g(m) = 3m^2$, then $f \times g : \mathbb{N} \times \mathbb{Z} \to \mathbb{N} \times \mathbb{N}$ is given by $f \times g(n, m) = (2n, 3m^2)$. △

Definition 1.14. Given $f : A \to B$ and $C \subseteq A$, *the restriction of f to C* is the function $f|_C : C \to B$ defined by $f|_C(x) = f(x)$ for $x \in C$. ◁

Definition 1.15. Given $f : X \to Y$ and $g : Y \to Z$ the *composition* of g and f is the function $g \circ f : X \to Z$ with rule $g \circ f(x) = g(f(x))$. ◁

Proposition 1.16. *Suppose (X and Y are nonempty and)* $f : X \to Y$.

(1) There exists a function $g : Y \to X$ such that $f \circ g = id_Y$ if and only if f is onto;
(2) There exists a function $g : Y \to X$ such that $g \circ f = id_X$ if and only if f is one-one;
(3) There exists a function $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$ if and only if f is one-one and onto;
(4) There exists at most one function $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$.

†

Proof. In each of items (i), (ii) and (iii) the existence of g is easily seen to imply the conclusion on f. Now suppose that f is onto. For $y \in Y$ the set $S_y = f^{-1}(Y)$ is nonempty. For each y choose $x_y \in S_y$ (using the axiom of choice). In this way we obtain a function $g : Y \to X$ defined by $g(y) = x_y$ satisfying $f \circ g = id_Y$. Now suppose f is one-one. Fix $a \in X$. Define $g : Y \to X$ as follows: given $y \in Y$, if there is an $x \in X$ such that $f(x) = y$, then, as f is one-one, there is only one such x. In this case, let $g(y) = x$. Otherwise define $g(y) = a$.

By items (i) and (ii), if there exists a g such that $g \circ f = id_X$ and $f \circ g = id_Y$, then f is one-one and onto. Now suppose that f is one-one and onto. By what has already been proved, there exist functions $g, h : Y \to X$ such that $f \circ g = id_Y$ and $h \circ f = id_X$. It follows that

$$g = h \circ f \circ g = h$$

proving both (iii) and (iv). □

Definition 1.17. *A function $f : X \to Y$ is* invertible *if there is a function $g : Y \to X$ such that*

$$\begin{aligned} g \circ f &= id_X \\ f \circ g &= id_Y. \end{aligned}$$

We call g the inverse *of f, written $g = f^{-1}$.* ◁

Remark 1.18. If $f : X \to Y$ is invertible, and $B \subseteq Y$, then $f^{-1}(B)$ could refer to either the inverse image of B under f, or the image of B under the function f^{-1}. Happily, these two sets are the same. ◇

Example 1.19. The function $h : [0, \infty) \to [0, \infty)$ given by $h(x) = x^2$ of Example 1.10 is one-one and onto (as we will see later in Proposition 2.13) and thus has an inverse. Of course this inverse $h^{-1} : [0, \infty) \to [0, \infty)$ is commonly denoted as $\sqrt{\ }$ so that $h^{-1}(x) = \sqrt{x}$. △

1.3. **Finite and Countable Sets.**

Definition 1.20. Two sets A and B are *equivalent*, denoted $A \sim B$ if there is a one-one onto function $f : A \to B$. ◁

Observe that $\sim$ is behaves like an equivalence relation; i.e., $A \sim A$; if $A \sim B$, then $B \sim A$; and finally if $A \sim B$ and $B \sim C$, then $A \sim C$.

Given a positive integer n, let J_n denote the set $\{1, 2, \dots, n\}$. To show that J_n is not equivalent to $\mathbb{N}$ note, if $f : J_n \to \mathbb{N}$, then $f(j) \le \sum_{\ell=1}^{n} f(\ell)$ for each $j \in J_n$ and so f is not onto.

Definition 1.21. Let A be a set.

(i) A is *finite* if it is either empty or there is an $n \in \mathbb{N}^+$ such that $A \sim J_n$;
(ii) A is *infinite* if it is not finite;
(iii) A is *countable* if $A \sim \mathbb{N}$;
(iv) A is *at most countable* if A is either finite or countable; and
(v) A is *uncountable* if it is not at most countable.

◁

Remark 1.22. Note, by the comments preceding the definition, that $\mathbb{N}$ is infinite. ◇

Proposition 1.23. *A set A is at most countable if and only if there exists a one-one mapping $f : A \to \mathbb{N}$.*

Likewise A is at most countable if and only if there exists an onto mapping $g : \mathbb{N} \to A$. †

You may wish to compare Proposition 1.23 with the Cantor–Schroeder–Bernstein Theorem (which is also goes by the names Schroeder–Bernstein Theorem and Cantor–Bernstein Theorem). See also Problem 1.3. Note that the proposition implies that a subset of an at most countable set is at most countable.

Sketch of proof. We may assume A is not finite. Let $B = f(A) \subseteq \mathbb{N}$. Since f is one-one, A and $B \subseteq \mathbb{N}$ are equivalent. Hence, it suffices to prove if $B \subseteq \mathbb{N}$ is infinite, then $B \sim \mathbb{N}$. To this end, note that since B is a nonempty subset of $\mathbb{N}$ it has a smallest element, n_0. Recursively define, for $k \in \mathbb{N}^+$,

$$n_k = \min(B \setminus \{n_0, \dots, n_{k-1}\}),$$

where the assumption that B is not finite is used so that the minimum is taken over a nonempty subset of $\mathbb{N}$. By construction the mapping $h : \mathbb{N} \to B$ defined by $h(k) = n_k$ is one-one. On the other hand, since $n_k \ge k$, given $m \in B$ the set $\{j : n_j \ge m\}$ is nonempty. Its smallest element k satisfies $n_k = m$. Hence h is onto and $B \sim \mathbb{N}$.

If $g : \mathbb{N} \to A$ is onto, then, by Problem 1.3 (which is a simple consequence of Proposition 1.16), there is a one-one mapping $f : A \to \mathbb{N}$ and from what is already proved $A \sim \mathbb{N}$. □

Proposition 1.24. *The sets $\mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$, and $\mathbb{Q}$ are all at most countable.* †

Proof. Verify that $f : \mathbb{N} \to \mathbb{Z}$ by $f(2m) = m$ and $f(2m+1) = -m-1$ is one-one and onto.

To prove $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$ first note that the function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by

$$F(m,n) = 2^n \, 3^m$$

is one-one, a fact which follows from prime factorization (the Fundamental Theorem of Arithmetic). Hence, by Proposition 1.23, $\mathbb{N} \times \mathbb{N}$ is at most countable.

Since $\mathbb{N}\times\mathbb{N}$ is at most countable, there exists an onto map $g : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$. Consequently the composition $(f \times id_{\mathbb{N}}) \circ g : \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ is onto. Thus, using Problem 1.3 and Proposition 1.23, to prove that $\mathbb{Q}$ is at most countable, it suffices to exhibit an onto mapping $h : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$, since then $h \circ (f \times id_{\mathbb{N}}) \circ g$ maps $\mathbb{N}$ onto $\mathbb{Q}$. Define h by $h(m,n) = \frac{m}{n+1}$. □

Remark 1.25. It is easy to see that $\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}$ are not finite. Hence both are countable.

An alternative argument that $\mathbb{N} \times \mathbb{N}$ is equivalent to $\mathbb{N}$ is to view the set $\mathbb{N} \times \mathbb{N}$ as an array $\{(m,n) : m,n \in \mathbb{N}\}$ and define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ by $f(0) = (0,0)$, $f(1) = (1,0)$, $f(2) = (0,1)$, $f(3) = (2,0)$ etc. It is easy to believe, but a bit painful to write a formula for f and check it is one-one and onto. ◇

Do Problems 1.4 and 1.6

Proposition 1.26. *The set $P(\mathbb{N})$ is not countable.* †

The proof is accomplished using Cantor's diagonalization argument.

Proof. It suffices to prove, if $f : \mathbb{N} \to P(\mathbb{N})$, then f is not onto.

Given such an f, let

$$B = \{n \in \mathbb{N} : n \notin f(n)\}.$$

We claim that B is not in the range of f. Arguing by contradiction, suppose $m \in \mathbb{N}$ and $f(m) = B$. If $m \notin B$, then $m \in f(m) = B$ a contradiction. On the other hand, if $m \in B$, then $m \notin f(m) = B$, also a contradiction. □

Do Problem 1.7.

1.4. Worked Examples.

Sample Problem 1.1. Suppose $f : X \to S$ and $D \subseteq S$. Show, $f(f^{-1}(D)) \subseteq D$ and give an example where strict inclusion holds.

Solution. Let $y \in f(f^{-1}(D))$ be given. By definition of image of a set under a function, there exists an $x \in f^{-1}(D)$ such that $y = f(x)$. By definition of inverse image, $x \in f^{-1}(D)$ means $f(x) \in D$. Thus $y \in D$ and the desired inclusion has been established.

To give an example where $f(f^{-1}(D)) \neq D$ (strict inclusion holds), let $X = \{0\}$, $S = \{0,1\}$, define $f : X \to S$ by $f(0) = 0$ and finally let $D = S$. In this case $f(f^{-1}(D)) = f(\{0\}) = \{0\} \neq D$.

Sample Problem 1.2. Suppose $f : X \to Y$ and $\mathcal{F} \subseteq P(X)$. Show,

$$f(\cup_{A\in\mathcal{F}} A) = \cup_{A\in\mathcal{F}} f(A).$$

Solution. A point y is in $f(\cup_{A\in\mathcal{F}} A)$ if and only if there is an $x \in \cup_{A\in\mathcal{F}} A$ such that $y = f(x)$ if and only if there is $B \in \mathcal{F}$ such that $x \in B$ and $y = f(x)$ if and only if there exists a $B \in \mathcal{F}$ such that $y \in f(B)$ if and only if $y \in \cup_{A\in\mathcal{F}} f(A)$.

1.5. Exercises.

Exercise 1.1. Define $f : \mathbb{R} \to \mathbb{R}^2$ by

$$f(x) = (\cos(x), \sin(x)).$$

Let

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

and

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 1\}.$$

Identify

(i) $f^{-1}(S)$;
(ii) $f^{-1}(\mathbb{D})$; and
(iii) $f^{-1}(f((-\frac{\pi}{2}, \frac{\pi}{2})))$.

Exercise 1.2. Consider the function $h = f \times g$ of Example 1.13 and let $6\mathbb{N}$ denote the set $\{6k : k \in \mathbb{N}\}$. Find the inverse image of the set $\{(j, k) : j \in \{2, 3, 4\}, \quad k \in 6\mathbb{N}\}$. Find the inverse image of the set $\{(j, k) : j \in \{0, 1, 2\}, \quad k \text{ is odd}\}$.

Exercise 1.3. Consider the sets $A_n = \{m \in \mathbb{N} : m \geq n\} \subseteq \mathbb{N}$ for $n \in \mathbb{N}$. Determine,

$$\cup_{n\in\mathbb{N}} A_n \quad \text{and} \quad \cap_{n\in\mathbb{N}} A_n.$$

1.6. Problems.

Problem 1.1. Suppose S is a set and $\emptyset \neq \mathcal{F} \subseteq P(S)$. Show

$$\widetilde{\cup_{A\in\mathcal{F}} A} = \cap_{A\in\mathcal{F}} \tilde{A}.$$

Problem 1.2. Suppose $f : X \to S$ and $\mathcal{F} \subseteq P(S)$. Show,

$$f^{-1}(\cup_{A\in\mathcal{F}} A) = \cup_{A\in\mathcal{F}} f^{-1}(A)$$
$$f^{-1}(\cap_{A\in\mathcal{F}} A) = \cap_{A\in\mathcal{F}} f^{-1}(A)$$

Show, if $A, B \subseteq X$, then $f(A \cap B) \subseteq f(A) \cap f(B)$. Give an example, if possible, where strict inclusion holds.

Show, if $C \subseteq X$, then $f^{-1}(f(C)) \supseteq C$. Give an example, if possible, where strict inclusion holds.

Problem 1.3. Let A be a nonempty set. Prove there exists an onto mapping $f : \mathbb{N} \to A$ if and only if there is a one-one mapping $g : A \to \mathbb{N}$.

Problem 1.4. Prove that an at most countable union of at most countable sets is at most countable; i.e., if S is a set, $\alpha : \mathbb{N} \to P(S)$ is a function such that each $A_j = \alpha(j)$ is at most countable, then

$$T = \cup_{j=0}^{\infty} A_j := \cup_{j \in \mathbb{N}} A_j$$

is at most countable.

Suggestion: For each j there is a function $g_j : \mathbb{N} \to A_j$. Define a function $F : \mathbb{N} \times \mathbb{N} \to T$ by $F(j,k) = g_j(k)$. Proceed.

Problem 1.5. Use induction to show, for $n \in \mathbb{N}^+$, that $P(J_n) \sim J_{2^n}$.

Problem 1.6. Show that the collection $\mathcal{F} \subseteq P(\mathbb{N}^+)$ of finite subsets of $\mathbb{N}$ is an at most countable set.

Problem 1.7. Suppose A is a non-empty set. Show there does not exist an onto mapping $f : A \to P(A)$; i.e., show $A \not\sim P(A)$.

Problem 1.8. Let A be a given nonempty set. Show, $2^A = \{f : A \to \{0,1\}\}$ is equivalent to $P(A)$.

2. The Real Numbers

We will take the view that we know what the real numbers are and we will simply *review* some important properties in this section.

Example 2.1. The square root of 2 is not rational; i.e., there is no rational number $s > 0$ such that $s^2 = 2$. △

2.1. The Least Upper Bound Property.

Definition 2.2. Let S be a subset of $\mathbb{R}$.

(i) The set S is *bounded above* if there is a $b \in \mathbb{R}$ such that $b \geq s$ for all $s \in S$.
(ii) Any $b \in \mathbb{R}$ such that $b \geq s$ for all $s \in S$ is an *upper bound* for S.

◁

Example 2.3. Informally identify the set of upper bounds for the the subsets of $\mathbb{R}$ in items (i), (ii) and (iv).

(i) $[0, 1)$;
(ii) $[0, 1]$;
(iii) $\mathbb{Q}$;
(iv) $\emptyset$.

For $\mathbb{Q}$, the natural guess is that there are no upper bounds, but the justification for this assertion needs to wait for Theorem 2.10. (See Remark 2.11.) △

Lemma 2.4. *Let S be a subset of $\mathbb{R}$ and suppose both b and b' are upper bounds for S. If b and b' both have the property that if $c \in \mathbb{R}$ is an upper bound for S, then $c \geq b$ and $c \geq b'$, then $b = b'$.* †

Definition 2.5. The *least upper bound* of a subset S of $\mathbb{R}$, if it exists, is a $b \in \mathbb{R}$ such that

(i) b is an upper bound for S; and
(ii) if $c \in \mathbb{R}$ is an upper bound for S, then $c \geq b$.

◁

Remark 2.6. Lemma 2.4 justifies the use of *the* (as opposed to *an*) in describing the least upper bound.

The condition (ii) can be replaced with either of the following conditions

(ii)′ if $c < b$, then there exists an $s \in S$ such that $c < s$; or
(ii)″ for each $\epsilon > 0$ there is an $s \in S$ such that $b - \epsilon < s$.

The notions of *bounded below, lower bound* and *greatest lower bound* are defined analogously. A set $S \subseteq \mathbb{R}$ is *bounded* if it bounded both above and below.

Least upper bound is often abbreviated lub. The term *supremum*, often abbreviated *sup*, is synonymous with lub. Likewise *glb* and *inf* for greatest lower bound and infimum. ◇

Example 2.7. Here is a list of examples.

(i) The least upper bound of $S = [0, 1) \subseteq \mathbb{R}$ is 1.
(ii) The least upper bound of $V = [0, 1] \subseteq \mathbb{R}$ is also 1.
(iii) The set $\mathbb{Q} \subseteq \mathbb{R}$ has no upper bound (see the comment in Example 2.3) and thus no least upper bound;
(iv) Every real number is an upper bound for the set $\emptyset \subseteq \mathbb{R}$. Thus $\emptyset$ has no least upper bound.
(v) Let $S = \{x \in \mathbb{Q} : 0 < x, \quad x^2 < 2\} \subseteq \mathbb{R}$. Since $1 \in S$, the set S is nonempty. It is easily seen that S is bounded above by 2.

△

Theorem 2.8. *Every nonempty subset of* $\mathbb{R}$ *which is bounded above has a least upper bound.*

In fact more is true as you likely learned in a previous course. Namely, that $\mathbb{R}$ is the unique, up to ordered field isomorphism, ordered field with the property that every nonempty subset which is bounded above has a least upper bound.

Remark 2.9. Returning to Example 2.7(v), Theorem 2.8 implies S has a least upper bound s and $2 > s \geq 1 > 0$. It takes effort to show $s^2 = 2$ and hence s is the (positive) square root of 2. A more general result is stated and proved as Proposition 2.13. ⋄

Do Exercise 2.3 and Problem 2.1.

2.2. **Archimedean Properties and Square Roots.**

Theorem 2.10 (*Archimedean properties*)**.** *Suppose* $x, y \in \mathbb{R}$.

(i) There is a natural number n *so that* $n > x$.
(ii) If $1 < x - y$, *then there is an integer* m *so that* $y < m < x$.
(iii) If $y < x$, *then there is a* $q \in \mathbb{Q}$ *such that* $y < q < x$.

Remark 2.11. The last part of the theorem is sometimes expressed as saying $\mathbb{Q}$ is *dense* in $\mathbb{R}$.

Item (i) can be restated as $\mathbb{N}$ is not bounded above. Since $\mathbb{N} \subseteq \mathbb{Q}$, it follows that $\mathbb{Q}$ is not bounded above, justifying the claim made in Example 2.3(iii). Likewise, $\mathbb{R}$ is not bounded below, thus justifying the claim in Example 2.7(iv). ⋄

Proof. We prove (i) by arguing by contradiction. Accordingly, suppose no such natural number exists. In that case x is an upper bound for $\mathbb{N}$. It follows that $\mathbb{N}$ has a lub α. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$. Hence $n + 1 \leq \alpha$ and thus $n \leq \alpha - 1$ for all $n \in \mathbb{N}$. Consequently, $\alpha - 1$ is an upper bound for $\mathbb{N}$, contradicting the least property of α. Hence $\mathbb{N}$ is not bounded above and there is an $n > x$, which proves item (i).

To prove (ii), it suffices to assume that $x > 0$ (why). By (i), the set $\{k \in \mathbb{N} : k \geq x\}$ is nonempty and does not contain 0. It has a least element $k > 0$. Thus $x - 1 \leq k - 1 < x$ and since $x - y > 1$, it follows that $y < k - 1 < x$.

Item (iii) is Problem 2.2. As a suggestion, note that, by item (i), there is a positive integer n so that $n(x-y) > 1$. Proceed. □

Example 2.12. Suppose $0 < a < 1$. Show the set $A = \{a^n : n \in \mathbb{N}\}$ is bounded below and its infimum is 0.

Since $a \geq 0$ each $a^n \geq 0$. Thus A is bounded below by 0. The set A is not empty. It follows that A has an infimum. Let $\alpha = \inf(A)$ and note $\alpha \geq 0$. Since $\alpha \leq a^n$ for $n = 0, 1, 2, \dots$, it follows that $\alpha \leq a^{n+1}$ for $n \in \mathbb{N}$ and therefore $\frac{\alpha}{a} \leq a^n$ for $n \in \mathbb{N}$. Thus, $\frac{\alpha}{a}$ is a lower bound for A. Hence $\frac{\alpha}{a} \leq \alpha$. Since $a < 1$ and $\alpha \geq 0$, $\alpha = 0$. △

Do Exercise 2.4 and Problems 2.3, 2.4, 2.5,

Proposition 2.13. *If $y \geq 0$ and $n \in \mathbb{N}^+$, then there is a unique nonnegative real number s such that $s^n = y$. In particular, the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^n$ is (one-one and) onto.* †

Of course, $s = y^{\frac{1}{n}}$ is the notation for this n-th root.

Proof outline. The uniqueness is straightforward based upon the fact that if $0 < a < b$, then $a^n < b^n$. It should not come as a shock that existence depends upon the existence of least upper bounds, Theorem 2.8.

Let

$$S = \{x \in \mathbb{R} : 0 < x \text{ and } x^n < y\}.$$

Show S is non-empty and bounded above. Hence S has a least upper bound, say s.

Show, if $0 < t$ and $y < t^n$, then t is an upper bound for S.

Show if $0 < t$ and $y < t^n$, then there is a v such that $0 < v < t$ such that $y < v^n$. Hence, $v < t$ and v is an upper bound for S. In particular, t does not satisfy the least property of least upper bound. Thus, $s^n \leq y$.

Finally, show if $0 < t$ and $t^n < y$, then there exists a v such that $0 < t < v$ such that $v^n < y$. Hence, t is not an upper bound for S. Thus $s^n \geq y$. Hence $s^n = y$. □

Remark 2.14. Since $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^n$ is both one-one and onto it has an inverse, $f^{-1} : [0, \infty) \to [0, \infty)$. This inverse is called the square root function and is commonly denoted by $\sqrt[n]{\ }$ or $x^{\frac{1}{n}}$. In particular, $f^{-1}(x) = x^{\frac{1}{n}}$.

Later, after more machinery is developed, a simpler proof will be given. It is convenient though to have existence of roots in the meantime. It is also good to emphasize again here that the existence of roots, say $\sqrt{2}$, depends on the existence of least upper bounds, and it is this property which distinguishes $\mathbb{R}$ from $\mathbb{Q}$. ⋄

2.3. **Exercises.**

Exercise 2.1. Find the glb and lub of the set $\{\frac{1}{n} : n \in \mathbb{N}^+\}$.

Exercise 2.2. Find the glb and lub of the set $\mathbb{Q} \cap (0,1)$.

Exercise 2.3. Let A be a nonempty set of real numbers which is bounded both above and below. Prove, $\sup(A) \geq \inf(A)$.

Exercise 2.4. Prove, if $A \subseteq B$ are subsets of $\mathbb{R}$ and A is nonempty and B is bounded above, then A and B have least upper bounds and

$$\sup(A) \leq \sup(B).$$

2.4. **Worked Example.**

Sample Problem 2.1. Suppose A and B are nonempty subsets of $\mathbb{R}$ which are bounded above. Prove $A \cup B$ has a least upper bound and further

$$\sup(A \cup B) = \max\{\sup(A), \sup(B)\}.$$

Solution. Note that the hypotheses imply A and B have least upper bounds. For notational convenience, let $\alpha = \max\{\sup(A), \sup(B)\}$.

Since A and B are nonempty, $A \cup B$ is nonempty. Given $x \in A \cup B$, either $x \in A$ and hence $x \leq \sup(A) \leq \alpha$ or $x \in B$ and hence $x \leq \sup(B) \leq \alpha$. In either case, $x \leq \alpha$ and thus α is an upper bound for $A \cup B$. It follows now that $A \cup B$ is nonempty and bounded above and hence has a least upper bound β. Further, $\beta \leq \alpha$, since α is an upper bound for $A \cup B$.

On the other hand, since $A \subseteq A \cup B$, Exercise 2.4 implies $\sup(A) \leq \beta$. By symmetry $\sup(B) \leq \beta$. Thus $\alpha \leq \beta$.

2.5. **Problems.**

Problem 2.1. Let A be a nonempty set of real numbers which is bounded above. Let $-A = \{-a : a \in A\} = \{x \in \mathbb{R} : -x \in A\}$. Show $-A$ is bounded below and $-\inf(-A) = \sup(A)$.

Problem 2.2. Prove item (iii) of Theorem 2.10.

Problem 2.3. Suppose $A \subseteq \mathbb{R}$ is nonempty and bounded above and $\beta \in \mathbb{R}$. Let

$$A + \beta = \{a + \beta : a \in A\}$$

Prove that $A + \beta$ has a supremum and

$$\sup(A + \beta) = \sup(A) + \beta.$$

Problem 2.4. Suppose $A \subseteq [0, \infty) \subseteq \mathbb{R}$ is nonempty and bounded above and $\beta > 0$. Let

$$\beta A = \{a\beta : a \in A\}.$$

Prove βA is nonempty and bounded above and thus has a supremum and

$$\sup(\beta A) = \beta \sup(A).$$

Problem 2.5. Suppose $A, B \subseteq [0, \infty)$ are nonempty and bounded above. Let

$$AB = \{ab : a \in A, \quad b \in B\}.$$

Prove that AB is nonempty and bounded above and

$$\sup(AB) = \sup(A)\sup(B).$$

Here is an outline of a proof. The hypotheses on A and B imply that $\alpha = \sup(A)$ and $\beta = \sup(B)$ both exist. Argue that AB is nonempty and bounded above by $\alpha\beta$ and thus

$$\sup(AB) \leq \alpha\beta.$$

Fix $a \in A$. From an earlier exercise,

$$\sup(aB) = a\sup(B) = a\beta.$$

On the other hand, $aB \subseteq AB$ and thus,

$$a\beta \leq \sup(AB)$$

for each $a \in A$. It follows that βA is bounded above by $\sup(AB)$ and thus,

$$\alpha\beta = \sup(\beta A) \leq \sup(AB).$$

Problem 2.6. Suppose $A, B \subseteq \mathbb{R}$ are nonempty and bounded above. Define

$$A + B = \{a + b : a \in A, \ b \in B\}.$$

Show that $A + B$ has a supremum and moreover, that

$$\sup(A + B) = \sup(A) + \sup(B).$$

Problem 2.7. Suppose X and Y are nonempty subsets of $\mathbb{R}$ and $x < y$ for each $x \in X$ and $y \in Y$. Prove that X has a supremum, that Y has an infimum, and that $\sup(X) \leq \inf(Y)$.

Problem 2.8. Given a positive real number y and positive integers m and n, show that

$$(y^{1/n})^m = (y^m)^{1/n}.$$

Likewise verify that

$$(y^m)^n = (y^n)^m$$

and

$$(y^{1/m})^{1/n} = (y^{1/n})^{1/m}$$

These results show that $y^{m/n}$ is unambiguously defined.

Problem 2.9. Suppose $f : [a, b] \to [\alpha, \beta]$ and $\varphi : [\alpha, \beta] \to \mathbb{R}$. Let $h = \varphi \circ f$. Show, if there is a $C > 0$ such that

$$|\varphi(s) - \varphi(t)| \leq C|s - t|$$

for all $s, t \in [\alpha, \beta]$, then

$$\begin{aligned}\sup\{h(x) : a \leq x \leq b\} - &\inf\{h(x) : a \leq x \leq b\} \\ &\leq C\left[\sup\{f(x) : a \leq x \leq b\} - \inf\{f(x) : a \leq x \leq b\}\right].\end{aligned}$$

Problem 2.10. Suppose $f : [0, 1] \to \mathbb{R}$ is increasing (meaning, if $0 \leq x \leq y \leq 1$, then $f(x) \leq f(y)$). Prove

$$\alpha = \inf\{t \in [0, 1] : f(t) \leq t\}$$

exists and $f(\alpha) = \alpha$. (Thus α is a fixed point of f. For a further challenge, show α is the smallest fixed point of f.)

3. Metric Spaces

3.1. Definitions and Examples.

Definition 3.1. A *metric space* (X, d) consists of a set X and function $d : X \times X \to \mathbb{R}$ such that, for $x, y, z \in X$,

(i) $d(x, y) \geq 0$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$; and
(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

We usually call the metric space X, and d the *metric*, or *distance function*. Thus, a distance function or metric on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the axioms above. Item (iv) is the *triangle inequality*. Items (i) and (ii) together are sometimes expressed by saying d is *positive definite*. Evidently (iii) is a symmetry axiom. ◁

Example 3.2. Here are some examples of metric spaces.

(i) Unless otherwise noted, $\mathbb{R}$ is the metric space with the distance function $d(x, y) = |x - y|$.
(ii) Let X be any nonempty set and define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. This function is the *discrete metric*.
(iii) On the vector space $\mathbb{R}^n$ the ℓ^1*-metric* is defined by

$$d_1(x, y) = \sum |x_j - y_j|.$$

(iv) On $\mathbb{R}^n$, define d_∞ by

$$d_\infty(x, y) = \max\{|x_j - y_j| : 1 \leq j \leq n\}.$$

This metric is the ℓ^∞ *metric* (or worst case metric). In particular $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$ are different metric spaces.
(v) Define, on the space of polynomials $\mathcal{P}$, the metric

$$d_1(p, q) = \int_0^1 |p - q| dt.$$

(vi) If (X, d) is a metric space and $Y \subseteq X$, then $(Y, d|_{Y \times Y})$ is a metric space and is called a *subspace* of X.

△

Do Problem 3.1.

Proposition 3.3. *Let (X, d) be a metric space.*

If $p, q, r \in X$, then

$$|d(p, r) - d(q, r)| \leq d(p, q).$$

If $p_1, \dots, p_n \in X$, then

$$d(p_1, p_n) \leq \sum_{j=1}^{n-1} d(p_j, p_{j+1}).$$

†

3.2. **Normed Vector Spaces.** Recall that $\mathbb{R}^n$ is the vector space of n-tuples of real numbers. Thus an element $x \in \mathbb{R}^n$ has the form,

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Vectors - elements of $\mathbb{R}^n$ - are added and multiplied by *scalars* (elements of $\mathbb{R}$) entrywise.

The set of polynomials $\mathcal{P}$ (in one variable with real coefficients) is a vector space under the usual operations of addition and scalar multiplication.

Definition 3.4. A *norm* on a vector space V over $\mathbb{R}$ is a function $\|\cdot\| : V \to \mathbb{R}$ satisfying

(i) $\|x\| \geq 0$ for all $x \in V$;
(ii) $\|x\| = 0$ if and only if $x = 0$;
(iii) $\|cx\| = |c|\,\|x\|$ for all $c \in \mathbb{R}$ and $x \in V$; and
(iv) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

The last condition is known as the *triangle inequality.* ◁

Example 3.5. The functions $\|\cdot\|_1$ and $\|\cdot\|_\infty$ mapping $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\|x\|_1 = \sum_{j=1}^{n} |x_j|$$

and

$$\|x\|_\infty = \max\{|x_j| : 1 \leq j \leq n\}$$

respectively are norms on $\mathbb{R}^n$. △

Proposition 3.6. *If $\|\cdot\|$ is a norm on a vector space V, then the function*

$$d(x, y) = \|x - y\|,$$

is a metric on V. †

Proof. With the exception of the triangle inequality, it is evident that d satisfies the axioms of a metric.

To prove that d satisfies the triangle inequality, let $x, y, z \in V$ be given and estimate, using the triangle inequality for the norm,

$$\begin{aligned} d(x, z) &= \|x - z\| \\ &= \|(x - y) + (y - z)\| \\ &\leq \|x - y\| + \|y - z\| \\ &= d(x, y) + d(y, z). \end{aligned}$$

□

Remark 3.7. In the case $(V, \|\cdot\|)$ is a normed vector space the default is to view V also as a metric space with the metric coming from the norm. ◇

Definition 3.8. Let V be a vector space over $\mathbb{R}$. A function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ is an *inner product* (or *scalar product*) on V if,

(i) $\langle x, x\rangle \geq 0$ for all $x \in V$;
(ii) $\langle x, x\rangle = 0$ if and only if $x = 0$;
(iii) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in V$;
(iv) $\langle cx + y, z\rangle = c\langle x, z\rangle + \langle y, z\rangle$.

◁

Example 3.9. On $\mathbb{R}^n$, the pairing,

$$\langle x, y\rangle = \sum_{j=1}^{n} x_j y_j$$

is an inner product. In the case of $n = 2, 3$ it is often called the *dot product.*

On $\mathcal{P}$, the space of polynomials, the pairing

$$\langle p, q\rangle = \int_0^1 pq\, dt$$

is an inner product.

△

Proposition 3.10 (Cauchy–Schwarz inequality)**.** *Suppose $\langle\cdot,\cdot\rangle$ is an inner product on a vector space V. For all $x, y \in V$ we have*

$$\langle x, y\rangle^2 \leq \langle x, x\rangle\, \langle y, y\rangle.$$

†

Proof. Take any two vectors $x, y \in V$ and let $t \in \mathbb{R}$ an arbitrary real number. By property (i) of inner products, we see that

$$\langle tx + y, tx + y\rangle \geq 0.$$

Expanding the left-hand side of the above inequality using properties (iii) and (iv) of inner products shows that

$$t^2\langle x, x\rangle + 2t\langle x, y\rangle + \langle y, y\rangle \geq 0.$$

Viewing x and y as fixed, the above inequality states that a certain quadratic in the variable t is always nonnegative. Therefore the discriminant of the quadratic (i.e., the quantity $b^2 - 4ac$ inside the quadratic formula) must be non-positive. Thus we see that

$$4\langle x, y\rangle^2 - 4\langle x, x\rangle\langle y, y\rangle \leq 0.$$

Upon simplification, this is precisely the inequality we sought to prove. □

Proposition 3.11. *If $\langle\cdot,\cdot\rangle$ is an inner product on a vector space V, then the function $\|\cdot\| : V \to \mathbb{R}$ defined by $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm on V.* †

Proof. Here is a proof of the triangle inequality. That the other axioms of a norm are satisfied is left to the gentle reader.

To verify the triangle inequality, let $x, y \in V$ be given and estimate, using linearity of the inner product in both variables and the Cauchy–Schwarz inequality (Proposition 3.10),

$$\begin{aligned}\|x+y\|^2 =&\langle x+y, x+y\rangle \\ =&\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \\ \leq&\|x\|^2 + 2\|x\|\,\|y\| + \|y\|^2 \\ =&(\|x\|+\|y\|)^2.\end{aligned}$$

□

Remark 3.12. In the case that V has an inner product, the norm $\|\cdot\|$ of Proposition 3.11 is, unless otherwise noted, understood to be *the norm* on V and $\|x\|$ the norm of a vector $x \in V$.

In the case of $\mathbb{R}^n$ with the (usual) inner product given in Example 3.9 the resulting norm is the *Euclidean norm* and metric is the *Euclidean distance* which will sometimes be written as d_2. *Euclidean space* refers to $(\mathbb{R}^n, d_2)$ (for some n). Note that $(\mathbb{R}^n, d_2)$ is, as a metric space, distinct from both $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$. Generally, $\mathbb{R}^n$ refers to Euclidean space unless indicated otherwise.

With this notation and understating of the meaning of $\|\cdot\|$, the Cauchy–Schwarz inequality says

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|.$$

◇

Do Problem 3.2.

3.3. **Open Sets.**

Definition 3.13. Let (X, d) be a metric space. A subset $U \subseteq X$ is *open* if for each $x \in U$ there is an $\epsilon > 0$ such that

$$N_\epsilon(x) := \{p \in X : d(p,x) < \epsilon\} \subseteq U.$$

The set $N_\epsilon(x)$ is the *ϵ-neighborhood* of x. More or less synonymously, an *open ball* is a set of the form $N_r(y)$ for some $y \in X$ and $r > 0$. ◁

Proposition 3.14. *Neighborhoods are open sets; i.e., if (X, d) is a metric space, $y \in X$ and $r > 0$, then the set*

$$N_r(y) = \{p \in X : d(p,y) < r\}$$

is an open set. †

Proof. We must show, for each $x \in N_r(y)$ there is an ϵ (depending on x) such that $N_\epsilon(x) \subseteq N_r(y)$. Accordingly, let $x \in N_r(y)$ be given. Thus, $d(x,y) < r$. Choose $\epsilon = r - d(x,y) > 0$. Suppose now that $p \in N_\epsilon(x)$ so that $d(x,p) < \epsilon$. Estimate, using the triangle inequality,

$$d(y,p) \leq d(y,x) + d(x,p) < d(y,x) + \epsilon = d(y,x) + (r - d(y,x)) = r.$$

Thus, $p \in N_r(y)$. We have shown $N_\epsilon(x) \subseteq N_r(y)$ and the proof is complete. □

Do Problem 3.3.

Example 3.15. In $\mathbb{R}^2$ with the Euclidean distance, show $E = \{(x_1, x_2) : x_1, x_2 > 0\}$ is an open set. △

Example 3.16. The set $[0, 1) \subset \mathbb{R}$ is not an open, since, for every $\epsilon > 0$, the set $N_\epsilon(0) = (-\epsilon, \epsilon)$ contains negative numbers and is thus not a subset of $[0, 1)$. △

Proposition 3.17. *Let (X, d) be a metric space.*

(i) $\emptyset, X \subseteq X$ are open;
(ii) if $\mathcal{F} \subseteq P(X)$ is a collection of open sets, then

$$\cup_{U \in \mathcal{F}} U$$

is open; and
(iii) if $n \in \mathbb{N}^+$ and $U_1, \dots, U_n \subseteq X$ are open, then

$$\cap_{j=1}^n U_j$$

is open.

†

Corollary 3.18. *A subset U of a metric space X is open if and only if it is a union of neighborhoods.* †

Example 3.19. Let $U_j = (-\frac{1}{j+1}, 1) \subset \mathbb{R}$ for $j \in \mathbb{N}$. The sets U_j are open in $\mathbb{R}$ (they are open balls). However, the set

$$[0, 1) = \cap_{j=0}^\infty U_j$$

is not open. Thus it is not possible to improve on the last item in the proposition. △

Example 3.20. The set $(-\infty, 0) = \cup_{n=0}^\infty (-2n, 0) = \cup_{n=0}^\infty N_n(-n)$ and is therefore open. We could of course easily checked this directly from the definition of open set. △

Example 3.21. The set

$$\mathbb{R}^2 \supseteq E = \{(x_1, x_2) : x_j > 0\} = \{x \in \mathbb{R}^2 : x_1 > 0\} \cap \{x \in \mathbb{R}^2 : x_2 > 0\}.$$

This provides yet another way to prove E is open. Namely, show that each of the sets on the right hand side above is open. △

Do Problem 3.4.

3.3.1. *Open sets and norms.*

Definition 3.22. Two norms $\|\cdot\|$ and $\|\cdot\|_*$ are *equivalent* if there exists constants $0 < c \leq C$ such that

$$c\|x\| \leq \|x\|_* \leq C\|x\|.$$

◁

Though there appears to be some asymmetry in the definition, the notion of equivalent norms is in fact an equivalence relation on the set of norms on a given vector space.

Proposition 3.23. *Suppose $\|\cdot\|$ and $\|\cdot\|_*$ are norms on a vector space X and let d and d_* denote the resulting metrics. If there is a constant $C > 0$ such that*

$$\|x\| \leq C\|x\|_*,$$

and $U \subseteq X$ is open in the metric space (X, d), then U is open in the metric space (X, d_*).

Hence, if the norms are equivalent, then a set is open in (X, d) if and only if it is open in (X, d_*). †

Proof. Let $N_r(z)$ and $N_r^*(z)$ denote the r-neighborhoods of the point z in the metric spaces (X, d) and (X, d_*) respectively.

Given $x, y \in V$, the hypothesis imply that

$$d(x, y) = \|x - y\| \leq C\|x - y\|_* = Cd_*(x, y).$$

Hence, $N^*_{\frac{r}{C}}(x) \subseteq N_r(x)$.

Now suppose U is open in X and $x \in U$. There is an $r > 0$ such that $N_r(x) \subseteq U$. Thus, with $\epsilon = \frac{r}{C}$,

$$N^*_\epsilon(x) \subseteq N_r(x) \subseteq U$$

and it follows that U is open in X^*. □

Proposition 3.24. *Let $U \subseteq \mathbb{R}^n$ be given. The following are equivalent,*

(i) U is open in $(\mathbb{R}^n, d_1)$;
(ii) U is open in $(\mathbb{R}^n, d_2)$; and
(iii) U is open in $(\mathbb{R}^n, d_\infty)$.

†

Proof. By Proposition 3.23, it suffices to prove that the norms $\|\cdot\|_j$ for $j = 1, 2, \infty$ are all equivalent.

Let $e \in \mathbb{R}^n$ denote the vector with each entry equal to 1 and, given $x \in \mathbb{R}^n$, let $|x|$ denote the vector obtained by taking entrywise absolute value. By the Cauchy–Schwarz inequality,

$$\|x\|_1 = \sum_{j=1}^n |x_j| = \langle |x|, e\rangle \leq \|x\|_2\|e\|_2 = \sqrt{n}\|x\|_2.$$

On the other hand,

$$\|x\|_2 = \sqrt{\sum_{j=1}^{n} x_j^2} \le \sqrt{\sum_{j=1}^{n} \|x\|_\infty^2} = \sqrt{n}\|x\|_\infty.$$

Finally,

$$\|x\|_\infty \le \sum_{j=1}^{n} |x_j|,$$

since, $|x_k| = \|x\|_\infty$ for some k. □

Example 3.25. Returning to the example of the set $E = \{(x,y) : x,y > 0\} \subseteq \mathbb{R}^2\}$ above, it is convenient to use the d_∞ metric to prove E is open; i.e., show that E is open in $(\mathbb{R}^2, d_\infty)$ and conclude that E is open in $\mathbb{R}^2$. △

3.3.2. *Relatively open sets.*

Definition 3.26. Suppose (Z,d) is a metric space and $X \subseteq Z$ so that $(X, d|_{X\times X})$ is also a metric space. A subset $U \subseteq X$ is *open relative to* X or is *relatively open*, if U is open in the metric space X. ◁

Example 3.27. Let $X = [0,\infty) \subseteq Z = \mathbb{R}$. The set $[0,1)$ is open in X, but not in Z. △

Proposition 3.28. *Suppose Z is a metric space and $U \subseteq X \subseteq Z$. The set U is open in X if and only if there is an open set W in Z such that $U = W \cap X$.* †

Proof. For notational convenience, given $p \in U$ and $r > 0$, let $N_r^X(p) = \{x \in X : d(x,p) < r\}$ and $N_r^Z(p) = \{x \in Z : d(x,p) < r\}$, the r-neighborhoods of p in X and Z respectively. Note that

$$N_r^X(p) = N_r^Z(p) \cap X.$$

First, suppose $W \subseteq Z$ is open (in Z) and $U = W \cap X \subseteq X$. Given $p \in U$, there is a $\delta_p > 0$ such that $N_{\delta_p}^Z(p) \subseteq W$ since $p \in W$ and W is open in Z. Moreover,

$$N_{\delta_p}^X(p) = N_{\delta_p}^Z(p) \cap X \subseteq W \cap X = U.$$

It follows that

$$U \subseteq \bigcup_{p\in U} N_{\delta_p}^Z(p) \cap X = \bigcup_{p\in U} N_{\delta_p}^X(p) \subseteq U.$$

Hence U is a union of neighborhoods (in X) and is thus open in X.

Now suppose $U \subseteq X$ is open relative to X. For each $p \in U$ there is an $\epsilon_p > 0$ such that $N_{\epsilon_p}^X(p) \subseteq U$. Let

$$W = \bigcup_{p\in U} N_{\epsilon_p}^Z(p)$$

and note that W is open in Z and

$$W \cap X = \left(\bigcup_{p\in U} N_{\epsilon_p}^Z(p)\right) \cap X = \bigcup_{p\in U} \left(N_{\epsilon_p}^Z(p) \cap X\right) = \bigcup_{p\in U} N_{\epsilon_p}^X(p) = U,$$

completing the proof. □

Returning to Example 3.27, note that $[0,1) = (-1,1) \cap X$.

3.4. **Closed Sets.**

Definition 3.29. Let (X,d) be a metric space. A subset $C \subseteq X$ is *closed* if $X \setminus C$ is open. ◁

Example 3.30. (i) In $\mathbb{R}$ the set $[0,\infty)$ is closed, since its complement, $(-\infty,0)$ is open.
(ii) The set $[0,1) \subseteq \mathbb{R}$ is neither open nor closed.
(iii) The set $\mathbb{Q} \subseteq \mathbb{R}$ is neither open nor closed.
(iv) The set $F = \{(x_1,x_2) \in \mathbb{R}^2 : x_1x_2 = 0\}$ is closed.
(v) The sets X and $\emptyset$ are both open and closed. They are *clopen*.
(vi) Every subset of a discrete metric space is clopen. (See Problem 3.4.)

△

Proposition 3.31. *Let (X,d) be a metric space and let $x \in X$ and $r \geq 0$ be given. The set*

$$\{p \in X : d(p,x) \leq r\}$$

is closed. †

Proof. The complement of $\{p \in X : d(p,x) \leq r\}$ is the set

$$U = \{p : d(p,x) > r\}$$

and it suffices to prove that U is open. Let $y \in U$ be given. Then $d(y,x) > r$. Let $\epsilon = d(y,x) - r > 0$. If $z \in N_\epsilon(y)$ so that $d(z,y) < \epsilon$, then,

$$\begin{aligned} d(x,z) \geq& d(x,y) - d(y,z) \\ >& d(x,y) - \epsilon \\ =& r. \end{aligned}$$

It follows that $N_\epsilon(y) \subseteq U$ and thus, since $y \in U$ was arbitrary, U is open. □

Corollary 3.32. *In a metric space, singleton sets are closed; i.e., if (X,d) is a metric space and $x \in X$, then $\{x\}$ is closed.* †

Proposition 3.33. *Let X be a metric space.*

(i) X and $\emptyset$ are closed;
(ii) if $C_1,\dots,C_n$ are closed subsets of X, then $\cup_1^n C_j$ is closed; and
(iii) if C_α, $\alpha \in J$ is a family of closed subsets of X, then

$$C = \cap_{\alpha \in J} C_\alpha$$

is closed.

†

Corollary 3.34. *A finite set F in a metric space X is closed.* †

Proposition 3.35. *If $C \subseteq \mathbb{R}$ is bounded above, nonempty, and closed, then C has a largest element.* †

Proof. The hypotheses imply $\sup(C)$ exist. It is of course an upper bound for C. Suppose β is an upper bound for C and $\beta \notin C$. Since C is closed, $\tilde{C}$ is open and therefore there is an $\epsilon > 0$ such that $N_\epsilon(\beta) \subseteq \tilde{C}$. In particular, if $\beta - \epsilon < x \leq \beta$, then $x \notin C$. Of course if $x > \beta$, then $x \notin C$. Hence, $\beta - \epsilon$ is an upper bound for C and thus $\beta > \sup(C)$ and the result follows. □

Example 3.36. Let $R = \mathbb{Q} \cap [0,1]$ denote the rational numbers in the interval $[0,1]$. Since $\mathbb{Q}$ is countable, so is R. Choose an enumeration $R = \{r_1, r_2, \dots\}$ of R. Fix $1 > \epsilon > 0$ and let

$$V_j = N_{\frac{\epsilon}{2^{j+1}}}(r_j)$$

and $V = \cup V_j$. Thus V is an open set which contains R.

The set $C = [0,1] \setminus V$ is closed because it is the intersection of the closed sets $[0,1]$ and $\tilde{V}$. On the other hand, its complement contains every rational in the interval $[0,1]$, but is also the union of intervals the sum[1] of whose lengths is at most

$$\sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon < 1.$$

Thus C is a closed subset of $[0,1]$ which contains no rational number, but is large in the sense that its complement can be covered by open intervals whose lengths sum to at most ϵ.

A heuristic is that open sets are nice and closed sets can be strange, while most sets are neither open nor closed. △

Do Problem 3.5.

3.5. **Interior, Closure, and Boundary.**

Definition 3.37. Let (X,d) be a metric space and $S \subseteq X$. The *closure* of S is

$$\overline{S} = \bigcap \{C \subseteq X : C \supseteq S \text{ and } C \text{ is closed}\}.$$

◁

Proposition 3.38. *Let S be a subset of a metric space X.*

(i) $S \subseteq \overline{S}$;
(ii) $\overline{S}$ is closed;
(iii) if K is any other set satisfying (i) and (ii), then $\overline{S} \subseteq K$.

Moreover, S is closed if and only if $S = \overline{S}$. †

Definition 3.39. Let (X,d) be a metric space and $S \subseteq X$. The *interior* of S is the set

$$S^\circ = \bigcup \{U \subseteq X : U \subseteq S \text{ and } U \text{ is open}\}.$$

◁

[1] Series are introduced in Problem 4.17 in the next section and will be treated in detail later, but this particular series should be familiar from Calculus II.

Proposition 3.40. *Let S be a subset of a metric space X.*

(i) $S^\circ \subseteq S$*;*
(ii) S° *is open;*
(iii) if $V \subseteq S$ *is an open set, then* $V \subseteq S^\circ$*.*

Moreover, S *is open if and only if* $S = S^\circ$*.* †

Definition 3.41. A point $x \in X$ is an *interior point* of S if there is an $\epsilon > 0$ such that $N_\epsilon(x) \subseteq S$. ◁

Do Problems 3.6, 3.7 and 3.8.

Definition 3.42. The *boundary* of a set S in a metric space X is $\partial S = \overline{S} \cap \overline{\tilde{S}}$. ◁

Do Problem 3.9

3.6. **Exercises.**

Exercise 3.1. Show if (X, d) is a metric space, then

$$d_*(x,y) = \frac{d(x,y)}{1 + d(x,y)}$$

is also metric on X. *Hint:* to verify the triangle inequality, you might want to first establish that for nonnegative real numbers a and b, if $a \le b$ then $a/(1+a) \le b/(1+b)$.

Exercise 3.2. Show that the functions in Example 3.5 are both norms on $\mathbb{R}^n$.

Exercise 3.3. Verify the claims made in Example 3.9.

Exercise 3.4. Show that the subset $S = \{(x,y) \in \mathbb{R}^2 : x \neq y\}$ is open.

Exercise 3.5. Verify that the discrete metric is indeed a distance function.

Exercise 3.6. Let X be a nonempty set and d the discrete metric. Fix a point $z \in X$. Is the closure of the set $N_1(z)$ equal to $\{x \in X : d(x,z) \le 1\}$?

Exercise 3.7. Show that the set

$$\{(x_1, x_2) : x_1, x_2 \ge 0\} \subset \mathbb{R}^2$$

is closed.

Show that the set

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 x_2 = 1\}$$

is closed.

Exercise 3.8. By Proposition 3.6,

$$d(f,g) = (\int_0^1 |f-g|^2 dt)^{\frac{1}{2}}$$

defines a metric on the space of polynomials $\mathcal{P}$. For $n \in \mathbb{N}$, let

$$p_n(t) = \sqrt{2n+1}\, t^n.$$

Find $d(p_n, p_m)$.

Exercise 3.9. Determine the interior, closure and boundary of an interval $(a, b]$ in $\mathbb{R}$. What is the boundary of $(0, 1]$ as a subset of the metric space $(0, \infty)$?

Exercise 3.10. Let S be a subset of a discrete metric space X. Determine the interior, boundary and closure of S.

Exercise 3.11. Determine $\partial\partial(0, 1]$.

Let $S = \{(x, 0) : 0 < x < 1\} \subseteq \mathbb{R}^2$. Find ∂S and $\partial\partial S$?

3.7. **Worked Examples.**

Sample Problem 3.1. Let X be a metric space. Show,

(i) if $S, T \subseteq X$, then $\overline{S \cup T} = \overline{S} \cup \overline{T}$; and
(ii) if $S_1, S_2, \cdots \subseteq X$, then
$$\overline{\cup_{j=1}^{\infty} S_j} \supseteq \cup_{j=1}^{\infty} \overline{S_j}.$$

Solution. Since the finite union of closed sets is again closed, $\overline{S} \cup \overline{T}$ is a closed set which contains both S and T and hence contains $S \cup T$. The inclusion $\overline{S \cup T} \subseteq \overline{S} \cup \overline{T}$ now follows from Proposition 3.38(iii).

Now suppose $S_1, S_2, \cdots \subseteq X$ and let, for notational ease, $R = \cup_{j=1}^{\infty} S_j$. For each j the inclusion $S_j \subseteq R$ implies $S_j \subseteq \overline{R}$. Hence, by Proposition 3.38(iii), $\overline{S_j} \subseteq \overline{R}$. Consequently, $\cup_{j=1}^{\infty} \overline{S_j} \subseteq \overline{R}$. Thus, item (ii) is now proved. Moreover, choosing $S_1 = S$ and $S_2 = T$ and $S_j = \emptyset$ for $j \geq 3$ proves the inclusion $\overline{S} \cup \overline{T} \subseteq \overline{S \cup T}$, and the proof of (i) is complete.

Though it was not asked for in the problem, here is an example where strict inclusion holds in (ii). Let $X = \mathbb{R}$ and $S_j = [\frac{1}{j}, 1]$ for $j \in \mathbb{N}^+$. In particular, each S_j is closed so that $\overline{S_j} = S_j$. Note that $\cup_{j=1} S_j = (0, 1]$ which is not a closed set (so this is an example where the union of closed sets is not closed).

Sample Problem 3.2. Fix $p \in \mathbb{R}^n$ and let $r > 0$ be given. Show that the closure of $N_r(p)$ is $B_r(p)$.

Solution. For notational purposes, let $S = N_r(p)$ and $T = B_r(p)$. Since T is closed and contains S, it follows from Proposition 3.38 that $\overline{S} \subseteq T$. To prove the reverse inclusion, we use the criteria that $q \in \overline{S}$ if and only if for each $\epsilon > 0$
$$N_\epsilon(q) \cap S \neq \emptyset.$$
Further we may assume that $q \in T \setminus S$. In this case, $d(p, q) = r$. Given $\epsilon > 0$, choose $s = \min\{\frac{\epsilon}{r}, 1\}$ and let
$$z = q + s(p - q).$$
Since
$$d(q, z) = \|z - q\| = s\|p - q\| < \epsilon,$$
$z \in N_\epsilon(q)$. On the other hand,
$$d(p, z) = \|z - p\| = \|(1 - s)(q - p)\| = (1 - s)r < r,$$
where $s < 1$ was used in the second equality. Hence $z \in S$ and the proof is complete.

3.8. **Problems.**

Problem 3.1. Suppose (X, d_X) and (Y, d_Y) are metric spaces. Define $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x, y), (a, b)) = d_X(x, a) + d_Y(y, b).$$

Prove d is a metric on $X \times Y$.

Problem 3.2. Show, if V is a vector space with an inner product, then the norm

$$\|v\| = \sqrt{\langle v, v \rangle} \tag{1}$$

satisfies the *parallelogram law* ,

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

Explain why this is called the parallelogram law.

Recall the norm $\| \cdot \|_1$ on $\mathbb{R}^n$ defined in Example 3.5. Does this norm come from an inner product?

Problem 3.3. Describe the neighborhoods in a discrete metric space (X, d).

Problem 3.4. Determine, with proof, the open subsets of the discrete metric space (X, d).

Problem 3.5. Given a metric space Z and $F \subseteq X \subseteq Z$ define F is *relatively closed* in X. Show, F is relatively closed in X if and only if there is a closed set $C \subseteq Z$ such that $F = C \cap X$.

Prove that the closure of $F \subseteq X$, as a subset of X, is $X \cap \overline{F}$, where $\overline{F}$ is the closure of F in Z. Conclude, if F is relatively closed, then $F = \overline{F} \cap X$.

Finally, show, if

(i) $A, B \subseteq Z$;
(ii) $X = A \cup B$; and
(iii) $\overline{A} \cap B = \emptyset$,

then $B = \tilde{\overline{A}} \cap X$ and hence is open relative to X.

Problem 3.6. Show,

$$I(S) = \{s \in S : s \text{ is an interior point of } S\} = S^\circ.$$

Problem 3.7. Prove that

$$\overline{S} = \widetilde{(\tilde{S})^\circ}.$$

Suggestion: Use the properties of closure and interior. For instance, note that $\tilde{\overline{S}}$ is open and contained in $\tilde{S}$.

Problem 3.8. Show $\overline{S}$ consists of those points $x \in X$ such that for every $\epsilon > 0$, $N_\epsilon(x) \cap S \neq \emptyset$. Equivalently, for every ϵ there exists a $y \in S$ such that $d(x, y) < \epsilon$.

Show also, if S is a non-empty subset of a metric space X and $x \in X$, then x is in $\overline{S}$ if and only if

$$\inf\{d(x, s) : s \in S\} = 0.$$

Problem 3.9. Prove that $x \in \partial S$ if and only if for every $\epsilon > 0$ there exists $s \in S$, $t \in \tilde{S}$ such that $d(x,s), d(x,t) < \epsilon$.

Prove S is closed if and only if S contains its boundary; and S is open if and only if S is disjoint from its boundary.

Problem 3.10. Show, in $\mathbb{R}^2$, if $x \in \mathbb{R}^2$ and $r > 0$, then the closure of

$$N_r(x) = \{y \in \mathbb{R}^2 : d(x,y) = \|x - y\| < r\}$$

is the set

$$\{y \in \mathbb{R}^2 : d(x,y) = \|x - y\| \leq r\}.$$

Is the corresponding statement true in all metric spaces?

Problem 3.11. Show if A and B are disjoint closed sets in a metric space X, then there exists disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. (This separation property is expressed by saying a metric space is *normal*.) Suggestion: by Problem 3.8, for each $a \in A$ there exists an $\epsilon_a > 0$ such that $N_{2\epsilon_a}(a) \cap B = \emptyset$. Likewise there exists, for each $b \in B$, an $\eta_b > 0$ such that $N_{2\eta_b}(b) \cap A = \emptyset$. Let $U = \cup_{a \in A} N_{\epsilon_a}(a)$ and define V similarly.

Problem 3.12. Prove Proposition 3.40.

Problem 3.13. Show that the closure of $\mathbb{Q}$ in $\mathbb{R}$ is all of $\mathbb{R}$. (Suggestion: Use Problem 3.7 and Theorem 2.10 item iii). Compare with Remark 2.11.

Problem 3.14. Show that the closure of $\tilde{\mathbb{Q}}$ (the irrationals) in $\mathbb{R}$ is all of $\mathbb{R}$. Combine this problem and Problem 3.13 to determine the boundary of $\mathbb{Q}$ (in $\mathbb{R}$).

Problem 3.15. Suppose (X,d) is a metric space and $x \in X$ and $r > 0$ are given. Show that the closure of $N_r(x)$ is a subset of the set

$$\{y \in X : d(x,y) \leq r\}.$$

Give an example of a metric space X, an $x \in X$, and an $r > 0$ such that the closure of $N_r(x)$ is **not** the set

$$\{y \in X : d(x,y) \leq r\}.$$

Compare with Problem 3.10.

Problem 3.16. Let (X,d) and d_* be as in Exercise 3.1. Do the metric spaces (X,d) and (X,d_*) have the same open sets?

Problem 3.17. Suppose d and d' are metrics on the set X and there is a constant C such that, for all $x, y \in X$,

$$d(x,y) \leq C d'(x,y).$$

Prove, if U is open in (X,d), then U is open in (X,d').

Thus, if there is also a constant C' such that

$$d'(x,y) \leq C' d(x,y),$$

then the metric spaces (X,d) and (X,d') have the same open sets.

4. Sequences

4.1. Definitions and Examples.

Definition 4.1. A *sequence* is a function a with domain of the form $\{n \geq n_0 : n \in \mathbb{Z}\}$ for some $n_0 \in \mathbb{Z}$. It is customary to write $a_n = a(n)$ to denote the value of this function at $n \in \mathbb{N}$ and (a_n) or $(a_n)_n$ or $(a_n)_{n=n_0}^{\infty}$ for this function.

If the a_n lie in the set X, then (a_n) is a *sequence from* X.

Suppose (X, d) is a metric space and $L \in X$. The sequence (a_n) (from X) *converges to* L, denoted

$$\lim_{n\to\infty} a_n = L,$$

if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(a_n, L) < \epsilon$ for all $n \geq N$. In this case, L is **the** *limit of the sequence*.

The sequence (a_n) *converges* if there exists an $L \in X$ such that (a_n) converges to L. A sequence which does not converge is said to *diverge*. ◁

Remark 4.2. From the (positive definite) axioms (items (i) and (ii) of Definition 3.1) of a metric, if x, y are points in a metric space (X, d) and if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$. ◇

The following proposition list some of the most basic properties of limits. The first justifies the terminology *the limit* (as opposed to *a limit*) above.

Proposition 4.3. *Let $(a_n)_{n=k}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be sequences from the metric space X.*

(i) If (a_n) converges, then its limit is unique;
(ii) if there is an N and an ℓ such that for $n \geq N$, $b_n = a_{n+\ell}$, then (a_n) converges if and only if (b_n) converges and moreover in this case the sequences have the same limit;
(iii) If there is a sequence (r_n) of real numbers which converges to 0, a real number C, and positive integer M such that, for $m \geq M$,

$$d(a_m, L) \leq C r_m,$$

then (a_n) converges to L; and
(iv) if $(a_n)_n$ is a sequence from $\mathbb{R}$ ($X = \mathbb{R}$), $c \in \mathbb{R}$, and (a_n) converges to L, then (ca_n) converges to cL.

†

Item (ii) says that we need not be concerned with keeping close track of k. In particular if $\ell \geq k$, then $(a_j)_{j=k}^{\infty}$ converges if and only if $(a_j)_{j=\ell}^{\infty}$ converges. Item (iv) follows immediately from Item (iii).

Example 4.4. By Theorem 2.10(i), the sequence $(\frac{1}{n})_{n=1}^{\infty}$ converges to 0 in $\mathbb{R}$; however it does not converge in the metric space $(0, 1]$ (as a subspace of $\mathbb{R}$ of course) as can be proved using item (i) of the previous proposition and the fact that the sequence converges to 0 (and not a point in $(0, 1]$) in $\mathbb{R}$.

The sequence $(\frac{n}{n+1})_n$ converges to 1 (in $\mathbb{R}$). △

Example 4.5. If $0 \leq a < 1$, then the sequence (a^n) converges to 0.

To prove this statement, recall that we have already shown (see Example 2.12) that $\inf(\{a^n : n \in \mathbb{N}\}) = 0$. Thus, given $\epsilon > 0$ there is an N such that $0 \leq a^N < \epsilon$. Hence, if $n \geq N$, then $|a^n - 0| \leq a^N < \epsilon$ and the proof is complete. △

Do Problems 4.1, 4.2, 4.3, and Exercise 4.2.

We will make repeated use of the following simple identity, valid for all real r and positive integers m,

$$1 - r^m = (1-r)(1 + r + r^2 + \dots r^{m-1}). \tag{2}$$

Proposition 4.6. *In (the metric space)* $\mathbb{R}$*,*

(i) if $\rho > 0$, then the sequence $(\rho^{\frac{1}{n}})$ converges to 1; and
(ii) the sequence $(n^{\frac{1}{n}})$ converges to 1.

†

Proof. To prove (i), first suppose $\rho > 1$. Using Equation (2) with $m = n$ and $r = \rho^{\frac{1}{n}}$ gives

$$\rho^{\frac{1}{n}} - 1 = \frac{\rho - 1}{\sum_{j=0}^{n-1} \rho^{\frac{j}{n}}}.$$

Consequently,

$$|\rho^{\frac{1}{n}} - 1| < \frac{\rho - 1}{n}.$$

Thus $(\rho^{\frac{1}{n}})$ converges to 1 by Proposition 4.3(iii) and Example 4.4.

If $0 < \rho < 1$, then $\sigma = \frac{1}{\rho} > 1$ and $(\sigma^{\frac{1}{n}})$ converges to 1. On the other hand,

$$|1 - \rho^{\frac{1}{n}}| = \rho^{\frac{1}{n}} |\sigma^{\frac{1}{n}} - 1| \leq |\sigma^{\frac{1}{n}} - 1|,$$

from which the result follows.

To prove (ii) note that the Binomial Theorem gives, for $x > 0$,

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j \geq \frac{n(n-1)}{2} x^2.$$

Thus, with $x = n^{\frac{1}{n}} - 1$,

$$n \geq \frac{n(n-1)}{2} (n^{\frac{1}{n}} - 1)^2.$$

Hence, for $n \geq 2$,

$$(\frac{2}{n-1})^{\frac{1}{2}} \geq |n^{\frac{1}{n}} - 1| \geq 0.$$

To complete the proof it suffices, by Proposition 4.3 (iii), to show that the sequence $((\frac{2}{n-1})^{\frac{1}{2}})$ converges to 0. Accordingly, given $\epsilon > 0$ choose $N \in \mathbb{N}^+$ such that $N \geq \frac{2}{\epsilon^2} + 1$ and observe if $n \geq N$, then $N \geq 2$ and

$$\epsilon \geq \sqrt{\frac{2}{N-1}} \geq \sqrt{\frac{2}{n-1}}.$$

□

Remark 4.7. The limit of a sequence depends only upon the notion of open sets. See Problem 4.4. In particular if (X, d) and (X, d_*) have the same open sets, then they have the same convergent sequences. ◇

4.2. **Sequences and Closed Sets.**

Proposition 4.8. *Suppose X is a metric space, $S \subseteq X$ and $L \in X$.*

(i) If for every $\epsilon > 0$, there exists a point $s \in S$ such that $d(s, L) < \epsilon$ (equivalently $N_\epsilon(L) \cap S \neq \emptyset$), then there is a sequence (s_n) from S which converges to L.
(ii) Conversely, if there exists a sequence (s_n) from S which converges to L, then for every $\epsilon > 0$ the set $N_\epsilon(L) \cap S \neq \emptyset$.
(iii) The set S is closed if and only if every sequence from S which converges in X actually converges in S.

†

Proof. To prove item (i), given $n \in \mathbb{N}^+$ there exists $s_n \in S$ such that $d(s_n, L) < 1/n$. It follows that (s_n) converges to L.

To prove item (ii), suppose there is a sequence (s_n) from S which converges to L. Given $\epsilon > 0$ there is an N such that $d(a_n, L) < \epsilon$ for all $n \geq N$. Hence $s_N \in N_\epsilon(L) \cap S$.

Now suppose that S is not closed, equivalently $\tilde{S}$ is not open. Thus there is a point $L \in \tilde{S}$ such that for every $\epsilon > 0$, $N_\epsilon(L) \not\subseteq \tilde{S}$. Equivalently, for every $\epsilon > 0$, there is some $s \in N_\epsilon(L) \cap S$. Hence, by item (i), there is a sequence from S which converges to L.

Finally, suppose S is closed and (a_n) is a sequence from S which converges to $L \in X$. Since $\tilde{S}$ is open, if $L \notin S$, then there is an $\epsilon > 0$ such that $N_\epsilon(L) \cap S = \emptyset$. In particular, $d(a_n, L) \geq \epsilon$ for all n and (a_n) does not converge to L. Hence $L \in S$ and item (iii) is established. □

Do Problems 4.5, 4.6, and 4.7.

Suppose X is a metric space and S is a subset of X. A point $p \in X$ is a *limit point of S* if there exists a sequence (s_n) from $S \setminus \{p\}$ which converges to p. Problem 4.5 gives an often useful alternate characterization of limit point and in Problem 4.6 you are asked to show that a set is closed if and only if it contains all its limit points. For examples of limit points, see Exercise 4.13.

4.3. **The Monotone Convergence Theorem for Real Numbers.**

For *numerical sequences*, that is sequences from $\mathbb{R}$, limits are compatible with the order structure on $\mathbb{R}$.

Proposition 4.9. *Suppose (a_n) and (b_n) are sequences from $\mathbb{R}$ and $c \in \mathbb{R}$.*

(i) If $a_n \leq b_n + c$ for all n and if both sequences converge, then

$$\lim_n a_n \leq \lim_n b_n + c.$$

(ii) If (a_n), (b_n), and (c_n) are all sequences from $\mathbb{R}$, if there is an N so that for $n \geq N$,

$$a_n \leq b_n \leq c_n$$

and if (a_n) and (c_n) converge to the same limit L, then (b_n) also converges to L.

†

The second part of the Proposition is a version of the *squeeze theorem* and in Problem 4.8 you are asked to provide a proof of it.

Proof of part (i). Let A and B denote the limits of (a_n) and (b_n) respectively. Let $\epsilon > 0$ be given. There is an N so that for $n \geq N$ both $|a_n - A| < \epsilon$ and $|b_n - B| < \epsilon$. Hence, $A - B - c = (A - a_n) + (a_n - b_n - c) + (b_n - B) < 2\epsilon$. □

Definition 4.10. A sequence (a_n) from $\mathbb{R}$ is increasing (synonymously *non-decreasing*) if $a_n \leq a_{n+1}$ for all n. The sequence is strictly increasing if $a_n < a_{n+1}$ for all n.

A sequence is *eventually increasing* if there is an N so that the sequence $(a_n)_{n=N}^{\infty}$ is increasing.

The notion of a *decreasing sequence* is defined analogously. A *monotone* sequence is a sequence which is either increasing or decreasing.

The numerical sequence $(a_n)_{n=n_0}^{\infty}$ is *bounded above* if there is a C such that $a_n \leq C$ for all $n \geq n_0$. Equivalently, (a_n) is bounded above if the set $\{a_n : n \geq n_0\}$ (the range of the sequence) is bounded above as a subset of $\mathbb{R}$. ◁

Definition 4.11. A sequence (a_n) of real numbers *diverges to infinity,* written

$$\lim_{n\to\infty} a_n = \infty,$$

if for each $C > 0$ there is an N such that $a_n > C$ for all $n \geq N$. ◁

If (a_n) diverges to infinity, then (a_n) diverges (does not converge). Indeed, given $L \in \mathbb{R}$ there exists an N such that $a_n \geq L + 1$ for all $n \geq N$. For a general result see Proposition 4.18.

Theorem 4.12 (monotone convergence for real numbers)**.** If (a_n) is an increasing sequence from $\mathbb{R}$, then $(a_n)_{n=n_0}^{\infty}$ converges if and only if it is bounded above. If it is bounded above then it converges to

$$\sup\{a_n : n \geq n_0\}$$

If it is not bounded above, then it diverges to infinity.

Remark 4.13. Generally, results stated for increasing sequences hold for eventually increasing sequences in view of Proposition 4.3(ii). ◇

Proof. Suppose the sequence is bounded above. The set $R = \{a_n : n \in \mathbb{N}\}$ (the range of the sequence) is nonempty and bounded above and therefore has a least upper bound. Let $A = \sup(R)$. Given $\epsilon > 0$ there is an $r \in R$ such that $A - \epsilon < r$. There is an N so that $r = a_N$. If $n \geq N$, then, since the sequence is increasing, $0 \leq A - a_n \leq A - a_N < \epsilon$. Hence (a_n) converges to A.

Suppose now that (a_n) is increasing and not bounded above. Since it is not bounded above, given C there is an N such that $a_N > C$. Since it is increasing, if $n \geq N$, then $a_n \geq a_N > C$ and thus (a_n) diverges to infinity.

□

Do Problem 4.9.

4.3.1. *The real numbers as infinite decimals.* Here is an informal discussion of infinite *decimal* (base ten) expansions. An *infinite decimal expansion* (base 10) is an expression of the form

$$a = a_0.a_1a_2a_3\cdots,$$

where $a_0 \in \mathbb{Z}$ and $a_j \in \{0, 1, 2, \ldots, 9\}$. Let

$$s_n = a_0 + \sum_{j=1}^{n} \frac{a_j}{10^j}$$

and note that the sequence (s_n) is increasing and bounded above by $a_0 + 1$. Thus the sequence (s_n) converges to some real number s and we identify a with this real number.

Conversely, given a real number s there is a smallest integer $m > s$. Let $a_0 = m - 1$. Recursively choose a_j so that, with $s_n = a_0.a_1\cdots a_n$, we have $0 \leq s - s_n \leq \frac{1}{10^n}$. In this case (s_n) converges to s and we can identify s with an infinite decimal expansion.

Note that a real number can have more than one decimal expansion. For example both $0.999\cdots$ and $1.000\cdots$ represent the real number 1.

Remark 4.14. We could also consider expansions with other bases, not just base 10. Base two, called *binary*, is common. Base three is called *ternary*. For $n \in \mathbb{N}$ with $n \geq 2$, expansions base n are called *n-ary*. ◇

Remark 4.15. Here is an informal argument that a rational number has a repeating infinite decimal expansion.

Suppose x is rational, $x = \frac{m}{n}$. Note that the Euclidean division algorithm produces a decimal representation of x. At each stage there are at most n choices of remainder. Hence, after at most n steps of the algorithm, we must have a repeat remainder. From there the decimal repeats. ◇

4.3.2. *An abundance of real numbers.*

Proposition 4.16. *The set $\mathbb{R}$ is uncountable.* †

Proof. It suffices to show if $f : \mathbb{N} \to \mathbb{R}$, then f is not onto. For notational ease, let $x_j = f(j)$.

Choose $b_0 > a_0$ such that $x_0 \notin I_0 := [a_0, b_0]$. Next choose $a_1 < b_1$ such that $a_0 \leq a_1 < b_1 \leq b_0$ and $x_1 \notin I_1 = [a_1, b_1]$. Continuing in this fashion, construct, by the principle of recursion, a sequence of intervals $I_j = [a_j, b_j]$ such that

(i) $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$;

(ii) $b_j - a_j > 0$; and
(iii) $x_j \notin I_k$ for $j \le k$.

Observe that the recursive construction of the sequences of endpoints (a_j) and (b_j) implies that $a_0 \le a_1 \le a_2 < \cdots < b_2 \le b_1 \le b_0$; i.e., (a_j) is increasing and is bounded above by each b_m. By Theorem 4.12 (a_j) converges to

$$y = \sup\{a_j : j \in \mathbb{N}\}.$$

In particular, $a_m \le y \le b_m$ for each m. Thus $y \in I_m$ for all m. On the other hand, for each k,

$$x_k \notin I_k$$

and so $y \neq x_k$. Hence y is not in the set $\{x_k : k \in \mathbb{N}\}$ which is the range of f. □

Do Problem 4.10.

4.4. **Limit Theorems.**

Definition 4.17. A sequence (a_n) from a metric space X is *bounded* if there exists an $x \in X$ and $R > 0$ such that $\{a_n : n \in \mathbb{N}\} \subseteq N_R(x)$. ◁

Proposition 4.18. *Convergent sequences are bounded.* †

Proof. Suppose (a_n) converges to L in the metric space X. Observe, with $\epsilon = 1$ there is an N such that if $n \ge N$, then $d(a_n, L) < 1$. Choosing

$$R = \max(\{d(a_j, L) : 0 \le j < N\} \cup \{1\}) + 1$$

gives $\{a_n : n \in \mathbb{N}\} \subseteq N_R(L)$. Hence $\{a_n : n \in \mathbb{N}\}$ is bounded. □

Proposition 4.19. *Let (a_n) be a sequence from $\mathbb{R}^g$ and write $a_n = (a_{1,n}, \dots, a_{g,n})$. The sequence (a_n) converges to $L = (L_1, \dots, L_g) \in \mathbb{R}^g$ if and only if $(a_{j,n})_n$ converges to L_j for each $1 \le j \le g$.* †

Proposition 4.20. *Let (a_n) and (b_n) be sequences from $\mathbb{R}^g$ and $c \in \mathbb{R}$. If (a_n) converges to A and (b_n) converges to B, then*

(i) $(a_n + b_n)$ converges to $A + B$;
(ii) (ca_n) converges to cA;
(iii) $(a_n \cdot b_n)$ converges to $A \cdot B$; and
(iv) if $g = 1$ and $b_n \neq 0$ for each n and $B \neq 0$, then $\frac{a_n}{b_n}$ converges to $\frac{A}{B}$.

†

Proof. Proofs of the first two items are routine and left to the reader.

To prove the third item, let $\epsilon > 0$ be given. Since the sequence (b_n) converges, it is bounded by say M. Since (a_n) and (b_n) converge to A and B respectively, there exists N_a and N_b such that if $n \ge N_a$, then

$$\|A - a_n\| \le \frac{\epsilon}{2(M+1)}$$

and likewise if $n \geq N_b$, then

$$\|B - b_n\| < \frac{\epsilon}{2(\|A\|+1)}.$$

Choose $N = \max\{N_a, N_b\}$. If $n \geq N$, then

$$\begin{aligned}\|A \cdot B - a_n \cdot b_n\| =& \|A \cdot (B - b_n) + (A - a_n) \cdot b_n\| \\ \leq& \|A\|\|B - b_n\| + \|A - a_n\|\|b_n\| \\ \leq& \|A\|\|B - b_n\| + \|A - a_n\|\, M \\ <& \epsilon.\end{aligned}$$

In view of item (iii), to prove the last statement, it suffices to prove it under the assumption that $a_n = 1$ for all n. Since (b_n) converges to $B \neq 0$, with $\epsilon = \frac{|B|}{2} > 0$ there is an M such that if $n \geq M$, then $|B - b_n| < \frac{|B|}{2}$. Using this inequality and rearranging $|B - b_n| + |b_n| \geq |B|$ gives $|b_n| \geq \frac{|B|}{2}$. For such n

$$|\frac{1}{B} - \frac{1}{b_n}| = \frac{|B - b_n|}{|B|\,|b_n|} \leq |B - b_n| \frac{2}{|B|^2}.$$

The remaining details are left to the gentle reader. □

Proposition 4.21. *In the metric space* $\mathbb{R}$, *if* $0 \leq r < 1$, *then both* (r^n) *and* (nr^n) *converge to* 0. †

Proof. That (r^n) converges to 0 is Example 4.5. To prove that (nr^n) converges to 0, note that, by Example 4.4, for n sufficiently large

$$\frac{n}{n+1} > r.$$

It follows that there is an N such that for $n \geq N$ the sequence (nr^n) is decreasing. Since it also bounded below by 0 it converges to some $L \geq 0$. Hence, using (r^n) converges to 0,

$$rL = rL + 0 = r \lim nr^n + \lim r^{n+1} = \lim (n+1) r^{n+1} = L.$$

Since $r \neq 1$, it follows that $L = 0$. □

Do Problem 4.11.

Proposition 4.22. *Suppose* (a_n) *is a sequence of nonnegative numbers,* $p, q \in \mathbb{N}^+$ *and* $r = \frac{p}{q}$. *If* (a_n) *converges to* L, *then* (a_n^r) *converges to* L^r. †

Proof. Item (iii) of Proposition 4.20 with $g = 1$ and $b_n = a_n$ shows that (a_n^2) converges to L^2. An induction argument now shows that (a_n^p) converges to L^p.

To show $(a_n^{\frac{1}{q}})$ converges to $L^{\frac{1}{q}}$, first observe that $L \geq 0$. Suppose $L > 0$. In this case, the identity,

$$(x^q - y^q) = (x - y) \sum_{j=0}^{q-1} x^j y^{q-1-j}$$

applied to $x = a_n^{\frac{1}{q}}$ and $y = L^{\frac{1}{q}}$ gives,

$$|a_n - L| = |a_n^{\frac{1}{q}} - L^{\frac{1}{q}}| \sum_{j=0}^{q-1} a_n^{\frac{j}{q}} L^{\frac{q-1-j}{q}} \geq |a_n^{\frac{1}{q}} - L^{\frac{1}{q}}| L^{\frac{q-1}{q}}.$$

From here the remainder of the argument is easy and left to the gentle reader. □

Have another look at Problem 4.11.

4.5. **Subsequences.**

Definition 4.23. Given a sequence (a_n) and an increasing sequence $n_1 < n_2 < \dots$ of natural numbers, the sequence $(b_j = a_{n_j})_j$ is a *subsequence* of (a_n).

Alternately, a sequence (b_j) is a subsequence of (a_n) if there is a strictly increasing function σ (from the domain of b to that of a) such that $b_j = a_{\sigma(j)}$. ◁

Example 4.24. The sequence $(\frac{1}{j^2})$ is a subsequence of $(\frac{1}{n})$ (choosing $n_j = j^2$ for $j \geq 1$).

The constant sequences (-1) and (1) are both subsequences of $((-1)^n)$. △

Proposition 4.25. *Suppose (a_n) is sequence in a metric space X. If (a_n) converges to $L \in X$, then every subsequence of (a_n) converges to L.* †

This proposition is an immediate consequence of Problem 4.1.

Note that Proposition 4.25 gives convenient criteria for a sequence to diverge. Namely, if (a_n) has a divergent subsequence, then (a_n) is divergent. Similarly, if (a_n) has subsequences (b_j) and (c_k) which converge to B and C respectively and $B \neq C$, then (a_n) is divergent.

Do Problem 4.12.

Proposition 4.26. *Let (x_n) be a sequence from a metric space X and let $y \in X$ be given. There exists a subsequence (x_{n_k}) of (x_n) such that $(x_{n_k})_k$ converges to y if and only if for every $\epsilon > 0$ the set*

$$J_\epsilon = \{n \in \mathbb{N} : d(y, x_n) < \epsilon\}$$

is infinite. †

Proof. Suppose for each ϵ the set J_ϵ is infinite. With $\epsilon = 1$ there is an n_1 such that $d(y, x_{n_1}) < 1$. Suppose now that $n_1 < n_2 \cdots < n_k$ have been constructed so that $d(y, x_{n_j}) < \frac{1}{j}$ for each $1 \leq j \leq k$. Since the set $\{n : d(y, a_n) < \frac{1}{k+1}\}$ is infinite, there exists a $n_{k+1} > n_k$ such that $d(y, a_{n_{k+1}}) < \frac{1}{k+1}$. Thus, by recursion, we have constructed a subsequence (a_{n_k}) which converges to y.

For the converse, suppose the subsequence $(x_{n_k})_k$ converges to y. Given $\epsilon > 0$ the set

$$I_\epsilon = \{n_k : d(x_{n_k}, y) < \epsilon\}$$

is infinite (since there is a K such that $d(a_{n_k}, y) < \epsilon$ for $k \geq K$). The observation $I_\epsilon \subseteq J_\epsilon$ completes the proof. □

4.6. **Limits Superior and Inferior.**

Proposition 4.27. *Given a bounded sequence* (a_n) *of real numbers, let*

$$\alpha_n = \sup\{a_j : j \geq n\}.$$

The sequence (α_n) *is decreasing and bounded below and hence converges.* †

The proof of the Proposition is left as an exercise (see Problem 4.13.)

Definition 4.28. The limit of the sequence (α_n) of Proposition 4.27 is called the *limsup* or *limit superior* of the sequence (a_n) and denoted $\limsup a_n$ or $\overline{\lim}\, a_n$. The *liminf* is defined analogously. ◁

In the case that (a_n) is not bounded above, define $\limsup a_n = \infty$. In the case that (a_n) is bounded above, but not necessarily below, the sequence (α_n) is defined and it is not bounded below if and only if (a_n) converges to $-\infty$. In this latter case, define the $\limsup a_n = -\infty$.

Do Problem 4.16.

Example 4.29. Here are some simple examples.

(i) The lim sup and lim inf of $(\sin(\frac{\pi}{2}n))$ are 1 and -1 respectively.
(ii) The lim sup and lim inf of the sequence $((-1)^n(1+\frac{1}{n}))$ are also 1 and -1 respectively.
(iii) The lim inf of the sequence $((1-(-1)^n)n)$ is 0. It has no lim sup. Alternately, the lim sup could be interpreted as ∞.

△

Observe that $\inf\{a_j : j \geq n\} \leq a_n \leq \sup\{a_j : j \geq n\}$ for each n. Do Exercise 4.11.

Proposition 4.30. *A bounded sequence* (a_n) *converges if and only if*

$$\limsup a_n = \liminf a_n$$

and in this case (a_n) *converges to this common value.* †

Proof. For notational purposes, let $\alpha_n = \sup\{a_j : j \geq n\}$ and let $\gamma_n = \inf\{a_j : j \geq n\}$.

Suppose

$$\lim \alpha_n = \limsup a_n = \liminf a_n = \lim \gamma_n$$

and let L denote this common value. Observe that $\gamma_n \leq a_n \leq \alpha_n$ for all n. Hence, by the Squeeze Theorem, Problem 4.8, (a_n) converges to L.

Now suppose (a_n) converges to L. Given $\epsilon > 0$, there is an N such that if $j \geq N$, then $|a_j - L| < \epsilon$. In particular, for $j \geq N$, we have $a_j \leq L + \epsilon$ and thus $\alpha_N \leq L + \epsilon$. Consequently, if $n \geq N$, then

$$L - \epsilon < a_n \leq \alpha_n \leq \alpha_N \leq L + \epsilon$$

and therefore $|\alpha_n - L| \leq \epsilon$. It follows that (α_n) converges to L and therefore

$$\limsup a_n = L.$$

By a symmetrical argument, we also have

$$\liminf a_n = L,$$

completing the proof. □

Do Problem 4.14.

Lemma 4.31. *If (a_n) is a bounded sequence of non-negative real numbers, and if (c_n) is a sequence of nonnegative reals which converges to the c, then $\limsup c_n a_n = c \limsup a_n$. (If (a_n) is unbounded (but still nonnegative) so that $\limsup a_n = \infty$, and $c > 0$, then $\limsup c_n a_n = \infty$ too.)* †

The proof is left to the gentle reader as Problem 4.15.

Proposition 4.32. *Suppose (a_n) is a bounded sequence and let $L = \limsup a_n$. Then the set $\{n : a_n \geq \rho\}$ is finite for all $\rho > L$, and infinite for all $\rho < L$.*

Conversely, if L' is a real number such that $\{n : a_n \geq \rho\}$ is finite for all $\rho > L'$, and infinite for all $\rho < L'$, then $L' = L$. †

Proof. Suppose $\rho > L$. With notation as above, with $\epsilon = \rho - L$, there is an N such that $\alpha_n - L < \rho - L$ for all $n \geq N$. Thus there is an N such that $\alpha_N < \rho$ and thus, $a_n < \rho$ for all $n \geq N$. Now suppose $\rho < L$. In this case, $\alpha_N \geq L > \rho$ for all N. In particular, there exists an element of the set $\{a_m : m \geq N\}$ which exceeds ρ; i.e., there is an $n \geq N$ such that $a_n > \rho$.

The proof of the converse is left as an exercise. See Problem 4.18. □

Proposition 4.33 (Optional)**.** *Suppose (a_n) is a bounded sequence of real numbers. Given $x \in \mathbb{R}$, let $J_x = \{n : a_n > x\}$ and let*

$$S = \{x \in \mathbb{R} : J_x \text{ is infinite}\}.$$

Then,

$$\limsup a_n = \sup(S).$$

†

Proof. For notational ease, let $\alpha_m = \sup\{a_n : n \geq m\}$ and let $\alpha = \limsup a_n$

Observe that J_x is infinite if and only if for each $n \in \mathbb{N}$ there is an $m \geq n$ such $m \in J_x$; i.e., there is an $m \geq n$ such that that $a_m > x$.

To prove α is an upper bound for S, let $x \in S$ be given. Given an integer n there is an $m \geq n$ such that $a_m > x$. Hence $\alpha_n > x$. It follows that $\alpha \geq x$.

To prove that α is the least upper bound of S, suppose $x < \alpha$. Given n, it follows that $x < \alpha_n$. Hence, x is not an upper bound for the set $\{a_j : j \geq n\}$ which means there is an $m \geq n$ such that $x < a_m \leq \alpha_n$. This shows J_x is infinite. Thus $x \in S$. It follows that $(-\infty, \alpha) \supseteq S$ and thus if β is an upper bound for S, then $\beta \geq \alpha$. Hence α is the least upper bound of S. □

4.7. **Worked Examples.**

Sample Problem 4.1. Let $x_0 = 2$ and define, recursively, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$. Show that (x_n) converges (in $\mathbb{R}$).

Solution. It suffices to show that the sequence is decreasing and bounded below. To see that (x_n) is bounded below, it is enough to observe that $x_n > 0$ and show that (x_n^2) is bounded below. To this end, note that, if $a > 0$ and $a^2 \geq 2$, then

$$\left(\frac{a}{2} + \frac{1}{a}\right)^2 - 2 = \left(\frac{a}{2} - \frac{1}{a}\right)^2 \geq 0.$$

Thus, arguing by induction, if $x_n > 0$, then $x_{n+1}^2 \geq 2$. Since $x_0 = 2$, we conclude $x_n^2 \geq 2$ for all n.

Now to see that (x_n) is decreasing, note that

$$x_n - x_{n+1} = \frac{x_n}{2} - \frac{1}{x_n} = \frac{x_n^2 - 2}{2x_n} \geq 0.$$

Sample Problem 4.2. Show, arguing directly from the definitions, that the *numerical sequence* (sequence from $\mathbb{R}$) (a_n) given by

$$a_n = \frac{n^2 + 2}{n^2 - n + 1}$$

converges to 1.

Solution. Observe that,

$$\frac{n^2}{2} - n + 1 = \frac{1}{2}[(n-1)^2 + 1] \geq 0.$$

Hence $n^2 - n + 1 \geq \frac{n^2}{2}$. Further, $n + 1 \leq 2n$ for $n \geq 1$.

Now, given $\epsilon > 0$ choose, by Theorem 2.10, an $N \in \mathbb{N}^+$ such that $N \geq \max\{1, \frac{4}{\epsilon}\}$. For $n \geq N$,

$$|a_n - 1| = \left|\frac{n+1}{n^2 - n + 1}\right| \leq \frac{2n}{\frac{n^2}{2}} = \frac{4}{n} \leq \frac{4}{N} < \epsilon.$$

4.8. **Exercises.**

Exercise 4.1. Show, arguing directly from the definitions, that the numerical sequences

$$a_n = \frac{2n-3}{n+5}, \quad n \geq 0;$$
$$b_n = \frac{n+3}{n^2 - n - 1} \quad n \geq 2$$

converge.

Exercise 4.2. By negating the definition of convergence of a sequence, state carefully what it means for the sequence (a_n) from the metric space (X, d) to not converge. The statement should start with: *For every* $L \in X$ *there exists ...* .

Exercise 4.3. Show that the numerical sequence $(a_n = (-1)^n)$ does not converge, arguing directly from the definitions. Suggestion: show if $L \neq 1$, then (a_n) does not converge to L; and if $L \neq -1$, then (a_n) does not converge to L.

Exercise 4.4. Consider the sequence (s_n) from $\mathbb{R}$ defined by

$$s_n = \sum_{j=1}^{n} j^{-2}.$$

Show by induction that

$$s_n \leq 2 - \frac{1}{n}.$$

Prove that the sequence (s_n) converges.

Exercise 4.5. Define a sequence from $\mathbb{R}$ as follows. Fix $r > 1$. Let $a_1 = 1$ and define recursively,

$$a_{n+1} = \frac{1}{r}(a_n + r + 1).$$

Show, by induction, that (a_n) is increasing and bounded above by $\frac{r+1}{r-1}$. Does the sequence converge?

Exercise 4.6. Return to Exercise 4.1, but now verify the limits using Theorem 4.20 together with a little algebra.

Exercise 4.7. Find the limit in Exercise 4.5.

Exercise 4.8. A point p in a metric space X is a *subsequential limit* of a sequence (a_n) if there is a subsequence $(a_{n_k})_k$ of (a_n) which converges to p. Let $\sigma : \mathbb{N} \to \mathbb{Q} \cap [0, 1]$ be a bijection. What are the subsequential limits of the sequence $(\sigma(n))$?

Exercise 4.9. Prove Proposition 4.3(iii). Show that Item (iv) is a consequence of Item (iii).

Exercise 4.10. Show, if (a_n) is a bounded sequence of real numbers, then $\limsup a_n = -\liminf(-a_n)$.

Exercise 4.11. Suppose (a_n) is a bounded sequence of real numbers. Prove

$$\liminf a_n \leq \limsup a_n.$$

Give an example which shows the inequality can be strict.

Exercise 4.12. Show, with out appealing to the existence of square roots, if (a_n) is a sequence of real numbers and (a_n^2) converges to 0, then (a_n) converges to 0. (Suggestion: prove the contrapositive.) Note that this exercise provides a route to complete the proof of Propsition 4.6(ii) without appealing to the existence of square roots.

Exercise 4.13. Let X be a set and $S \subseteq X$. Let S' denote the set of limit points of S (in X). Determine carefully the sets S' for the sets S below.

(i) S is finite;
(ii) $X = \mathbb{R}$ and $S = \mathbb{Z}$;
(iii) $X = \mathbb{R}$ and $S = (0, 1)$;
(iv) $X = \mathbb{R}$ and $S = [0, 1]$;
(v) $X = \mathbb{R}$ and $S = \{\frac{1}{n} : n \in \mathbb{N}^+\}$;
(vi) $S = X = \{\frac{1}{n} : n \in \mathbb{N}^+\}$.
(vii) $X = \mathbb{R}$ and $S = [0, 1] \cup \{2\}$.

4.9. Problems.

Problem 4.1. Suppose (a_n), a sequence in a metric space X, converges to $L \in X$. Show, if $\sigma : \mathbb{N} \to \mathbb{N}$ is one-one, then the sequence $(b_n = a_{\sigma(n)})_n$ also converges to L.

Problem 4.2. Suppose (a_n) is a sequence from $\mathbb{R}$. Show that if (a_n) converges to L then the sequence (of Cesaro means) (s_n) defined by

$$s_n = \frac{1}{n+1} \sum_{j=0}^{n} a_j$$

also converges to L. Does the converse also hold?

Problem 4.3. Suppose (a_n) and (b_n) are sequences from a metric space X. Show, if both sequences converge to $L \in X$, then (c_n), defined by $c_{2n} = a_n$ and $c_{2n+1} = b_n$, also converges to L.

Problem 4.4. Suppose d and d' are both metrics on X and that the metric spaces (X, d) and (X, d') have the same open sets. Show, the sequence (a_n) from X converges in (X, d) if and only if it converges in (X, d') and then to the same limit.

Problem 4.5. Let S be a subset of a metric space X.

Prove that $y \in X$ is a limit point of S if and only if for every $\epsilon > 0$ there exists a point $s \in S$ such that $s \neq y$ and $d(s, y) < \epsilon$ (equivalently $N_\epsilon(y) \cap (S \setminus \{y\}) \neq \emptyset$).

Prove that S is closed if and only if S contains all its limit points. (Often this limit point criteria is taken as the definition of closed set.)

Problem 4.6. Let S' denote the set of limit points of a subset S of a metric space X. (See Problem 4.5.) Prove that S' is closed.

Problem 4.7. Show, if C is a subset of $\mathbb{R}$ which has a supremum, say α, then there is a sequence (c_n) from C which converges to α. Use this fact, plus Proposition 4.8, to give another proof of Proposition 3.35.

Problem 4.8. [A squeeze theorem] Suppose (a_n), (b_n), and (c_n) are sequences of real numbers. Show, if $a_n \leq b_n \leq c_n$ for all n and both (a_n) and (c_n) converge to L, then (b_n) converges to L.

Problem 4.9. Suppose (a_n) is a sequence of positive real numbers and assume

$$L = \lim \frac{a_{n+1}}{a_n}$$

exists. Show, if $L < 1$, then (a_n) converges to 0 by completing the following outline (or otherwise).

(i) Choose $L < \rho < 1$.
(ii) Show there is an M so that if $m \geq M$, then $a_{m+1} \leq \rho a_m$;
(iii) Show $a_{M+k} \leq \rho^k a_M$ for $k \in \mathbb{N}$;
(iv) Show $a_n \leq \rho^n \frac{a_M}{\rho^M}$ for $n \geq M$;
(v) Complete the proof.

Give an example where (a_n) converges to 0 and $L = 1$; and give an example where (a_n) does not go to 0, but $L = 1$.

Prove, if $0 \leq L < 1$, and p is a positive integer, then $(n^p a_n)$ converges to 0 too.

Problem 4.10. A subset S of a metric space X is *dense* in X if $\overline{S} = X$. Use Theorem 2.10 to prove for each real number r there is a sequence (q_n) of rational numbers converging to r. Use Proposition 4.8 to conclude that the closure of $\mathbb{Q}$ (in $\mathbb{R}$) is $\mathbb{R}$; i.e., $\mathbb{Q}$ is dense in $\mathbb{R}$. (See Remark 2.11.)

Problem 4.11. Let $a_0 = \sqrt{2}$ and define, recursively, $a_{n+1} = \sqrt{a_n + 2}$. Prove, by induction, that the sequence (a_n) is increasing and is bounded above by 2. Does the sequence converge? If so, what should the limit be?

Problem 4.12. Suppose (a_n) and (b_k) are sequences in a metric space X and $L \in X$. Show,

(i) if there is an $\eta > 0$ such that $d(b_k, L) \geq \eta$ for all k, then no subsequence of (b_k) converges to L; and
(ii) if every subsequence of (a_n) has a further subsequence which converges to L, then (a_n) converges to L. (Suggestion: prove the contrapositive using part (i).)
(iii) Give an example of a sequence (a_n) which doesn't converge, but such that every subsequence of (a_n) has a subsequence which does converge.

Problem 4.13. Prove Proposition 4.27.

Problem 4.14. Suppose both (a_n) and (b_n) are bounded sequences of real numbers. Prove,

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

[Hint: Use $\{a_j + b_j : j \geq n\} \subseteq \{a_j + b_k : j, k \geq n\}$ and $\sup(S + T) = \sup(S) + \sup(T)$ (under appropriate hypotheses as in Problem 2.6) to show

$$\begin{aligned} \sup(\{a_j + b_j : j \geq n\}) &\leq \sup(\{a_j + b_k : j, k \geq n\}) \\ &= \sup\{a_j : j \geq n\} + \sup\{b_k : k \geq n\}. \end{aligned}$$

Give an example which shows the inequality can be strict.

Problem 4.15. Prove Lemma 4.31.

Problem 4.16. Let (a_n) be a bounded sequence of real numbers. Prove there is a subsequence $(a_{n_j})_j$ which converges to $y = \limsup a_n$. Here is one way to proceed. Show, for each $\epsilon > 0$ the set $\{n : |y - a_n| < \epsilon\}$ is infinite and apply Proposition 4.26.

Show if z is a subsequential limit (See Exercise 4.8) of (a_n), then $z \le \limsup a_n$. Hence $\limsup a_n$ is the largest subsequential limit of the sequence (a_n).

Problem 4.17. Given a sequence $(a_j)_{j=0}^{\infty}$ of real numbers, let

$$s_m = \sum_{j=0}^{m} a_j.$$

The expression $\sum_{n=0}^{\infty} a_n$ is called a *series* and the sequence (s_n) is its *sequence of partial sums*. If the sequence (s_n) converges, then the series is said to *converge* and if, moreover, (s_n) converges to L, then the series converges to L written

$$\sum_{n=0}^{\infty} a_n = L = \lim_{m\to\infty} s_m.$$

In particular, the expression $\sum_{n=0}^{\infty} a_n$ is used both for the sequence (s_n) and the limit of this sequence, if it exists.

Show, if $a_n \ge 0$, then the series either converges or diverges to ∞ depending on whether the partial sums form a bounded sequence or not.

Show, if $0 \le r < 1$, then, for each m,

$$(1-r)\sum_{n=0}^{m} r^n = 1 - r^{m+1}$$

and thus,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Problem 4.18. Suppose $\{a_n : n \ge n_0\}$ is a bounded sequence of real numbers and $L \in \mathbb{R}$. Show if for every $\rho > L$ there is an N so that for all $n \ge N$, $a_n < \rho$ and for every $\rho < L$ and for every N there is an $n \ge N$ such that $a_n \ge \rho$, then $L = \limsup a_n$. See Proposition 4.32.

5. Cauchy sequences and completeness

Definition 5.1. A sequence (a_n) in a metric space (X, d) is *Cauchy* if for every $\epsilon > 0$ there is an N such that if $m, n \geq N$ then $d(a_m, a_n) < \epsilon$. ◁

Proposition 5.2. *Convergent sequences are Cauchy; i.e., if (a_n) is a convergent sequence in a metric space X, then (a_n) is Cauchy.* †

Proof. Suppose that $\lim a_n = L$ and let $\epsilon > 0$ be arbitrary. There is an N such that $d(a_n, L) < \epsilon/2$ for all $n \geq N$. Therefore for $m, n \geq N$ we have (by the triangle inequality)

$$d(a_m, a_n) \leq d(a_m, L) + d(L, a_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired. □

Proposition 5.3. *Cauchy sequences are bounded.* †

Proof. Let (a_n) be a Cauchy sequence. Setting $\epsilon = 1$ in the definition of Cauchy sequences, we see that there is an N such that if $m, n \geq N$ then $d(a_m, a_n) < 1$. Setting $x = a_m$, we see that

$$\{a_n \,:\, n \geq N\} \subseteq N_1(x).$$

Now if we define

$$R = \max\left(\{d(a_j, x) : 0 \leq j < N\} \cup \{1\}\right) + 1$$

then we have

$$\{a_n \,:\, n \in \mathbb{N}\} \subseteq N_R(x),$$

as desired. □

Let X be a metric space and $L \in X$. Problem 5.1 asks you to prove, if (a_n) is a Cauchy sequence from X and (a_n) has a subsequence $(a_{n_k})_k$ that converges to L then (a_n) itself converges to L.

Definition 5.4. A metric space (X, d) is *complete* if every Cauchy sequence in X converges (in X). ◁

Example 5.5. Cauchy sequences in a discrete metric space are eventually constant and hence converge. Thus, a discrete metric space is complete. △

Example 5.6. The metric space $(\mathbb{Q}, d_1)$ is an example of an incomplete space. Exercise 5.2 gives further examples of incomplete spaces. △

Proposition 5.7. *If (a_n) is a bounded sequence of real numbers, then there is a subsequence $(a_{n_k})_k$ of (a_n) which converges to $\limsup a_n$.* †

Proof. (See Problem 4.16.) Let $L = \limsup a_n$ and let $\epsilon > 0$ be given. By Proposition 4.32, the set

$$\{n : a_n > L - \epsilon\}$$

is infinite and the set

$$\{n : a_n > L + \epsilon\}$$

is finite. Hence, the set

$$\{n : |a_n - L| < \epsilon\}$$

is infinite. By Proposition 4.26, there is a subsequence $(a_{n_k})_k$ of (a_n) which converges to L. □

Theorem 5.8. *The metric space* $(\mathbb{R}, d_1)$ *is complete.*

Proof. Let (a_n) be a given Cauchy sequence from $\mathbb{R}$. By Proposition 5.3, this sequence is bounded. Hence it has a limsup α and a subsequence converges to α by Proposition 5.7. By Problem 5.1 the sequence itself converges to α. □

Proposition 5.9. *A closed subset of a complete metric space is complete.* †

Proof. Apply Proposition 4.8. □

Proposition 5.10. *A complete subset of a metric space is closed.* †

Proof. Apply Proposition 4.8. □

Definition 5.11. A sequence $(a_n)_{n=n_0}^{\infty}$ from a metric space X is *super Cauchy* if there exists an N and a constant $0 \le k < 1$ such that

$$d(a_{n+1}, a_n) \le k d(a_n, a_{n-1}) \tag{3}$$

for all $n \ge N$. ◁

The following result is a version of the *contraction mapping principle.*

Proposition 5.12. *If* (a_n) *is super Cauchy, then* (a_n) *is Cauchy. In particular, super Cauchy sequences in a complete metric space converge.* †

Do Exercise 5.1 and Problem 5.2.

Proof. We assume, without loss of generality, that $(a_n)_{n=0}^{\infty}$ satisfies $d(a_{n+2}, a_{n+1}) \le k d(a_{n+1}, a_n)$ for all $n \ge 0$.

First observe, by Equation (2),

$$\sum_{j=0}^{n} k^j \le \frac{1}{k-1}.$$

Next note that, by iterating the inequality of Equation (3),

$$d(a_{m+1}, a_m) \le k^m d(a_1, a_0)$$

for all m. Thus, for $\ell \geq 0$,

$$\begin{aligned} d(a_{n+\ell}, a_n) &\leq \sum_{j=0}^{\ell-1} d(a_{n+j+1}, a_{n+j}) \\ &\leq \sum_{j=0}^{\ell-1} k^{n+j} d(a_1, a_0) \\ &= k^n d(a_1, a_0) \sum_{j=0}^{\ell-1} k^j \\ &\leq k^n \frac{d(a_1, a_0)}{1-k}. \end{aligned}$$

Now, given $\epsilon > 0$, using Example 4.5, choose N such that if $n \geq N$, then $k^n \frac{d(a_1,a_0)}{1-k}\epsilon$. Then for $m \geq n \geq N$ and choosing ℓ such that $m = n + \ell$, it follows that $d(a_m, a_n) < \epsilon$ and the sequence (a_n) is Cauchy. □

Example 5.13. For $n \in \mathbb{N}^+$, let

$$s_n = \sum_{j=2}^{n} \frac{1}{j}.$$

Note that

$$s_{2^n} = \sum_{k=0}^{n-1} \sum_{j=2^k+1}^{2^{k+1}} \frac{1}{j} \geq \frac{n}{2}$$

and thus (s_n) is not a bounded sequence and is therefore not Cauchy.

On the other hand,

$$|s_{n+2} - s_{n+1}| = \frac{1}{n+2} < \frac{1}{n+1} = |s_{n+1} - s_n|.$$

△

5.1. **Worked Examples.**

Sample Problem 5.1. Choose $x_0 > 0$. Show that the sequence (from $\mathbb{R}$) defined recursively by

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

converges.

Solution. If suffices to prove that the sequence is Cauchy since $\mathbb{R}$ is complete. Hence, it suffices to prove that the sequence is super Cauchy by Proposition 5.12. To this end, first observe that, an induction argument shows that $x_n > 0$ for all n. Hence,

$$x_{n+1}x_n = \frac{x_n^2}{2} + 1 \geq 1.$$

Thus,

$$\left|\frac{1}{2} - \frac{1}{x_{n+1}x_n}\right| \leq \frac{1}{2}.$$

Since

$$|x_{n+2} - x_{n+1}| = \left|\left(\frac{1}{2} - \frac{1}{x_{n+1}x_n}\right)(x_{n+1} - x_n)\right| \leq k|x_{n+1} - x_n|$$

with $k = \frac{1}{2} < 1$, the conclusion follows.

See Newton's method. Note too that using properties of limits it is not hard to see that the limit is the square root of two.

Sample Problem 5.2. Suppose $U_1, U_2, \dots$ is a sequence of open sets in a nonempty complete metric space X. Show, if, for each j, the closure of U_j is all of X, then

$$\bigcap_1^\infty U_j \neq \emptyset.$$

This fact is a version of the Baire Category Theorem.

Solution. Observe that for each $x \in X$, $r > 0$, and $j \in \mathbb{N}^+$, that $N_r(x) \cap U_j \neq \emptyset$ by Problem 3.7. Let, as usual, $B_r(x) = \{y \in X : d(x,y) \leq r\}$ (the closed ball of radius r with center x).

Pick a point $x_1 \in U_1$. There is an $r_1 \leq 1$ such that $B_{r_1}(x_1) \subseteq U_1$. There is a point $x_2 \in N_{\frac{r_1}{2}}(x) \cap U_2$. There is an $0 < r_2 < \frac{r_1}{2}$ such that $B_{r_2}(x_2) \subseteq U_2$. In particular, $B_{r_2}(x_2) \subseteq B_{r_1}(x_1)$. Continuing in this fashion constructs a sequence of nonempty sets $B_j = B_{r_j}(x_j)$ such that

(i) $d(p,q) \leq 2^{-j}$ for each $j \in \mathbb{N}$ and $p,q \in B_j$,
(ii) $B_1 \supseteq B_2 \supseteq \cdots$; and
(iii) $B_j \subseteq U_j$.

In view of (i) and (ii), Problem 5.3 implies $\bigcap B_j$ is not empty. On the other hand (iii) implies that $\bigcap B_j \subseteq \bigcap U_j$ and the conclusion follows.

5.2. **Exercises.**

Exercise 5.1. Define a sequence of real numbers recursively as follows. Let $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1 + a_n}.$$

Show (a_n) is not monotonic (that is neither increasing or decreasing). Show that $a_n \geq 1$ for all n and then use Proposition 5.12 to show that (a_n) is Cauchy. Conclude that the sequences converges and find its limit.

Exercise 5.2. Suppose y is a limit point (see Problem 4.5) of the metric space X. Show $Y = X \setminus \{y\}$ is not complete.

Exercise 5.3. Show directly that the sequence $((-1)^n)$ is not Cauchy and conclude that it doesn't converge. Compare with Exercise 4.2.

Exercise 5.4. Suppose (a_n) is an increasing sequence of real numbers. Show, if (a_n) has a bounded subsequence, then (a_n) converges; and (a_n) diverges to infinity if and only if (a_n) has an unbounded subsequence.

5.3. **Problems.**

Problem 5.1. Suppose (x_n) is a Cauchy sequence in a metric space X. Show, if (x_n) has a subsequence (x_{n_k}) which converges to some $y \in X$, then (x_n) converges to y.

Problem 5.2. Fix $A > 0$ and define a sequence from $\mathbb{R}$ as follows. Let $a_0 = 1$. For $n \geq 1$, recursively define

$$a_{n+1} = A + \frac{1}{a_n}.$$

Show, for all $n \geq 1$, $a_n \geq A$ and $a_n a_{n+1} \geq 1 + A^2$. Use Proposition 5.12 to prove that (a_n) converges. What is the limit?

Problem 5.3. The *diameter* of a set S in a metric space X is

$$\operatorname{diam}(S) = \sup\{d(s,t) : s,t \in S\}.$$

(In the case that the set of values $d(s,t)$ is not bounded above this supremum is interpreted as plus infinity.)

Prove, if X is a complete metric space, $S_1 \supseteq S_2 \supseteq \dots$ is a nested decreasing sequence of nonempty closed subsets of X, and the sequence $(\operatorname{diam}(S_n))_n$ converges to 0, then

$$\bigcap_{n=1}^{\infty} S_n$$

contains exactly one point.

Show that this result fails if any of the hypotheses - completeness, closedness of the S_n, or that the diameters tend to 0 - are omitted.

Problem 5.4. Show that the sequence (a_n) from Exercise 5.1 is not eventually monotone. As a suggestion, first show that, for each n, $a_{n+1} \neq a_n$ as otherwise a_n would be irrational.

Problem 5.5. Given metric spaces (X, d_X) and (Y, d_Y) let Z denote the metric space built from X and Y as in Problem 3.1. Show, if X and Y are complete, then so is $X \times Y$.

6. Compact Sets

6.1. **Definitions and Examples.**

Definition 6.1. A subset K of a metric space X is *compact* if for every family $\mathcal{U}$ of subsets of X such that

(i) each $U \in \mathcal{U}$ is open; and
(ii) $K \subseteq \bigcup_{U \in \mathcal{U}} U$,

there exists a finite subset $\mathcal{V} \subseteq \mathcal{U}$ such that

(iii) $K \subseteq \bigcup_{U \in \mathcal{V}} U$.

A family $\mathcal{U}$ satisfying conditions (i) and (ii) is an *open cover* of K. A subset $\mathcal{V} \subseteq \mathcal{U}$ satisfying (iii) is a *subcover* (of the open cover $\mathcal{U}$). ◁

Remark 6.2. Compactness is often expressed as: the subset K of a metric space X is *compact* provided every open cover of K has (or admits) a finite subcover.

Often it is convenient to view covers as an indexed family of sets. In this case an open cover of S consists of an index set I and a collection of open sets $\mathcal{U} = \{U_j : j \in I\}$ whose union contains S. A subcover is then a collection $\mathcal{V} = \{U_j : j \in J\}$, for some subset J of I. A set K is compact if for each collection $\{U_j : j \in I\}$ such that

$$K \subseteq \bigcup_{j \in I} U_j,$$

there is a finite subset $F \subseteq I$ such that

$$K \subseteq \bigcup_{j \in F} U_j.$$

◇

Example 6.3. Consider the set $K = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, 0\right\}$ as a subset of the metric space $\mathbb{R}$.

Let $\mathcal{U}$ be a given open cover of K. There is then a $U_0 \in \mathcal{U}$ such that $0 \in U_0$. Since U_0 is open, there is an $\epsilon > 0$ such that $N_\epsilon(0) \subseteq U_0$. Since $\frac{1}{n}$ converges to 0, there is an N such that if $n \geq N$, then $\frac{1}{n} \in N_\epsilon(0) \subseteq U_0$. For each $j = 1, 2, \dots, N-1$ there is a $U_j \in \mathcal{U}$ such that $\frac{1}{j} \in U_j$. It follows that $\mathcal{V} = \{U_0, \dots, U_{N-1}\} \subseteq \mathcal{U}$ is a finite subcover (of K). Thus K is compact. △

Do Problems 6.1 and 6.2.

Example 6.4. Let $S = (0, 1] \subseteq \mathbb{R}$ and consider the indexed family of sets $U_j = \left(\frac{1}{j}, 2\right)$ for $j \in \mathbb{N}^+$. It is readily checked that

$$S \subseteq \bigcup_{j=1}^{\infty} U_j$$

and of course each U_j is open. Thus $\mathcal{U} = \{U_j : j \in \mathbb{N}^+\}$ is an open cover of S.

Now suppose $F \subseteq \mathbb{N}^+$ is a finite set. In particular, there is an N such that $F \subseteq \{1, \ldots, N\}$ and therefore,

$$\bigcup_{j \in F} U_j \subseteq \bigcup_{j=1}^{N} U_j = (1/N, 2)\,.$$

Thus there is no finite subset F of the index set $\mathbb{N}^+$ such that $K \subseteq \bigcup \{U_j : j \in F\}$ and we conclude that S is not compact. △

Do Problem 6.3 which says that a subset K of a discrete metric space X is compact if and only if K is finite. In particular, if the set K in Example 6.3 is considered with the discrete metric, then it is not compact.

Theorem 6.5. *Closed bounded intervals in $\mathbb{R}$ are compact.*

Proof. Let $[a, b]$ be a given closed bounded interval and let $\mathcal{U}$ be an open cover of $[a, b]$.

If $b = a$ then $[a, b] = \{a\}$. There must be a set $U \in \mathcal{U}$ with $a \in U$ (since $\mathcal{U}$ is a cover of $[a, b]$), so $\mathcal{V} = \{U\}$ is a finite subcover of $[a, b]$ and we are done.

Now we may assume that $b > a$. Define

$$S = \{x \in [a, b] : [a, x] \text{ has a finite subcover from } \mathcal{U}\}.$$

As an initial observation, note that if $a \leq t \leq s$ and $s \in S$, then $t \in S$. By definition S is bounded above by b. There is a $U \in \mathcal{U}$ such that $a \in U$ and hence $[a, a] \subseteq U$. Thus S is not empty. Thus $\sup(S)$ exists and $a \leq \sup(S) \leq b$. In fact, since U is open and contains a, there is $b > c > a$ such that $[a, c] \subseteq U$. Therefore, $b \geq \sup(S) \geq c > a$.

There is a $U_0 \in \mathcal{U}$ such that $\sup(S) \in U_0$ since $\sup(S) \in [a, b]$ and $\mathcal{U}$ is an open cover of $[a, b]$. Because U_0 is open and $\sup(S) > a$, there is an $\epsilon > 0$ such that $a < \sup(S) - \epsilon$ and $N_\epsilon(\sup(S)) \subseteq U_0$. There is an $s \in S$ such that $\sup(S) - \frac{\epsilon}{2} < s \leq \sup(S)$. Since $s \in S$, there is a finite subcover $\mathcal{V} \subseteq \mathcal{U}$ of $[a, s]$; i.e., $\mathcal{V}$ is finite and

$$[a, s] \subseteq \bigcup \{U : U \in \mathcal{V}\}.$$

It follows that

$$\begin{aligned} [a, \sup(S)] &\subseteq \left[a, \sup(S) + \frac{\epsilon}{2}\right] \\ &\subseteq [a, s] \cup \left[\sup(S) - \frac{\epsilon}{2}, \sup(S) + \frac{\epsilon}{2}\right] \\ &\subseteq \bigcup \{U : U \in \mathcal{V}\} \cup \{U_0\}. \end{aligned}$$

Consequently, if $a \leq t \leq b$ and $t \leq \sup(S) + \frac{\epsilon}{2}$, then $t \in S$. Hence $\sup(S) \in S$ and $\sup(S) \geq b$. Thus $b = \sup(S) \in S$ and the proof is complete. □

Theorem 6.6. *If Y is a metric space and $K \subseteq X \subseteq Z$, then K is compact in X if and only if K is compact in Z.*

Remark 6.7. The proposition says that compactness is intrinsic and thus, unlike for open and closed sets, we can speak of compact sets without reference to a larger ambient metric space. ⋄

Proof. First suppose K is compact in X. To prove that K is compact in Z, let $\mathcal{U} \subseteq P(Z)$ an open (in Z) cover of K be given. Let $\mathcal{W} = \{U \cap X : U \in \mathcal{U}\}$. By Proposition 3.28, $\mathcal{W}$ consists of relatively (in X) open sets. Thus $\mathcal{W} \subseteq P(X)$ is an open (in X) cover of K. Hence there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\{U \cap X : U \in \mathcal{V}\}$ covers K. It follows that $\mathcal{V}$ is a finite subset of $\mathcal{U}$ which covers K and hence K is compact as a subset of Z.

Conversely, suppose K is compact in Z. To prove that K is compact in X, let $\mathcal{U} \subseteq P(X)$ be a given open (in X) cover of K. By Proposition 3.28, for each $U \in \mathcal{U}$ there exists an open in Z set W_U such that $U = X \cap W_U$. The collection $\mathcal{W} = \{W_U : U \in \mathcal{U}\} \subseteq P(Z)$ is an open cover of X. Hence there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\{W_U : U \in \mathcal{V}\}$ covers K. It follows that $\mathcal{V}$ is a finite subset of $\mathcal{U}$ which covers K. Hence K is compact in X. □

Do Problems 6.4 and 6.6.

6.2. **Compactness and Closed Sets.** We have already defined what it means for a sequence in an arbitrary metric space to be bounded. Use use a similar definition for subsets.

Definition 6.8. A subset B of a metric space X is *bounded* if there exists an $x \in X$ and a real number $R > 0$ such that $B \subseteq N_R(x)$. ◁

Equivalently, B is bounded if for every $y \in X$ there is a $C > 0$ such that $B \subseteq N_C(y)$. An easy exercise is to show that a subset B of $\mathbb{R}$ is bounded if and only if it is bounded both above and below if and only if there exists a C such that $|b| \leq C$ for all $b \in B$ (equivalently $B \subseteq [-C, C]$). In particular, this definition of bounded for a subset B of $\mathbb{R}$ agrees with our earlier definition of bounded.

Proposition 6.9. *All compact sets are closed and bounded.* †

Proof. Suppose K is a compact subset of a metric space X. First we show that K is closed. If $K = X$ then K is closed, as desired. Suppose now that $K \neq X$, so we can choose a point $y \notin K$. For all natural numbers $n \geq 1$ define

$$V_n = \{x \in X : d(x, y) > 1/n\}.$$

The sets V_n are open and

$$\bigcup_{n=1}^{\infty} V_n \supseteq X \setminus \{y\} \supseteq K.$$

Therefore the family $\mathcal{V} = \{V_n : n \geq 1\}$ is an open cover of K. Since K is compact, there is an $N \in \mathbb{N}$ so that

$$V_N = \bigcup_{n=1}^{N} V_n \supseteq K.$$

It follows that, for each $x \in K$, $d(x, y) > 1/N$. Hence $N_{1/N}(y) \subseteq \tilde{K}$ and so $\tilde{K}$ is open and thus K is closed.

To prove that K is bounded, fix $x_0 \in X$ and let $W_n = \{x \in X : d(x_0, x) < n\}$. Then

$$K \subseteq X = \bigcup_{n=1}^{\infty} W_n.$$

By the compactness of K, there is an N so that $K \subseteq W_N$ and thus K is bounded. □

Proposition 6.10. *Every closed subset of a compact set is compact.* †

Proof. Suppose X is a metric space, $C \subseteq K \subseteq X$, K is compact, and C is closed.

To prove that C is compact, let $\mathcal{U}$ be a given open cover of C. Then $\mathcal{W} = \mathcal{U} \cup \{\tilde{C}\}$ is an open cover of K. Hence some finite subset $\mathcal{G}$ of $\mathcal{W}$ covers K. Without loss of generality, there is a finite subset $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{G} = \mathcal{V} \cup \{\tilde{C}\}$. In particular,

$$C \subseteq K \subseteq \bigcup_{U \in \mathcal{V}} U \cup \tilde{C}.$$

Intersecting with C gives us that

$$C = K \cap C \subseteq \bigcup_{U \in \mathcal{V}} (U \cap C) \subseteq \bigcup_{U \in \mathcal{V}} U.$$

Hence $\mathcal{V}$ is a finite subset of $\mathcal{U}$ that covers C and the proof is complete. □

Corollary 6.11. *All closed and bounded subsets of* $\mathbb{R}$ *are compact. Therefore, a subset of* $\mathbb{R}$ *is compact if and only if it is closed and bounded.* †

Proof. Suppose $K \subseteq \mathbb{R}$ is both closed and bounded. Since K is bounded, there is a positive real M such that $K \subseteq [-M, M]$. Now K is a closed subset of the compact set $[-M, M]$ and is hence itself compact. □

It turns out that this corollary is true with $\mathbb{R}$ replaced by $\mathbb{R}^g$, a result which is called the Heine–Borel Theorem. A proof, based upon the Lebesgue number Lemma, and the concomitant fact that compactness and sequential compactness are the same for a metric space, is in Subsection 6.4 below.

Remark 6.12. If X is an infinite set with the discrete metric, then X is complete (hence closed) and bounded, but not compact. Hence, in general, complete (or closed) and bounded does not imply compact.

For another example, consider the metric d (see Exercise 3.1) on $\mathbb{R}$ defined by

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

The metric space $(\mathbb{R}, d)$ is bounded as

$$\mathbb{R} \subseteq N_1^d(0) = \{x \in \mathbb{R} : d(x, 0) < 1\}.$$

On the other hand the open sets in $(\mathbb{R}, d)$ are the same as the ordinary open sets in $\mathbb{R}$ (with the usual metric $|x-y|$) and thus both metric spaces have the same compact sets. To see that $\mathbb{R}$ is not compact, consider the collection,

$$\mathcal{U} = \{(-n, n) : n \in \mathbb{N}^+\}.$$

This is an open cover of $\mathbb{R}$, but it admits no finite subcover. Hence $\mathbb{R}$ is a closed and bounded subset of the metric space $(\mathbb{R}, d)$ which is not compact.

While these examples may seem a bit contrived, we will encounter other more natural metric spaces for which closed or complete and bounded is not the same as compact. (See for instance Problems 6.8 and 6.7.) $\diamond$

6.3. **Sequential Compactness.**

Definition 6.13. A subset K of a metric space X is *sequentially compact* if every sequence in K has a subsequence which converges in K; i.e., if (a_n) is a sequence from K, then there exists $p \in K$ and a subsequence $(a_{n_j})_j$ of (a_n) which converges to p. $\triangleleft$

Remark 6.14. The notion of sequentially compact does not actually depend upon the larger metric space X, just the metric space K. $\diamond$

Proposition 6.15. *If X is sequentially compact, then X is complete.* †

Problem 6.9 asks you to provide a proof of this Proposition.

Proposition 6.16. *Let X be a metric space. If X is compact, then X is sequentially compact.* †

Proof. Let $(s_n)_{n=n_0}^{\infty}$ be a given sequence from X. If there is an $s \in X$ such that for every $\epsilon > 0$ the set $J_\epsilon(s) = \{n : s_n \in N_\epsilon(s)\}$ is infinite, then, by Proposition 4.26, (s_n) has a convergent subsequence (that converges to s).

Arguing by contradiction, suppose for each $s \in X$ there is an $\epsilon_s > 0$ such that the set

$$J(s) = \{n : s_n \in N_{\epsilon_s}(s)\}$$

is finite. The collection $\{N_{\epsilon_s}(s) : s \in X\}$ is an open cover of X. Since X is compact there is a finite subset $F \subseteq X$ such that $\mathcal{V} = \{N_{\epsilon_t}(t) : t \in F\}$ is a cover of X; i.e.,

$$X \subseteq \bigcup_{t \in F} N_{\epsilon_t}(t).$$

For each n there is a $t \in F$ such that $s_n \in N_{\epsilon_t}(t)$ and thus

$$\{n \in \mathbb{Z} : n \geq n_0\} = \bigcup_{t \in F} J_{\epsilon_t}(t).$$

But then, for some $u \in F$, the set $J_{\epsilon_u}(u)$ is infinite, a contradiction. $\square$

Do Problem 6.10.

Proposition 6.17. *If X is compact, then X is complete.* †

Corollary 6.18. *The metric space $\mathbb{R}$ is complete.* †

Proof. Suppose (a_n) is a Cauchy sequence from $\mathbb{R}$. It follows that (a_n) is bounded and hence there is a number $R > 0$ such that each a_n is in the interval $I = [-R, R]$. Since I is compact, it is complete by Proposition 6.17. Hence (a_n) converges in I and thus in $\mathbb{R}$. □

The remainder of this section is devoted to proving the converse of Proposition 6.16.

Lemma 6.19 (Lebesgue number lemma)**.** *If K is a sequentially compact metric space and if $\mathcal{U}$ is an open cover of K, then there is a $\delta > 0$ such that for each $x \in K$ there is a $U \in \mathcal{U}$ such that $N_\delta(x) \subseteq U$.* †

Proof. We argue by contradiction. Accordingly, suppose for every $n \in \mathbb{N}^+$ there is an $x_n \in K$ such that, for each $U \in \mathcal{U}$, $N_{\frac{1}{n}}(x_n)$ is not a subset of U. The sequence (x_n) has a subsequence $(x_{n_k})_k$ which converges to some $w \in K$ because K is sequentially compact. There is a $W \in \mathcal{U}$ such that $w \in W$. Hence there is an $\epsilon > 0$ such that $N_\epsilon(w) \subseteq W$. Choose k so that $\frac{1}{n_k} < \frac{\epsilon}{2}$ and also so that $d(x_{n_k}, w) < \frac{\epsilon}{2}$. Then $N_{\frac{1}{n_k}}(x_{n_k}) \subseteq N_\epsilon(w) \subseteq W$, a contradiction. □

Definition 6.20. A metric space X is *totally bounded* if, for each $\epsilon > 0$, there exists a finite set $F \subseteq X$ such that

$$X = \bigcup_{x \in F} N_\epsilon(x).$$

◁

Proposition 6.21. *If X is sequentially compact then X is totally bounded.* †

Proof. We prove the contrapositive. Accordingly, suppose X is not totally bounded. Then there exists an $\epsilon > 0$ such that for every finite subset F of X,

$$X \neq \bigcup_{x \in F} N_\epsilon(x).$$

Choose $x_1 \in X$. Choose $x_2 \notin N_\epsilon(x_1)$. Recursively choose

$$x_{n+1} \notin \bigcup_{1}^{n} N_\epsilon(x_j).$$

The sequence (x_n) has no convergent subsequence since, for $j \neq k$, $d(x_k, x_j) \geq \epsilon$. Thus X is not sequentially compact. □

Proposition 6.22. *If X is sequentially compact then X is compact.* †

Proof. Let $\mathcal{U}$ be a given open cover of X. From the Lebesgue Number Lemma, there is a $\delta > 0$ such that for each $x \in X$ there is a $U \in \mathcal{U}$ such that $N_\delta(x) \subseteq U$.

Since X is totally bounded, there exists a finite set $F \subseteq X$ so that

$$X = \bigcup_{x \in F} N_\delta(x).$$

For each $x \in F$, there is a $U_x \in \mathcal{U}$ such that $N_\delta(x) \subseteq U_x$. Hence,

$$X = \bigcup_{x \in F} U_x;$$

i.e., $\{U_x : x \in F\} \subseteq \mathcal{U}$ is an open cover of X. Hence X is compact. □

6.4. **The Heine–Borel Theorem.**

Lemma 6.23. *Cubes in $\mathbb{R}^g$ are compact.* †

Proof for the case $g = 2$. Either an induction argument or an argument similar to the proof below for $g = 2$ handles the case of general d.

Consider the cube $C = [a, b] \times [c, d]$. It suffices to prove that every sequence (z_n) from C has a subsequence which converges in C; i.e., that C is sequentially compact. To this end, let $(z_n) = (x_n, y_n)$ be a given sequence from C. Since $[a, b]$ is compact, there is a subsequence $(x_{n_k})_k$ of (x_n) which converges to some $x \in [a, b]$. Similarly, since $[c, d]$ is compact the sequence $(y_{n_k})_k$ has a subsequence $(y_{n_{k_j}})_j$ which converges to a $y \in [c, d]$. It follows that $(z_{n_{k_j}})_j$ converges to $z = (x, y) \in C$. □

Theorem 6.24 (Heine–Borel)**.** *A subset K of $\mathbb{R}^g$ is compact if and only if it is closed and bounded.*

Proof. We have already seen that compact implies closed and bounded in any metric space. Suppose now that K is closed and bounded. There is a cube C such that $K \subseteq C \subseteq \mathbb{R}^g$. The cube C is compact and K is a closed subset of C and is therefore compact by Proposition 6.10. □

Do Problem 6.13.

Corollary 6.25. $\mathbb{R}^g$ *is complete.* †

The proof is similar to that of Corollary 6.18. The details are left as an exercise for the gentle reader.

6.5. **Worked Example.**

Sample Problem 6.1. Show that if K is a compact metric space, then it contains an at most countable subset S such that the closure of S is K. In other words, a compact metric space contains a countable dense set. A metric space which contains an at most countable dense set is called *separable.*

Solution. Since K is compact, it is totally bounded by Propositions 6.16 and 6.21. For each $n \in \mathbb{N}^+$, there exists a finite set F_n such that

$$K \subseteq \bigcup_{x \in F_n} N_{1/n}(x).$$

Let

$$S = \bigcup_{n=1}^{\infty} F_n.$$

Since S is a countable union of finite sets, S is at most countable. Let T denote the closure of S. Given $y \in K$, for each $n \in \mathbb{N}^+$ there is an $x_n \in F_n$ such that $y \in N_{1/n}(x_n)$. Thus, $d(x_n, y) < 1/n$. It follows that (x_n) is a sequence from S, and hence from T, which converges to y. Thus, by Proposition 4.8, $y \in T$. Hence $T = X$ and the proof is complete.

6.6. **Exercises.**

Exercise 6.1. Let X be a metric space. Show, if there is an $r > 0$ and a sequence (x_n) from X such that $d(x_n, x_m) \geq r$ for $n \neq m$, then X is not compact.

Exercise 6.2. Suppose X has the property that each closed bounded subset of X is compact. Show that X is complete.

Exercise 6.3. Let X be a (nonempty) metric space. Show, if X is totally bounded, then X is bounded. Give an example where X is bounded, but not totally bounded.

6.7. **Problems.**

Problem 6.1. Prove that if X is a metric space and $(a_n)_{n=1}^{\infty}$ is a sequence in X which converges to $L \in X$, then $\{a_1, a_2, \dots, L\}$ is compact.

Problem 6.2. Prove a finite subset of a metric space X is compact. More generally, prove that a finite union of compact sets is compact.

Problem 6.3. Show, a subset K of a discrete metric space X is compact if and only if it is finite. In particular, if X is infinite, then X is complete (and thus closed) and bounded, but not compact.

Problem 6.4. [*The finite intersection property (fip)*] Suppose X is a compact metric space and $\mathcal{F} \subseteq P(X)$. Show that if each $C \in \mathcal{F}$ is closed and for each finite subset $F \subseteq \mathcal{F}$ the set

$$\bigcap_{C \in F} C \neq \emptyset,$$

then in fact

$$\bigcap_{C \in \mathcal{F}} C \neq \emptyset.$$

As a corollary, show if $C_1 \supseteq C_2 \supseteq$ is a nested decreasing sequence of non-empty compact sets in a metric space X, then $\cap C_j$ is non-empty too. (This fact is referred to as the *nested intersection property.*)

Show the result fails if X is not assumed compact. On the other hand, even if X is not compact, the result is true if it assumed that there is a $D \in \mathcal{F}$ which is compact. Compare with Problem 5.3.

Problem 6.5. Suppose $(S_n)_{n=1}^{\infty}$ is a sequence of non-empty compact subsets of a metric space X that is nested decreasing; that is, $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. Show that $S = \cap S_n$ is connected. Show that if $U \subseteq X$ is open and $S \subseteq U$, then there is a j such that $S_j \subseteq U$.

Problem 6.6. Prove that any open cover of $\mathbb{R}$ has an *at most countable subcover*.

More generally, prove, if there exists a sequence $K_1, K_2, \ldots$ of compact subsets of a metric space X such that $X = \cup K_j$, then every open cover of X has an at most countable subcover.

Problem 6.7. Let ℓ^∞ denote the set of bounded sequences $a = (a(n))$ of real numbers. The function $d : \ell^\infty \times \ell^\infty \to \mathbb{R}$ defined by

$$d(a, b) = \sup\{|a(n) - b(n)| : n \in \mathbb{N}\}$$

is a metric on ℓ^∞.

Let e_j denote the sequence from ℓ^∞ (so a sequence of sequences) with $e_j(j) = 1$ and $e_j(k) = 0$ if $k \neq j$. Find $d(e_j, e_\ell)$.

Let 0 denote the zero sequence in ℓ^∞. Is

$$B = \{a \in \ell^\infty : d(a, 0) \leq 1\}$$

closed? Is it bounded? Is it compact?

As a challenge, show that ℓ^∞ is complete.

Problem 6.8. This problem assumes Problem 4.17. Let ℓ^2 denote the set of sequences $(a(n))$ of real numbers such that

$$\sum_{n=0}^{\infty} |a(n)|^2$$

converges (to a finite number). Use the Cauchy Schwarz inequality to show, if $a, b \in \ell^2$, then

$$\langle a, b \rangle := \sum_{0}^{\infty} a(j)b(j)$$

converges and that $\langle a, b \rangle$ is an inner product on ℓ^2. Let

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$$

denote the resulting metric.

Let e_j denote the sequence with $e_j(j) = 1$ and $e_j(k) = 0$ if $j \neq k$. What is $d(e_j, e_k)$? Does the sequence (of sequences) (e_j) have a convergent subsequence? Let 0 denote the zero sequence. Is the set

$$B = \{x \in \ell^2 : d(x, 0) \leq 1\}$$

closed? Is it bounded? Is it compact?

As a challenge, prove that ℓ^2 is complete.

Problem 6.9. Prove Proposition 6.15. (See Problem 5.1.)

Problem 6.10. Suppose K is a nonempty compact subset of a metric space X and $x \in X$.

(i) Give an example of an $x \in X$ for which there exists distinct points $p, r \in K$ such that, for all $q \in K$,
$$d(p,x) = d(r,x) \leq d(q,x).$$
(ii) Show, there is a point $p \in K$ such that, for all other $q \in K$,
$$d(p,x) \leq d(q,x).$$
[Suggestion: As a start, let $S = \{d(x,y) : y \in K\}$ and show there is a sequence (q_n) from K such that the numerical sequence $(d(x,q_n))$ converges to $\inf(S)$.]

(iii) Let $X = \mathbb{R} \setminus \{0\}$ and $K = (0,1]$. Is there a point $x \in X$ with no closest point in K? Is K closed, bounded, complete, compact?

(iv) Let $E = \{e_0, e_1, \dots\}$ be a countable set. Define a metric d on E by $d(e_j, e_k) = 1$ for $j \neq k$ and $j, k \neq 0$; $d(e_j, e_j) = 0$ and $d(e_0, e_j) = 1 + 1/j$ for $j \neq 0$. Show d is a metric on E. Let $K = \{e_1, e_2, \dots\}$ and $x = e_0$. Is there a closest point in K to x? Is K closed, bounded, complete, compact?

Problem 6.11. Suppose B is a compact subset of a metric space X and $a \notin B$. Show there exists disjoint open sets U and V such that $a \in U$ and $B \subseteq V$. Suggestion, first use Problem 6.10 to show there is an $\eta > 0$ such that for each $b \in B$ $N_\eta(b) \cap N_\eta(a) = \emptyset$.

Problem 6.12. Show if A and B are disjoint compact sets in a metric space X, then there exists disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Suggestion, by the previous problem, for each $a \in A$ there exists disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_b$.

Problem 6.13. Show that K compact can be replaced by K closed in Problem 6.10 in the case that $X = \mathbb{R}^g$.

Problem 6.14. Given metric spaces (X, d_X) and (Y, d_Y) let Z denote the metric space built from X and Y as in Problem 3.1. Show, if X and Y are compact, then so is $X \times Y$.

7. Connected sets

Definition 7.1. A metric space X is *disconnected* if there exists sets $U, V \subseteq X$ such that

(i) U and V are open;
(ii) $U \cap V = \emptyset$;
(iii) $X = U \cup V$; and
(iv) $U \neq \emptyset \neq V$;

The metric space X is *connected* if it is not disconnected.

A *subset S of X is connected* if the metric space (subspace) S is connected. $\triangleleft$

Do Exercise 7.1.

Remark 7.2. A metric space X is connected if and only if the only subsets of X which are both open and closed are X and $\emptyset$.

By Proposition 3.28, subsets U_0 and V_0 of S are open relative to S if and only if there exists subsets U, V of X which are open (in X) such that $U_0 = U \cap S$ and $V_0 = V \cap S$. Thus, a subset S of a metric space X is connected if and only if given subsets U and V of X such that

(i) U and V are open;
(ii) $U \cap S \cap V = \emptyset$; and
(iii) $S \subseteq U \cup V$

it follows that either $U \cap S$ or $V \cap S$ is empty (alternately, $S \subseteq U$ or $S \subseteq V$).

Note, if U, V satisfy (ii) and (iii), then $\tilde{V} \cap S = U \cap S$. $\diamond$

Problem 7.1 gives an alternate condition for a subset S of a metric space X to be connect in terms of subsets of X. Do also Problem 7.2.

Proposition 7.3. *A nonempty subset I of $\mathbb{R}$ is connected if and only if $x, y \in I$ and $x < z < y$ implies $z \in I$.*

In particular, intervals in $\mathbb{R}$ are connected. †

Proof. Suppose I has the property that $x, y \in I$ and $x < z < y$ implies $z \in I$. Arguing by contradiction, suppose $U, V \subseteq \mathbb{R}$ satisfy conditions (i)-(iv) of Remark 7.2. In particular, $U \cap I \neq \emptyset \neq V \cap I$. Choose $u \in U \cap I$ and $v \in V \cap I$. Without loss of generality, $u < v$. By hypothesis $[u, v] \subseteq I$. Consider $A = U \cap [u, v]$ and $B = V \cap [u, v]$ and observe that $A \cup B = [u, v]$ and $A \cap B = \emptyset$. Hence $\tilde{B} \cap [u, v] = A$. Since also $\tilde{B} = \tilde{V} \cup \widetilde{[u, v]}$, it follows that $A = \tilde{V} \cap [u, v]$. In particular, A is closed and bounded. Thus A has a largest element $a \in A$. Since $v \in B$, we find $a < v$. Since U is open, there is an ϵ such that $v - a > \epsilon > 0$ and $(a - \epsilon, a + \epsilon) = N_\epsilon(a) \subseteq U$. In particular, $(a, a + \epsilon) \subseteq U \cap [u, v] = A$. But then $a + \frac{\epsilon}{2} \in A$, a contradiction.

To prove the converse, suppose there exists $x, y \in I$ and $z \notin I$ such that $x < z < y$. In this case, let $U = (-\infty, z)$ and $V = (z, \infty)$. Then $U \cap V = \emptyset$, U and V are open, $U \cap I$ and $V \cap I$ are nonempty, and $I \subseteq U \cup V$, thus I is not connected. $\square$

Do Problem 7.3.

Proposition 7.4. *If $\mathcal{C}$ is a nonempty collection of connected subsets of a metric space X and if $A \cap B \neq \emptyset$ for each $A, B \in \mathcal{C}$, then $\Gamma = \cup\{C : C \in \mathcal{C}\}$ is connected.* †

Proof. Arguing by contradiction, suppose there $U, V \subseteq X$ open sets such that $U \cap \Gamma \cap V = \emptyset$, $\Gamma \subseteq U \cup V$ and $\Gamma \cap U \neq \emptyset \neq \Gamma \cap V$. From this last condition, there exists $C_U, C_V \in \mathcal{C}$ such that $C_U \cap U \neq \emptyset$ and $C_V \cap V \neq \emptyset$. Now U, V are open; $C_U \subseteq U \cup V$; and $U \cap C_U \cap V \subseteq U \cap \Gamma \cap V = \emptyset$. Thus, since C_U is connected, either $C_U \cap U = \emptyset$ or $C_U \cap V = \emptyset$. It follows that $C_U \cap V = \emptyset$ and hence $C_U \subseteq U$. By symmetry, $C_V \subseteq V$ and thus,

$$C_U \cap C_V \subseteq U \cap \Gamma \cap V = \emptyset,$$

contradicting the assumption that the intersection of any two sets in $\mathcal{C}$ is nonempty. □

Do Problem 7.4 and Exercise 7.6.

Corollary 7.5. *Given a point x in a subset S of a metric space X there is a largest connected set C_x containing x and contained in S; i.e.,*

(i) $x \in C_x \subseteq S$,
(ii) $C_x \subseteq X$ *is connected; and*
(iii) *if* $x \in D \subseteq S$ *and* $D \subseteq X$ *is connected, then* $D \subseteq C_x$.

†

The set C_x of Corollary 7.5 is called the *connected component* containing x.

Proof. Note that $\{x\}$ is connected. Let $\mathcal{C}$ denote the collection of connected sets containing x and contained in S and apply Proposition 7.4 to conclude that $\Gamma = \cup\{C : C \in \mathcal{C}\}$ is connected. By construction, if D is connected and $x \in D$, then $D \subseteq \Gamma$. □

Do Problems 7.5, 7.6 and 7.7.

7.1. **Exercises.**

Exercise 7.1. Show singleton sets are connected, but finite sets with more than one element are not.

Exercise 7.2. Determine the connected subsets of a discrete metric space.

Exercise 7.3. Let $I = [0, 1] \subseteq \mathbb{R}$. If $0 < x < 1$, is $I \setminus \{x\}$ connected?

Let $S \subseteq \mathbb{R}^2$ denote the unit circle, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. If $x \in S$, is $S \setminus \{x\}$ connected? If $x \neq y$ are both in S, is $S \setminus \{x, y\}$ connected?

Let $R \subseteq \mathbb{R}^2$ denote the unit square $R = [0, 1] \times [0, 1]$. If $F \subseteq R$ is finite, is $R \setminus F$ connected?

Exercise 7.4. Let $K = \{\frac{1}{n} : n \in \mathbb{N}^+\} \subseteq \mathbb{R}$ and let

$$C = (K \times [0,1]) \cup ([0,1] \times \{0\}) \subseteq \mathbb{R}^2.$$

Draw a picture of C. Is it connected?

Let $D = C \cup \{(0,1)\}$. Is D connected? (See Problem 7.2) Can you draw a path from $(0,0)$ to $(0,1)$ without leaving D?

Exercise 7.5. Show if A, B, C are connected subsets of X and $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$, then $A \cup B \cup C$ is connected. A more general statement, requiring a more elaborate proof, can be found in Problem 7.4.

Exercise 7.6. Must the intersection of two connected sets be connected?

7.2. Problems.

Problem 7.1. Prove, $S \subseteq X$ is disconnected if and only if there exists subsets $A, B \subseteq X$ such that

(i) both A and B are nonempty;
(ii) $A \cup B = S$;
(iii) $\overline{A} \cap B = \emptyset$; and
(iv) $A \cap \overline{B} = \emptyset$.

Here the closures are taken with respect to X. [Suggestion: Given A and B, let $U = \widetilde{\overline{B}}$ and $V = \widetilde{\overline{A}}$.]

Problem 7.2. Show, if S is a connected subset of a metric space X, then $\overline{S}$ is also connected. In fact, each $S \subseteq T \subseteq \overline{S}$ is connected.

Problem 7.3. Suppose $I \subseteq \mathbb{R}$ is open. Prove that I is also connected if and only if either

(1) I is an open interval;
(2) there is an $a \in \mathbb{R}$ such that $I = (a, \infty)$;
(3) there is a $b \in \mathbb{R}$ such that $I = (-\infty, b)$; or
(4) I is empty or all of $\mathbb{R}$.

The term *open interval* is expanded to refer to a set of any of the above forms.

Problem 7.4. Prove the following stronger variant of Proposition 7.4. Suppose $\mathcal{C}$ is collection of connected subsets of a metric space X and $B \in \mathcal{C}$. Show, if for each $A \in \mathcal{C}$, $A \cap B \neq \emptyset$, then $\Gamma = \cup\{C : C \in \mathcal{C}\}$ is connected. [Suggestion: Consider the collection $\mathcal{D} = \{C \cup B : C \in \mathcal{C}\}$

Problem 7.5. Let X be a metric space. For each $x \in X$, let C_x denote the connected component containing x. Prove that the collection $\{C_x : x \in X\}$ is a partition of X; i.e.,

(i) if $x, y \in X$ then either $C_x = C_y$ or $C_x \cap C_y = \emptyset$; and
(ii) $X = \cup_{x \in X} C_x$.

Problem 7.6. Prove, if $O \subseteq \mathbb{R}$ is open, then each connected component of O is open; i.e., if $U \subseteq O$ is connected in $\mathbb{R}$ and if $U \subseteq V \subseteq O$ is connected implies $U = V$, then U is open.

Problem 7.7. Prove that every open subset O of $\mathbb{R}$ is a disjoint union of open intervals (in the sense of Problem 7.3). Further show that this union is at most countable by noting that each component must contain a rational.

Problem 7.8. Given $c \in \mathbb{R}$ show that

$$S_c = \{(x, y) \in \mathbb{R}^2 : |x| = 1\} \cup \{(x, y) \in \mathbb{R}^2 : |x| < 1, y \geq c\}$$

is connected.

Show that

$$S = \cap_{n=1}^{\infty} S_n$$

is not connected.

Problem 7.9. Suppose $(S_n)_{n=1}^{\infty}$ is a sequence of non-empty compact connected subsets of a metric space X that is nested decreasing; that is, $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$. Show, $S = \cap S_n$ is connected. show, if $U \subseteq X$ is open and $S \subseteq U$, then there is a j such that $S_j \subseteq U$. (Suggestion: Use problem 6.5.)

8. Continuous Functions

8.1. **Definitions and Examples.**

Definition 8.1. Suppose X, Y are metric spaces, $a \in X$ and $f : X \to Y$. The function f is *continuous at* a if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $d_X(a, x) < \delta$, then $d_Y(f(a), f(x)) < \epsilon$.

The function f is *continuous* if it is continuous at each point $a \in X$. ◁

Example 8.2. (i) Constant functions are continuous.

(ii) For a metric space X, the identity function $id : X \to X$ given by $id(x) = x$ is continuous.

(iii) If $f : X \to Y$ is continuous and $Z \subseteq X$, then $f|_Z : Z \to Y$ is continuous.

(iv) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$ is nowhere continuous.

To prove this last statement, given $x \in \mathbb{R}$, choose $\epsilon_0 = \frac{1}{2}$.

(v) The function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ if $x \notin \mathbb{Q}$ and $f(x) = \frac{1}{q}$, where $x = \frac{p}{q}$, $p \in \mathbb{N}$, $q \in \mathbb{N}^+$, and $\gcd(p, q) = 1$, is continuous precisely at the irrational points.

Lets prove that f is continuous at irrational points, leaving the fact that it is not continuous at each rational point as an easy exercise.

Suppose $x \notin \mathbb{Q}$ ($x \in [0, 1]$) and let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}^+$ so that $\frac{1}{N} < \epsilon$. Let

$$\delta = \min\{|x - \frac{m}{n}| : m, n \leq N,\ m, n \in \mathbb{N}^+\}.$$

This minimum exists and is positive since it is a minimum over a finite set and 0 is not an element of the set (since $x \notin \mathbb{Q}$). If $|x - y| < \delta$ and $y \in [0, 1]$, then either $y \notin \mathbb{Q}$ in which case $|f(x) - f(y)| = |0 - 0| = 0$; or $y \in \mathbb{Q}$ and $y = \frac{p}{q}$ (in reduced form) where $q > N$ in which case $|f(x) - f(y)| = \frac{1}{q} < \epsilon$.

(vi) If X is a metric space and $a \in X$, then the function $f : X \to \mathbb{R}$ given by $f(x) = d(a, x)$ is continuous.

Fix x and let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. If $d(x, y) < \delta$, then

$$|f(x) - f(y)| = |d(x, a) - d(a, y)| \leq d(x, y) < \delta = \epsilon.$$

(vii) Given $\gamma \in \mathbb{R}^g$, the function $p_\gamma : \mathbb{R}^g \to \mathbb{R}$ defined by

$$p_\gamma(x) = \langle x, \gamma \rangle$$

is continuous.

△

Do Problems 8.1 and 8.2.

Proposition 8.3. *A function $f : X \to Y$ is continuous if and only if $f^{-1}(U) \subseteq X$ is open for every open set $U \subseteq Y$.* †

Note that the result doesn't change if Y is replaced by any Z with $f(X) \subseteq Z \subseteq Y$.

Proof. Suppose f is continuous and $U \subseteq Y$ is open. To prove $f^{-1}(U)$ is open, let $x \in f^{-1}(U)$ be given. Since U is open and $f(x) \in U$, there is an $\epsilon > 0$ such that $N_\epsilon(f(x)) \subseteq U$. Since f is continuous at x, there is a $\delta > 0$ such that if $d_X(x, z) < \delta$, then $d_Y(f(x), f(z)) < \epsilon$. Thus, if $z \in N_\delta(x)$, then $f(z) \in N_\epsilon(f(x)) \subseteq U$ and consequently $z \in f^{-1}(U)$. Hence $N_\delta(x) \subseteq f^{-1}(U)$. We have proved that $f^{-1}(U)$ is open.

Conversely, suppose that $f^{-1}(U)$ is open in X whenever U is open in Y. Let $x \in X$ and $\epsilon > 0$ be given. The set $U = N_\epsilon(f(x))$ is open and thus $f^{-1}(U)$ is also open. Since $x \in f^{-1}(U)$, there is a $\delta > 0$ such that $N_\delta(x) \subseteq f^{-1}(U)$; i.e., if $d_X(x, z) < \delta$, then $f(z) \in U$ which means $d_Y(f(x), f(z)) < \epsilon$. Hence f is continuous at x; and thus f is continuous. □

Corollary 8.4. *A function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed (in X) for every closed set C (in Y).* †

Do Problems 8.4 and 8.5. See also Problem 3.8.

Proposition 8.5. *Suppose X, Y, Z are metric spaces, $f : X \to Y$ and $g : Y \to Z$. If both f and g are continuous, then so is $h = g \circ f : X \to Z$.* †

Proof. Let V an open subset of Z be given. Since g is continuous, $U = g^{-1}(V)$ is open in Y. Since f is continuous, $f^{-1}(U)$ is open in X. Thus, $h^{-1}(V) = f^{-1}(U)$ is open and hence h is continuous. □

There are local versions of Propositions 8.5 and 8.3 (See Problems 8.7 and 8.6). Here is a sample whose proof is left to the reader.

Proposition 8.6. *Suppose X, Y, Z are metric spaces, $f : X \to Y$ and $g : Y \to Z$. If f is continuous at a and g is continuous at $b = f(a)$, then $h = g \circ f$ is continuous at a.* †

8.2. **Continuity and Limits.**

Definition 8.7. Let S be a subset of a metric space X. Recall, from Problem 4.5, a point $p \in X$ is a *limit point* of S if there exists a sequence (s_n) from $S \setminus \{p\}$ which converges to p.

A point $p \in S$ is an *isolated point* of S if p is not a limit point of S. ◁

Proposition 8.8. *Let S be a subset of the metric space X and suppose $p \in X$. The following are equivalent.*

(i) The point p is a limit point of S.
(ii) For every $\delta > 0$, the set $(S \setminus \{p\}) \cap N_\delta(p) \neq \emptyset$.
(iii) For every $\delta > 0$, the set $S \cap N_\delta(p)$ is infinite.

Furthermore, a point $p \in S$ is an isolated point of S if and only if the set $\{p\}$ is an open set in S; i.e., open relative to S. †

The proof of this proposition is left as an exercise. See Proposition 4.8 and Exercise 8.1 and compare with Problems 4.5 and 4.6.

Example 8.9. (i) If $S \neq \emptyset$ is an open set in $\mathbb{R}^g$, then every point of S is a limit point of S. In fact, as an exercise, show in this case the set of limit points of S is the closure of S.

(ii) The set $\mathbb{Z}$ in $\mathbb{R}$ has no limit points.

(iii) The only limit point of the set $\{\frac{1}{n} : n \in \mathbb{N}^+\}$ is 0.

△

Definition 8.10. Let X and Y be metric spaces and let $a \in X$ and $b \in Y$. Suppose a is a limit point of X and either $f : X \to Y$ or $f : X \setminus \{a\} \to Y$. The function f has *limit b as x approaches a*, written

$$\lim_{x \to a} f(x) = b,$$

if for every $\epsilon > 0$ there is a δ such that if $0 < d_X(a, x) < \delta$, then $d_Y(b, f(x)) < \epsilon$. ◁

Remark 8.11. The limit b, if it exists, is unique. ◇

Proposition 8.12. *Let X, Y be metric spaces and $a \in X$.*

(i) Suppose $f : X \to Y$ and $a \in X$ is a limit point of X. The function f is continuous at a if and only if $\lim_{x \to a} f(x)$ exists and equals $f(a)$.

(ii) Suppose $Z = X$ or $Z = X \setminus \{a\}$ and $b \in Y$. If $f : Z \to Y$ and $\lim_{x \to a} f(x)$ exists and equals b, then the function $g : X \to Y$ defined by $g(x) = f(x)$ for $x \neq a$ and $g(a) = b$ is continuous at a.

(iii) If a is not a limit point of X and $h : X \to Y$, then h is continuous at a.

†

Proposition 8.13. *Suppose $a \in X$ and $f : W \to Y$, where $W = X$ or $W = X \setminus \{a\}$. If $\lim_{x \to a} f(x) = b$ and if $g : Y \to Z$ is continuous at b, then $\lim_{x \to a} g \circ f(x) = g(b)$.* †

Proof. The function $h : X \to Y$ defined by $h(x) = f(x)$ if $x \neq a$ and $h(a) = b$ is continuous at a by Proposition 8.12. Hence $g \circ h$ is continuous at a by Proposition 8.6. It follows from Proposition 8.12 that

$$\lim_{x \to a} g \circ f(x) = \lim_{x \to a} g \circ h(x) = g(h(a)) = g(b),$$

completing the proof. □

For a variation on this composition law for limits, see Problem 8.8.

The following Proposition gives a sequential formulation of limit.

Proposition 8.14. *Suppose X is a metric space, a is a limit point of X, and $f : Z \to Y$ where Z is either X or $X \setminus \{a\}$. The limit $\lim_{x \to a} f(x)$ exists and equals $b \in Y$ if and only if for every sequence (a_n) from $X \setminus \{a\}$ which converges to a, $(f(a_n))$ converges to b.*

Moreover, if $f : X \to Y$ and $a \in X$, the following are equivalent:

(i) *f is continuous at a.*

(ii) *For every sequence (a_n) from $X \setminus \{a\}$ converging to a, $(f(a_n))$ converges to $f(a)$.*

(iii) *For every sequence (a_n) from X converging to a the sequence $(f(a_n))$ converges to $f(a)$.*

†

Proof. The "moreover" part of the proposition readily follows from the first part and Proposition 8.12(i).

To prove the the first part of the lemma, suppose $\lim_{x\to a} f(x) = b$ and (a_n) is a sequence from $X \setminus \{a\}$ which converges to a. To see that $(f(a_n))$ converges to b, let $\epsilon > 0$ be given. There is a $\delta > 0$ such that if $0 < d_X(a, x) < \delta$, then $d_Y(b, f(x)) < \epsilon$. There is an N so that if $n \geq N$, then $0 < d_X(a, a_n) < \delta$. Hence, if $n \geq N$, then $d_Y(b, f(a_n)) < \epsilon$ and thus $(f(a_n))$ converges to b.

Conversely, suppose $\lim_{x\to a} f(x) \neq b$. Then there is an $\epsilon_0 > 0$ such that for each n there exists a_n such that $0 < d_X(a, a_n) < \frac{1}{n}$, but $d_Y(b, f(a_n)) \geq \epsilon_0$. The sequence (a_n) is a sequence from $X \setminus \{a\}$ which converges to a, but $(f(a_n))$ does not converge to b. ☐

8.3. **Continuity of Rational Operations.**

Proposition 8.15. *Let X be a metric space and $a \in X$ be a limit point of X. Suppose $f : Y \to \mathbb{R}^k$ where Y is either X or $X \setminus \{a\}$. Write $f = (f_1, \dots, f_k)$ with $f_j : X \to \mathbb{R}$.*

The limit $\lim_{x\to a} f(x)$ exists and equals $A = (A_1, \dots, A_k) \in \mathbb{R}^k$ if and only if the limit $\lim_{x\to a} f_j(x)$ exist and equals A_j. In particular, if $f : X \to \mathbb{R}^k$, then f is continuous at a if and only if each f_j is continuous at a. †

Proof. First suppose the limits $\lim_{x\to a} f_j(x)$ exists and equal A_j. Let (a_n) be a given sequence from $X \setminus \{a\}$ which converges to a. By Proposition 8.14 the sequences $(f_j(a_n))$ converges to A_j. By Proposition 4.19, the sequence $A_n = f(a_n)$ converges to A. Another application of Proposition 8.14 implies that $\lim_{x\to a} f(x)$ exists and equals A.

Conversely, suppose $\lim_{x\to a} f(x) = A$ and let (a_n) be a given sequence from $X \setminus \{a\}$ which converges to a. It follows, from Proposition 8.14, that $(f(a_n))$ converges to A. Hence, by Proposition 4.19, the sequences $(f_j(a_n))$ converge to A_j. Another application of Proposition 8.14 implies the limits $\lim_{x\to a} f_j(x)$ exist and equal A_j. ☐

Proposition 8.16. *Suppose $a \in X$ is a limit point of the metric space X, W is either X or $X \setminus \{a\}$ and $f, g : W \to \mathbb{R}^k$. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and equal A and B respectively, then*

(i) $\lim\limits_{x\to a} f(x) \cdot g(x) = A \cdot B$;
(ii) $\lim\limits_{x\to a} (f + g)(x) = A + B$;
(iii) if $k = 1$, $g(x)$ is never 0, and $B \neq 0$, then $\lim\limits_{x\to a} 1/g(x) = 1/B$.

†

Proof. To prove item (i), suppose (a_n) is a sequence in $X \setminus \{a\}$ which converges to a. From Proposition 8.14, $(f(a_n))$ and $(g(a_n))$ converge to A and B respectively. Hence $(f(a_n) \cdot g(a_n))$ converges to $A \cdot B$, by Proposition 4.20. Finally, another application of Proposition 8.14 completes the proof.

The proofs of the other items are similar. ☐

Corollary 8.17. *If $f, g : X \to \mathbb{R}^k$ are continuous at a, then so are $f \cdot g$ and $f + g$. If $k = 1$ and g is never 0, then $1/g$ is continuous at a.* †

Example 8.18. For each j, the function $\pi_j : \mathbb{R}^d \to \mathbb{R}$ given by $\pi_j(x) = x_j$ is continuous since it can be expressed as

$$\pi_j(x) = \langle x, e_j \rangle = x \cdot e_j,$$

where e_j is the j-th standard basis vector of $\mathbb{R}^d$; i.e., e_j has a 1 in the j-th entry and 0 elsewhere.

If $p(x_1, \ldots, x_d)$ and $q(x_1, \ldots, x_d)$ are polynomials, then the rational function

$$r(x) = \frac{p(x)}{q(x)}$$

is continuous (wherever it is defined). △

Do Problems 8.9 and 8.10.

8.4. **Continuity and Compactness.**

Proposition 8.19. *If $f : X \to Y$ is continuous and X is compact, then $f(X)$ is compact; i.e., the continuous image of a compact set is compact.* †

Proof. Let $\mathcal{W}$ be a given open cover of $f(X)$. By continuity of f, the set

$$\mathcal{U} = \{f^{-1}(U) : U \in \mathcal{W}\} \subseteq P(X)$$

is an open cover of X. Since X is compact, there is a finite subset $\mathcal{F} \subseteq \mathcal{W}$ such that $\{f^{-1}(U) : U \in \mathcal{F}\}$ is a cover of X.

Using the fact that $f(f^{-1}(B)) \subseteq B$ and Worked Example 1.2, it follows that

$$\bigcup_{U \in \mathcal{F}} U \supseteq \bigcup_{U \in \mathcal{F}} f(f^{-1}(U)) = f\left(\bigcup_{U \in \mathcal{F}} f^{-1}(U)\right) \supseteq f(X).$$

Thus $\{U : U \in \mathcal{F}\} \subseteq \mathcal{U}$ is a finite subset of $\mathcal{U}$ which covers $f(X)$. Hence $f(X)$ is compact. □

Do Problem 8.11.

Corollary 8.20 (Extreme Value Theorem)**.** If $f : X \to \mathbb{R}$ is continuous and X is non-empty and compact, then there exists $x_0 \in X$ such that $f(x_0) \geq f(x)$ for all $x \in X$; i.e., f has a maximum on X. †

Proof. By the previous proposition, the set $f(X)$ is a compact subset of $\mathbb{R}$. It is also non-empty. In view of Proposition 3.35, non-empty compact subsets of $\mathbb{R}$ have a largest element; i.e., there is an $M \in f(X)$ such that $M \geq f(x)$ for all $x \in X$. Since $M \in f(X)$, there is an $x_0 \in X$ such that $M = f(x_0)$. □

Return to Problem 6.10.

Corollary 8.21. *If X is compact, and if $f : X \to Y$ is one-one, onto and continuous, then f^{-1} is continuous.* †

Proof. Let $C \subseteq X$, a closed set, be given. Since X is compact $C \subseteq X$ is closed, C is compact by Proposition 5.9. Since f is continuous and C is compact, $f(C)$ is compact by Proposition 8.19. Thus $f(C)$ is closed in Y. Since $(f^{-1})^{-1}(C) = f(C)$ is closed, it follows, from Corollary 8.4 that f^{-1} is continuous. □

Example 8.22. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and define $f : [0, 2\pi) \to \mathbb{T}$ by $f(t) = \exp(it) = (\cos(t), \sin(t))$. Then f is continuous and invertible, but f^{-1} is not continuous at 1.

In fact, if $g : \mathbb{T} \to [0, 2\pi)$ is continuous, then it is not onto since its image $g(\mathbb{T})$ will then be a compact, and hence proper, subset of $[0, 2\pi)$. Consequently there does not exists a one-one onto continuous map $f : [0, 2\pi) \to \mathbb{T}$. (See Problem 8.14.) △

8.5. **Uniform Continuity and Compactness.**

Definition 8.23. A function $f : X \to Y$ is *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in X$ and $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$.

Given $S \subseteq X$, f is *uniformly continuous on* S if $f|_S : S \to Y$ is uniformly continuous. ◁

See Exercise 8.7.

Proposition 8.24. *If $f : X \to Y$ is continuous on X and if X is compact, then f is uniformly continuous on X.* †

Proof. Let $\epsilon > 0$ be given. For each $x \in X$ there is a $r_x > 0$ such that if $d_X(x, y) < r_x$, then $d(f(x), f(y)) < \frac{\epsilon}{2}$.

The collection $\mathcal{U} = \{N_{\frac{r_x}{2}}(x) : x \in X\}$ is an open cover of X. Since X is compact, there is a finite subset $F \subseteq X$ such that $\mathcal{V} = \{N_{\frac{r_x}{2}}(s) : s \in F\}$ is a cover of X.

Let $\delta = \frac{1}{2} \min\{r_x : x \in F\}$. Suppose $y, z \in X$ and $d_X(y, z) < \delta$. There is an $x \in F$ such that $y \in N_{\frac{r_x}{2}}(x)$; i.e., $d_X(x, y) < \frac{r_x}{2}$. Hence

$$d_X(x, z) \leq d_X(x, y) + d_X(y, z) < \frac{r_x}{2} + \delta \leq r_x.$$

Consequently,

$$d_Y(f(y), f(z)) \leq d_Y(f(y), f(x)) + d_Y(f(x), f(z)) < \epsilon,$$

completing the proof. □

See Problem 8.25 for an alternate proof of Proposition 8.24.

Example 8.25. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not uniformly continuous.

Choose $\epsilon_0 = 1$. Given $\delta > 0$, let $x = \frac{2}{\delta}$ and $y = \frac{2}{\delta} + \frac{\delta}{2}$. Then $|x - y| < \delta$, but,

$$|f(y) - f(x)| = 2 + \frac{\delta^2}{4} \geq \epsilon_0 = 1.$$

On the other hand, the function from Problem 8.1 is uniformly continuous. △

Do Problems 8.13 and 8.12.

8.6. Continuity and Connectedness.

Proposition 8.26. *If $f : X \to Y$ is continuous and X is connected, then $f(X)$ is connected.* †

Proof. Suppose U and V are open subsets of $f(X)$ such that $f(X) = U \cup V$ and $U \cap V = \emptyset$.

The sets $A = f^{-1}(U)$ and $B = f^{-1}(V)$ are open, $X = A \cup B$ and $A \cap B = \emptyset$ (since if $x \in A \cap B$, then $f(x) \in U \cap V$). Hence, without loss of generality, $A = X$. Hence, $f(A) = f(X) = f(f^{-1}(U)) \subseteq U$ and $V = \emptyset$. It follows that $f(X)$ is connected. □

Example 8.27. Returning to Example 8.22, there does not exist a one-one onto continuous mapping $f : [0, 2\pi] \to \mathbb{T}$. If there were, then $g = f^{-1}$ would be a continuous one-one mapping of $\mathbb{T}$ onto $[0, 2\pi]$. Let $z = f(\pi)$ and $Z = \mathbb{T} \setminus \{z\}$. Now Z is connected and $g|_Z : Z \to [0, \pi) \cup (\pi, 2\pi]$ is one-one and onto. But then $g|_Z(Z) = [0, \pi) \cup (\pi, 2\pi]$ is connected which is a contradiction. △

Do Problems 8.14, 8.15, and 8.16.

Corollary 8.28 (Intermediate Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then there is a point $a < c < b$, such that $f(c) = 0$.* †

Definition 8.29. Let S denote a subset of $\mathbb{R}$. A function $f : S \to \mathbb{R}$ is increasing (synonymously *non-decreasing*) if $x, y \in S$ and $x \leq y$ implies $f(x) \leq f(y)$. The function is strictly increasing if $x, y \in S$ and $x < y$ implies $f(x) < f(y)$. ◁

Corollary 8.30. *If $f : [a, b] \to \mathbb{R}$ is continuous and increasing, then $f([a, b]) = [f(a), f(b)]$.* †

Example 8.31. Returning to the discussion in Subsection 2.2, fix a positive integer n and let $f : [0, \infty) \to [0, \infty)$ denote the function with rule $f(x) = x^n$. To show that f is onto, let $y \in [0, \infty)$ be given. With b the larger of 1 and y, consider $g = f|_{[0,b]} : [0, b] \to \mathbb{R}$. Since $f(b) \geq y$, it follows that y is in the interval $[0, g(b)]$. By Corollary 8.30, y is in the range of g and hence in the range of f. The conclusion is then that positive numbers have n-th roots. △

8.6.1. *More on connectedness - optional.* The following property of a metric space X is sometimes expressed by saying that X is *completely normal.* It is evidently stronger than the statement that disjoint closed sets can be separated by disjoint open sets, a property known as normality. Compare with Problem 6.12.

Proposition 8.32. *If A, B are subset of a metric space such that $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, then there exists $U, V \subseteq X$ such that*

(i) U and V are open;
(ii) $A \subseteq U$, $B \subseteq V$; and

(iii) $U \cap V = \emptyset$.

†

Proof. If either A or B is empty, then the result is immediate. Accordingly, suppose that $A \neq \emptyset$ and $B \neq \emptyset$ and of course that $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. By Problem 8.1, the function $f : X \to \mathbb{R}$ given by

$$f(x) = d(x; B) - d(x; A)$$

is continuous. Observe, if $x \in A$, then $x \notin \overline{B}$ and hence $d(x; A) = 0$, but $d(x; B) > 0$ by Problem 3.8. Thus, $f(x) > 0$ for $x \in A$. Similarly, $f(x) < 0$ for $x \in B$. Let $U = f^{-1}(0, \infty)$ and $V = f^{-1}(-\infty, 0)$. It follows that U and V are open, $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. Thus U and V satisfy conditions (i)–(iv). □

Remark 8.33. Proposition 8.32 gives another characterization of connected subsets S of a metric space X. Namely, S is not connected if and only if there exist nonempty, open, disjoint subsets U, V of X such that $S \subseteq U \cup V$. ◇

8.7. The Completion of a Metric Space — Optional.

Definition 8.34. Let X, Y be metric spaces. A mapping $f : X \to Y$ is *isometric* or an *isometry*, if

$$d_Y(f(a), f(b)) = d_X(a, b)$$

for all $a, b \in X$. ◁

Evidently, an isometry f is continuous.

Definition 8.35. The metric spaces X and Y are *equivalent* if there is a isometric bijection $f : X \to Y$. ◁

The notion of equivalent metric spaces satisfies the axioms of an equivalence relation.

Proposition 8.36. *Let X be a given metric space. Suppose Y, Z are complete metric spaces and $f : X \to Y$ and $g : X \to Z$ are isometric. If $f(X)$ and $g(X)$ are dense in Y and Z respectively, then Y and Z are equivalent.* †

Sketch of proof. Given $y \in Y$, there is a sequence (x_n) from X such that $(y_n = f(x_n))$ converges to y. It follows, using the fact that $d(f(x_n), f(x_m)) = d(g(x_n), g(x_m))$, that $(z_n = g(x_n))$ is Cauchy in Z and hence converges to some $z \in Z$. Define $h : Y \to Z$ by $h(y) = z$. □

Definition 8.37. *Given X, a metric space Y is* the completion *of X if there exists an isometry $f : X \to Y$ such that $f(X)$ is dense in Y. (The proposition above justifies the use of* the *instead of* a.*)* ◁

8.8. **Exercises.**

Exercise 8.1. Prove Proposition 8.8

Exercise 8.2. Show that $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous, but not uniformly so.

Exercise 8.3. Fix $n \in \mathbb{N}^+$. Use the Intermediate Value Theorem 8.28 along with Corollary 8.21 to argue that the function $\sqrt[n]{\ }$ exists (as a mapping from $[0, \infty)$ to $[0, \infty)$ and is continuous.

Exercise 8.4. Use Exercise 8.3 to show that if the sequence (a_n) of nonnegative real numbers converges to A and $r = m/n$ (for $m, n \in \mathbb{N}^+$) is a positive rational number then (a_n^r) converges to A^r.

Exercise 8.5. Give an alternate proof of the statement of Problem 6.10 using Example 8.2(f) and Corollary 8.20.

Exercise 8.6. Suppose $f : X \to \mathbb{R}^k$, the point $a \in X$ is a limit point of X and $A \in \mathbb{R}^k$. Show, if

$$\lim_{x \to a} [f(x) - A] = 0$$

(that is that the indicated limit exists and is 0), then

$$\lim_{x \to a} f(x) = A.$$

Exercise 8.7. Verify that the functions in parts (a), (b), (f) and (g) of Example 8.2 are actually uniformly continuous.

Exercise 8.8. Formulate and prove a squeeze theorem for limits of functions.

8.9. **Problems.**

Problem 8.1. Let A be a nonempty subset of a metric space X. Define $f : X \to [0, \infty)$ by $f(x) = \inf\{d(x, a) : a \in A\}$. Prove that f is continuous.

Problem 8.2. Let X be a metric space and Y a discrete metric space.

(i) Determine all continuous functions $f : Y \to X$.
(ii) Determine all continuous functions $g : \mathbb{R} \to Y$;

Problem 8.3. Suppose (X is not empty and) $f : X \to Y$ is continuous. Show, if X is connected and for each $x \in X$ there is an $r > 0$ such that $f(y) = f(x)$ for all $y \in N_r(x)$, then f is constant.

Problem 8.4. Prove Corollary 8.4.

Problem 8.5. Show, if $f : X \to \mathbb{R}$ is continuous, then the zero set of f,

$$Z(f) = \{x \in X : f(x) = 0\}$$

is a closed set.

Show that the set

$$\{(x, y) : xy = 1\} \subseteq \mathbb{R}^2$$

is a closed set. Compare with Exercise 3.7.

Problem 8.6. Prove the following local version of Proposition 8.3.

Suppose $f : X \to Y$ and $a \in X$. The function f is continuous at a if and only if for every open set U containing $b = f(a)$, there is an open set V containing a so that $V \subseteq f^{-1}(U)$.

Problem 8.7. Prove Proposition 8.6.

Problem 8.8. Suppose X is a metric space, $a \in X$ is a limit point of X and $f : X\setminus\{a\} \to Y$. Show, if

(i) $\lim_{x\to a} f(x)$ exists and equals b;
(ii) $g : Z \to X$ is continuous at c;
(iii) $g(c) = a$; and
(iv) $g(z) \neq a$ for $z \neq c$,

then

$$\lim_{z\to c} f \circ g(z) = b.$$

Problem 8.9. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = \sin(\frac{1}{x})$. Show

(i) f does not have a limit at 0;
(ii) show $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $g(x) = xf(x)$ has a limit at 0;
(iii) more generally, show if $h : \mathbb{R} \to \mathbb{R}$ is continuous at 0 and $h(0) = 0$, then hf (with domain $\mathbb{R} \setminus \{0\}$ of course) has a limit at 0.

Problem 8.10. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Is f continuous at $0 = (0,0)$?

Define $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Is g continuous at $0 = (0,0)$?

Problem 8.11. Suppose X and Y are metric spaces and let Z denote the metric space $Z = X \times Y$ with distance function

$$d((x_1,y_1),(x_2,y_2)) = \max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}.$$

Prove, if $f : X \to Y$ is continuous, then $F : X \to Z$ defined by $F(x) = (x, f(x))$ is also continuous.

Prove, if f is continuous and X is compact, then the graph of f,

$$\operatorname{graph}(f) = \{(x, f(x)) \in Z : x \in X\} \subseteq Z$$

is compact.

As a challenge, show, if the graph of f is compact, then f is continuous. As a suggestion, consider the function $H : \text{graph}(f) \to X$ defined by $H(x, f(x)) = x$.

Problem 8.12. Prove if $f : X \to Y$ is uniformly continuous and (a_n) is a Cauchy sequence from X, then $(f(a_n))$ is Cauchy in Y.

Problem 8.13. Given a metric space Y, a point $L \in Y$, and $f : [0, \infty) \to Y$, f *has limit* $L \in Y$ *at infinity,* written,

$$\lim_{x\to\infty} f(x) = L,$$

if for every $\epsilon > 0$ there is a $C > 0$ such that if $x > C$, then $d_Y(f(x), L) < \epsilon$.

Prove, if $f : [0, \infty) \to Y$ is continuous and has a limit at infinity, then f is uniformly continuous.

Problem 8.14. A function $f : X \to Y$ is a *homeomorphism* if it is one-one and onto and both f and f^{-1} are continuous.

Suppose $f : X \to Y$ is a homeomorphism. Show, if $Z \subseteq X$, then $f|_Z : Z \to f(Z)$ is also a homeomorphism. In particular, if Z is connected, then so is $f(Z)$.

Problem 8.15. Does there exist a continuous onto function $f : [0, 1] \to \mathbb{R}$?

Does there exist a continuous onto function $f : (0, 1) \to (-1, 0) \cup (0, 1)$?

Problem 8.16. Suppose $f : [0, 1] \times [0, 1] \to \mathbb{R}$. Prove, if f is continuous, then f is not one-one. (See Exercise 7.3.)

Problem 8.17. Let $I = (c, d)$ be an interval and suppose $a \in I$. Let E denote either I or $I \setminus \{a\}$ and suppose $f : E \to \mathbb{R}$. We say f *has a limit as* x *approaches* a *from the right (above)* if the function $f|_{(a,d)} : (a, d) \to \mathbb{R}$ has a limit at a. The limit, if it exists, is denoted,

$$\lim_{x\to a^+} f(x) = \lim_{a<x\to a} f(x).$$

The limit from the left (below) is defined similarly.

Show f has a limit at a if and only if both the limits from the right and left at a exist and are equal.

Problem 8.18. Suppose $f : (c, d) \to \mathbb{R}$ is monotone increasing and $c < a < d$. Show, f has a limit from the left at a and this limit is

$$\sup\{f(t) : c < t < a\}.$$

Problem 8.19. Suppose $f : [a, b] \to [c, d]$ is one-one and onto and (strictly) monotone increasing. Prove f is continuous.

Problem 8.20. A function $f : X \to X$ is a *contraction mapping* if there is an $0 \le r < 1$ such that

$$d(f(x), f(y)) \le r d(x, y)$$

for all $x, y \in X$.

A point p is a *fixed point* of f if $f(p) = p$.

Prove that a contraction mapping can have at most one fixed point.

Prove, if f is a contraction mapping and X is complete, then f has a (unique) fixed point. In fact, show, for any point $x \in X$, the sequence (x_n) defined recursively by $x_0 = x$ and $x_{n+1} = f(x_n)$ converges to this fixed point. (See Proposition 5.12.)

Problem 8.21. Suppose K is compact and $f : K \to K$. Show, if f is continuous, then the function $g : K \to [0, \infty)$

$$g(x) = d(f(x), x)$$

attains its infimum (achieves a minimum). Show further that if this minimum is taken at $z \in X$, then

$$d(f(f(z)), f(z)) \geq d(f(z), z).$$

Show that x is a fixed point of f if and only if $g(x) = 0$.

Suppose now that f satisfies

$$d(f(x), f(y)) < d(x, y)$$

for all $x \neq y$ in K.

Prove f has a unique fixed point.

Show by example, that the hypothesis that K is compact can not be dropped.

Problem 8.22. Suppose $f : X \to Y$ maps convergent sequences to convergent sequences; i.e., if (a_n) converges in X, then $(f(a_n))$ converges in Y.

Show, if (a_n) converges to a, and (b_n) is the sequence defined by $b_{2n} = a_n$ and $b_{2n+1} = a$, then (b_n) converges to a. Now prove that $f(b_n)$ converges to $f(a)$.

Prove f is continuous.

Problem 8.23 (Pasting Lemma)**.** Suppose $f : X \to Y$ and $X = S \cup T$, where S and T are closed. Show, if the restriction of f to both S and T is continuous, then f is continuous. The same is true if both S and T are open. (Suggestion: in the case S and T are closed, fix a closed set $C \subseteq Y$ and express $f^{-1}(C) = (f^{-1}(C) \cap S) \cup (f^{-1}(C) \cap T)$ in terms of $f|_S$ and $f|_T$.)

Problem 8.24. Show, if $f : X \to X$ is continuous, X is compact, and f does not have a fixed point, then there is an $\epsilon > 0$ such that $d(x, f(x)) \geq \epsilon$ for all $x \in X$.

Problem 8.25. Fill in the following outline of an alternate proof of Proposition 8.24. Arguing by contradiction, suppose f is not uniformly continuous. Hence there is an $\epsilon_0 > 0$ such that for each $n \in \mathbb{N}$ there exists $x_n, y_n \in X$ such that $d_X(x_n, y_n) < \frac{1}{n}$, but $d_Y(f(x_n), f(y_n)) \geq \epsilon_0$. There exists a point $z \in X$ and some choice of positive integers $n_1 < n_2 < \dots$, such that $(x_{n_k})_k$ and $(y_{n_k})_k$ both converge to z. Explain why this last statement contradicts the assumption that f is continuous.

9. Sequences of Functions and the Metric Space $C(X)$

9.1. Sequences of Functions.

Definition 9.1. Let X be a set and Y a metric space. A sequence (f_n) of functions $f_n : X \to Y$ *converges pointwise* if there exists an $f : X \to Y$ such that for each $x \in X$ the sequence $(f_n(x))$ converges to $f(x)$ in Y; i.e., if for every $x \in X$ and every $\epsilon > 0$ there is an N such that for every $n \geq N$, $d_Y(f_n(x), f(x)) < \epsilon$. The function f is the *limit* of the sequence, written

$$\lim_{n\to\infty} f_n = f \quad \text{(pointwise)},$$

and (f_n) is said to *converge pointwise* to f. ◁

Example 9.2. (i) Let $f : [0,1] \to \mathbb{R}$ denote the function defined by $f(1) = 1$ and $f(x) = 0$ for $0 \leq x < 1$. The sequence $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$ converges pointwise to f.

(ii) The sequence $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \frac{x}{1+nx^2}$$

converges pointwise to the zero function.

△

Definition 9.3. Let X be a set and Y a metric space. A sequence (f_n) of functions $f_n : X \to Y$ is *uniformly convergent* or *converges uniformly* if there exists $f : X \to Y$ such that for every $\epsilon > 0$ there is an N such that $d_Y(f_n(x), f(x)) < \epsilon$ for every $x \in X$ and every $n \geq N$.

In this case (f_n) *converges uniformly to* f. ◁

Remark 9.4. If (f_n) converges uniformly to f, then (f_n) converges pointwise to f. On the other hand, if (f_n) converges pointwise, but not uniformly, to f, then (f_n) does not converge uniformly. ◇

Example 9.5. (i) The sequence in item (i) of Example (9.2) above does not converge uniformly to its pointwise limit f (and thus does not converge uniformly). To prove this statement, choose $\epsilon_0 = 1/2$. Given N, choose $n = N$ and $x_N = (1/2)^{1/N}$. Then $|f_N(x_N) - f(x_N)| = 1/2 \geq \epsilon_0$.

(ii) The sequence from item (ii) of Example (9.2) does converge uniformly to the zero function f. To prove this claim, let $\epsilon > 0$ be given. Choose, using the uniform convergence of the sequence, $N \in \mathbb{N}^+$ such that $1/N < \epsilon^2$. Suppose $n \geq N$ and let $x \in [0,1]$ be given. If $0 \leq x < \epsilon$, then

$$|f_n(x) - f(x)| = f_n(x) \leq x < \epsilon.$$

On the other hand, if $1 \geq x \geq \epsilon$, then

$$|f_n(x) - f(x)| \leq \frac{x}{nx^2} \leq \frac{1}{n\epsilon} < \epsilon.$$

(iii) Let $h_n : [0,1] \to [0,1]$ be the piecewise linear function whose graph consists of line segments joining $(0,0)$ to $(1/2n, 1)$, then $(1/2n, 1)$ to $(1/n, 0)$, and finally $(1/n, 0)$ to $(1, 0)$. The sequence (h_n) converges pointwise to 0, but doesn't converge uniformly to 0. This sequence is sometimes called the *moving hats*.

△

Theorem 9.6. *Suppose X, Y are metric spaces and (f_n) is a sequence $f_n : X \to Y$. If each f_n is continuous and if (f_n) converges uniformly to f, then f is continuous.*

Proof. Let x and $\epsilon > 0$ be given. Choose N such that if $n \geq N$ and $y \in X$, then $d_Y(f_n(y), f(y)) < \epsilon$. Since f_N is continuous at x, there is a $\delta > 0$ such that if $d_X(x, y) < \delta$, then $d_Y(f_N(x), f_N(y)) < \epsilon$. Thus, if $d_X(x, y) < \delta$, then

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(y)) + d_Y(f_N(y), f(y)) \\ &< 3\epsilon, \end{aligned}$$

proving the theorem. □

Example 9.7. Example (9.5)(iii) shows that a sequence of continuous functions can converge pointwise to a continuous function, but not uniformly; i.e., pointwise convergence of a sequence of continuous functions to a continuous function does not imply uniform convergence.

The sequence $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = x^n$ cannot converge uniformly, since the limit f fails to be continuous at 1. Note that the same is true for any subsequence. △

9.2. **The Metric Space $C(X)$.**

Definition 9.8. Suppose X is a set and Y is a metric space. A function $f : X \to Y$ is *bounded* if the range of f is bounded. ◁

Remark 9.9. Suppose X is a set and Y is a metric space. If $f, g : X \to Y$ are bounded functions, then, as is easily proved, the set $\{d_Y(f(x), g(x)) : x \in X\}$ is a bounded set of real numbers and hence has a supremum. In the case that f is continuous and X is compact, then f is bounded. Further, if both f and g are continuous (and still X is compact), then because that mapping $D : X \to \mathbb{R}$ defined by $D(x) = d_Y(f(x), g(x))$ is continuous, the supremum is attained. ◇

Definition 9.10. Given a compact metric space X and a metric space Y, let $C(X, Y)$ denote the set of continuous functions from X to Y.

The function $d : C(X, Y) \times C(X, Y) \to \mathbb{R}$ defined by

$$d(f, g) = \sup\{d_Y(f(x), g(x)) : x \in X\}$$

is called the *uniform metric*. ◁

Remark 9.11. (i) In the case that Y is either $\mathbb{R}$ (resp. $\mathbb{C}$) it is customary to write $C(X)$ instead of $C(X, \mathbb{R})$ (resp. $C(X, \mathbb{C})$).

(ii) Note that $C(X)$ is a vector space (under pointwise addition and scalar multiplication).

(iii) The metric d on $C(X)$ is induced by the (supremum or max) norm,

$$\|f\|_\infty = \max\{|f(x)| : x \in X\}$$

on $C(X)$.

◇

Proposition 9.12. *Suppose X and Y are metric spaces and X is compact.*

(i) The uniform metric d is a metric on $C(X, Y)$.
(ii) A sequence (f_n) from $C(X, Y)$ converges to f in the metric space $(C(X, Y), d)$ if and only if (f_n) converges to f uniformly.
(iii) If Y is complete, then $C(X, Y)$ is also complete.

†

Definition 9.13. Suppose X is a compact metric space and Y is a metric space. A sequence (f_n) of functions $f_n : X \to Y$ is *uniformly Cauchy* if it is Cauchy in $C(X, Y)$. ◁

An easy variation of Proposition 9.12(ii) is the fact that (f_n) is uniformly Cauchy if and only if for every $\epsilon > 0$ there is an N such that $d(f_n(x), f_m(x)) < \epsilon$ for all $n, m \geq N$ and all $x \in X$.

Proof. We will prove that d satisfies the triangle inequality, the other properties of a metric being easily verified.

Accordingly, let $f, g, h \in C(X, Y)$ be given. Given $x \in X$, the triangle inequality in Y and the definition of d gives,

$$d_Y(f(x), h(x)) \leq d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \leq d(f, g) + d(g, h).$$

Hence, $d(f, g) + d(g, h)$ is an upper bound for the set $\{d_Y(f(x), h(x)) : x \in X\}$ and therefore, $d(f, h) \leq d(f, g) + d(g, h)$.

Now suppose (f_n), a sequence from $C(X, Y)$, converges to $f \in C(X, Y)$ in the metric space $(C(X, Y), d)$. Given $\epsilon > 0$ there is an N such that if $n \geq N$, then $d(f_n, f) < \epsilon$. In particular, for all $x \in X$, $d_Y(f_n(x), f(x)) \leq d(f_n, f) < \epsilon$ and (f_n) converges to f uniformly.

Conversely, suppose (f_n) converges to $f \in C(X, Y)$ uniformly. Given $\epsilon > 0$ there is an N such that if $n \geq N$, then, for all $x \in X$, $d_Y(f_n(x), f(x)) < \epsilon$. Hence, for $n \geq N$, $d(f_n, f) \leq \epsilon$ (in fact strict inequality holds) and (f_n) converges to f in $C(X, Y)$.

Finally, suppose that Y is complete and that (f_n), a sequence from $C(X, Y)$, is uniformly Cauchy. Then, for each $x \in X$, the sequence $(f_n(x))$ is Cauchy in Y. Since Y is complete, $(f_n(x))$ converges to some $f(x)$. Thus, there is a function $f : X \to Y$ such that (f_n) converges pointwise to f.

Now, let $\epsilon > 0$ be given. There is an N such that if $n, m \geq N$, then $d(f_n, f_m) < \epsilon$. In particular, $d_Y(f_n(x), f_m(x)) < \epsilon$ for each $x \in X$ and $m, n \geq N$. Given $n \geq N$ and $x \in X$, choose $m \geq N$ such that $d(f_m(x), f(x)) < \epsilon$ (using pointwise convergence) and observe that

$$d_Y(f_n(x), f(x)) \leq d_Y(f_n(x), f_m(x)) + d_Y(f_m(x), f(x)) < 2\epsilon.$$

Hence (f_n) converges uniformly to f. Since each f_n is continuous, so is f, by Theorem 9.6. Thus $f \in C(X, Y)$ and (f_n) converges to f. □

Remark 9.14. The unit ball in $C([0,1])$ is the set

$$B = \{f \in C([0,1]) : d(0,f) \leq 1\} = \{f \in C([0,1]) : \|f\|_\infty \leq 1\}.$$

The set B is closed (it is a closed ball in $C([0,1])$) and hence complete as $C([0,1])$ is complete. It is also bounded - it is contained in the r-neighborhood of 0 for any $r > 1$. But it is not compact since $f_n(x) = x^n$ is a sequence from B which has no uniformly convergent subsequence. See Problem 9.8 for a more general statement. ◇

9.3. **Exercises.**

Exercise 9.1. Show that the sequence of continuous real-valued functions (f_n) defined on the interval $(-1,1)$ by $f_n(x) = x^n$ does not converge uniformly, even though it does converge pointwise to a continuous function.

Exercise 9.2. Show, for a fixed $0 < a < 1$, the sequence of real-valued functions (f_n) defined on the interval $[-a, a]$ by $f_n(x) = x^n$ does converge uniformly.

Exercise 9.3. Does the sequence of functions (f_n) defined on the interval $[0,1]$ by

$$f_n(x) = \frac{nx}{1 + nx^2}$$

converge pointwise? Does it converge uniformly?

Exercise 9.4. Same as Exercise 9.3, but with

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

Exercise 9.5. Show, if (f_n) and (g_n) are uniformly convergent sequences of functions mapping the set X into $\mathbb{R}$, then the sequence $(h_n = f_n + g_n)$ converges uniformly too.

Exercise 9.6. Consider the sequence (f_n) where $f_n : \mathbb{R} \to \mathbb{R}$ is defined by $f_n(x) = x + \frac{1}{n}$. Show f_n converges uniformly. Does the sequence (f_n^2) converge uniformly? Compare with Exercise 9.5.

9.4. **Problems.**

Problem 9.1. Let X be a set and suppose $g : X \to \mathbb{R}$. Define $f_n : X \to \mathbb{R}$ by $f_n(x) = g(x)^n$.

(i) Show, (f_n) converges pointwise if and only if the range of g lies in $(-1, 1]$ and in this case find the pointwise limit; and
(ii) Show, (f_n) converges uniformly if and only if there exists $0 < a < 1$ such that the range of g lies in $[-a, a] \cup \{1\}$.

Problem 9.2. Suppose $f : [0,1] \to \mathbb{R}$ is continuous and let $g_n : [0,1] \to \mathbb{R}$ denote the function $g_n(t) = t^n f(t)$. Show, if (g_n) converges uniformly, then $f(1) = 0$; and conversely, if $f(1) = 0$, then (g_n) converge uniformly. [Note that it suffices to assume that f is bounded and continuous at 1.]

Problem 9.3. Here is an application of Problem 8.20. Suppose $g : [0,1] \to \mathbb{R}$ is continuous and

$$\int_0^1 |g(t)|dt < 1.$$

Show, for $k \in \mathbb{R}$, that the mapping $F : C([0,1]) \to C([0,1])$ defined by

$$F(f)(x) = kx + \int_0^x g(t)f(t)dt$$

is a contractive mapping (see Problem 8.20).

Show that the equation $F(f) = f$ has a unique solution.

Note that this solution satisfies $f' = k + gf$ and $f(0) = 0$ (though a proof will have to wait until after a discussion of differentiation of course).

Problem 9.4. Show, if (f_n) converges to f in $C(X,Y)$, then (f_n) is *equicontinuous* ; i.e., given $\epsilon > 0$ there is a $\delta > 0$ such that, for every n, if $d(x,y) < \delta$, then $d(f_n(x), f_n(y)) < \epsilon$. (Thus the collection $\{f_n : n \in \mathbb{N}\}$ is *uniformly uniformly continuous.*) (Recall, implicit is the assumption that X is compact.)

Note that the conclusion holds if (f_n) is a Cauchy sequence from $C(X,Y)$, not necessarily convergent with a slight modification of the proof for the case above.

Problem 9.5. Let X be a compact metric space. A subset C of $C(X)$ is equicontinuous if for each $\epsilon > 0$ there is a $\delta > 0$ such that for every $f \in C$ and $x, y \in X$ with $d(x,y) < \delta$,

$$|f(x) - f(y)| < \epsilon.$$

If C is compact, then C is closed and bounded (in the metric space $C(X)$). Show that C is also equicontinuous.

It turns out that the converse is true too, but more challenging to prove. Namely, if C is closed, bounded and equicontinuous, then C is compact.

Problem 9.6. Show, if (f_n) converges to f in $C(X,Y)$ and if (z_n) is a sequence from X which converges to z, then $(f_n(z_n))$ converges to $f(z)$. Use this fact to show that the sequence (h_n) from Example 9.5 (iii) does not converge uniformly. See also Exercise 9.4.

Note the following variant of this problem. If (f_n) is a Cauchy sequence from $C(X,Y)$ and if (z_n) is Cauchy from X, then $(f_n(z_n))$ is Cauchy in Y.

Problem 9.7. Prove the following partial converse to Problem 9.6. Suppose X is compact, (f_n) is a sequence from $C(X,Y)$ and $f \in C(X,Y)$. Show, if $(f_n(x_n))$ converges to $f(x)$ for each $x \in X$ and sequence (x_n) converging to x, then (f_n) converges to f uniformly. Give an example to show that the assumption that X is compact is needed.

Problem 9.8. Given a point a in a compact metric space X, let

$$g(x) = \frac{1}{1 + d(a,x)}.$$

Let $f_n : X \to \mathbb{R}$ denote the sequence of functions $f_n(x) = g(x)^n$ (for $n \geq 1$).

Recall that a point $a \in X$ is an *isolated point* of X if the set $\{a\}$ is open in X.

(i) Find the pointwise limit f of (f_n);
(ii) Explain why each f_n is continuous;
(iii) Prove that f is continuous if and only if a is an isolated point of X;
(iv) Prove, if a is not an isolated point of X, then the unit ball of $C(X)$ is not compact;
(v) Prove, if the closed unit ball of $C(X)$ is compact, then every point of X is an isolated point of X;
(vi) Prove the closed unit ball of $C(X)$ is compact if and only if X is a finite set.

Problem 9.9. Given a set X and metric space Y, let $B(X,Y)$ denote the bounded functions from X to Y. Prove that $d(f,g) = \sup\{d_Y(f(x),g(x)) : x \in X\}$ defines a metric on $B(X,Y)$ and that most of Proposition 9.12 holds with $B(X,Y)$ in place of $C(X,Y)$.

10. Differentiation

This section treats the derivative of a real-valued function defined on an open set in $\mathbb{R}$. Subsections 10.1 and 10.2 contain the definitions and basic properties respectively. Subsections 10.3 and 10.4 treat the Mean Value Theorem and Taylor's Theorem. The brief subsection 10.5 introduces the derivative of a vector-valued function of a real variable.

10.1. Definitions and Examples.

Definition 10.1. Suppose that $D \subseteq \mathbb{R}$, that $a \in D$ is a limit point of D, and that $f : D \to \mathbb{R}$. The function f is *differentiable at a* provided the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case the value of the limit is called the *derivative of f at a* and is denoted $f'(a)$.

If f is differentiable at each point in D, then f is *differentiable* and in this case $f' : D \to \mathbb{R}$ is called the *derivative* of f. ◁

Example 10.2. (i) Fix $c \in \mathbb{R}$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = c$ is differentiable and $f' = 0$.
(ii) The identity function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x$ is is differentiable and $g'(x) = 1$.
(iii) The function $\varphi : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^{3/2}$ is differentiable at 0.
(iv) The function $\psi : [0, \infty) \to \mathbb{R}$ defined by $\psi(t) = t^{1/2}$ is not differentiable at 0.

△

Do Exercise 10.1.

10.2. Basic Properties.

Proposition 10.3. *Suppose $D \subseteq \mathbb{R}$ and $a \in D$ is an accumulation point of D. If $f : D \to \mathbb{R}$ is differentiable at a, then f is continuous at a.* †

Proof. Since the

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and $\lim_{x \to a} x - a = 0$, it follows that

$$\lim_{x \to a} (f(x) - f(a)) = \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \left(\lim_{x \to a} x - a \right) = 0.$$

An application of Exercise 8.6 shows that the limit $\lim_{x \to a} f(x)$ exists and equals $f(a)$. □

Proposition 10.4 (Additivity)**.** Suppose $D \subseteq \mathbb{R}$ and $a \in D$ is an accumulation point of D. If $f, g : D \to \mathbb{R}$ are differentiable at a, then $f + g$ is differentiable at a and $(f + g)'(a) = f'(a) + g'(a)$. †

The proof is left to the reader.

Proposition 10.5 (Product Rule)**.** *Suppose $D \subseteq \mathbb{R}$ and $a \in D$ is an accumulation point of D. If $f, g : D \to \mathbb{R}$ are differentiable at a, then*

(i) *fg is differentiable at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.* †

Proof. Consider

$$\begin{aligned} f'(a)g(a) + f(a)g'(a) &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a} g(x) + \lim_{x\to a} \frac{g(x)-g(a)}{x-a} f(a) \\ &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} g(x) + \lim_{x\to a} \frac{g(x)-g(a)}{x-a} f(a) \\ &= \lim_{x\to a} \frac{f(x)g(x)-f(a)g(a)}{x-a} \end{aligned}$$

□

Proposition 10.6 (Quotient Rule)**.** Suppose that $D \subseteq \mathbb{R}$ and $a \in D$ is an accumulation point of D. If $f : D \to \mathbb{R}$ is differentiable at a and f is never 0 on D then $1/f$ is differentiable at a and

$$\left(\frac{1}{f}\right)'(a) = -\frac{f'(a)}{f(a)^2}.$$

†

The proof of the quotient rule is left to the reader. Note that the "usual" quotient rule is obtained by combining Propositions 10.6 and 10.5. The proofs of the chain rule and inverse function theorem, while similar to the proofs Propositions 10.4, 10.5, and 10.6, are more subtle.

Proposition 10.7 (Chain Rule)**.** *Suppose $U, V \subseteq \mathbb{R}$ and that $a \in U$ and $b \in V$ are accumulation points of U and V respectively. If $f : U \to V$ and $g : V \to \mathbb{R}$ are differentiable at a and $b = f(a)$ respectively, then $h = g \circ f$ is differentiable at a and $h'(a) = g'(f(a))f'(a)$.* †

Proof. The function

$$F(y) = \begin{cases} \dfrac{g(y)-g(b)}{y-b} & \text{if } y \neq b, \\ & \\ g'(b) & \text{if } y = b \end{cases}$$

is continuous at b and f is continuous at a (since g and f are differentiable at b and a respectively). Thus, $F \circ f$ is continuous at a by Proposition 8.13 and hence

$$\begin{aligned} g'(b)f'(a) &= \left(\lim_{x\to a} F(f(x))\right)\left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right) \\ &= \lim_{x\to a} F(f(x))\left(\frac{f(x)-f(a)}{x-a}\right) \\ &= \lim_{x\to a}\left(\frac{g(f(x))-g(f(a))}{f(x)-f(a)}\right)\left(\frac{f(x)-f(a)}{x-a}\right) \\ &= \lim_{x\to a}\frac{g(f(x))-g(b)}{x-a}, \end{aligned}$$

where the fact that the limit of a product is a product of the limits, provided they both exist, has been used in the second equality. □

Recall Corollary 8.30.

Proposition 10.8 (Inverse Function Theorem I)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous, strictly increasing, and differentiable at $c \in [a,b]$. If $f'(c) \neq 0$ then $f^{-1} : [f(a), f(b)] \to [a,b]$ is differentiable at $f(c)$ and*

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}.$$

†

Proof. The function

$$F(x) = \begin{cases} \dfrac{x-c}{f(x)-f(c)} & \text{if } x \neq c, \\ \dfrac{1}{f'(c)} & \text{if } x = c. \end{cases}$$

is defined and continuous, including at c, by the assumption that f is differentiable at c and Proposition 8.16(iii). Since the function $f^{-1}(y)$ is also continuous (by Corollary 8.21) and the composition of continuous functions is continuous, it follows that

$$\lim_{y\to f(c)} F(f^{-1}(y)) = F(f^{-1}(f(c))) = F(c).$$

Noting that

$$F(f^{-1}(y)) = \frac{f^{-1}(y)-c}{y-f(c)} \quad \text{and} \quad F(f^{-1}(f(c))) = \frac{1}{f'(c)}$$

completes the proof. □

Example 10.9. The product rule and induction show that if $n \in \mathbb{N}^+$ and $f(x) = x^n$, then f is differentiable and $f'(x) = nx^{n-1}$. Using the quotient rule, the same formula holds for $n \in \mathbb{Z}$, $n \neq 0$.

Given $n \in \mathbb{N}^+$, the function $g : (0,\infty) \to (0,\infty)$ defined by $g(x) = x^{1/n}$ is the inverse of $f(x) = x^n$. Hence, by the inverse function theorem, g is differentiable and

$$g'(x^n) = \frac{1}{f'(x)} = \frac{1}{nx^{n-1}}$$

Thus $g'(x) = (1/n)x^{(1/n)-1}$. That this last formula holds for negative integers follows from the quotient rule.

Finally, if $q = m/n$ is rational with $m, n \in \mathbb{Z}$, $n > 0$, and $f(x) = x^q$ then an application of the chain rule shows that f is differentiable and $f'(x) = qx^{q-1}$. △

10.3. **The Mean Value Theorem.** Suppose X is a metric space. A function $f : X \to \mathbb{R}$ has a *local maximum* at $a \in X$ if there is an open set V such that $a \in V \subseteq X$ and if $x \in V$, then $f(a) \geq f(x)$.

Proposition 10.10. *Suppose W is an open subset of $\mathbb{R}$, $a \in W$, and $f : W \to \mathbb{R}$. If f is differentiable at a and if f has a local maximum at a, then then $f'(a) = 0$.* †

Proof. There is an open set $a \in U \subseteq W$ such that $f(x) \leq f(a)$ for all $x \in U$. By assumption, the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. Since U is open in $\mathbb{R}$, there is a sequence (x_n) from U converging to a with $x_n > a$. It follows, as $f(x_n) - f(a) < 0$, that

$$f'(a) = \lim_{n \to \infty} \frac{f(x_n) - f(a)}{x_n - a} \leq 0.$$

Likewise, choosing a sequence (x_n) from U converging to a with $x_n < a$ shows $f'(a) \geq 0$. Hence $f'(a) = 0$. □

Proposition 10.11 (Rolle's Theorem)**.** *Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and f is differentiable at each point in (a, b). If $f(a) = 0 = f(b)$, then there is a point $a < c < b$ such that $f'(c) = 0$.* †

Proof. Since f is continuous and real valued on a compact set f has both a maximum and minimum. Since $f(a) = f(b)$, at least one of either the maximum or minimum occurs at a point $a < c < b$ (of course it is possible for both the maximum and minimum to occur at the endpoints in which case f is identically 0). By Proposition 10.10, $f'(c) = 0$. □

Theorem 10.12 (Mean Value Theorem, Cauchy's Version)**.** *If $f, g : [a, b] \to \mathbb{R}$ are continuous, and if both are differentiable at each point in (a, b), then there is a c with $a < c < b$ so that $(f(b) - f(a))g'(c) = f'(c)(g(b) - g(a))$.*

Proof. Let $F(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a))$. Then $F(a) = F(b) = 0$ and F satisfies the hypotheses of Rolle's Theorem. Hence there is a $a < c < b$ such that $F'(c) = 0$; i.e., $f'(c)(g(b) - g(a)) = (f(b) - f(a))g'(c)$. □

Choosing $g(x) = x$ in the Cauchy Mean Value Theorem captures the usual Mean Value Theorem.

Corollary 10.13 (Mean Value Theorem, MVT)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous, and if f is differentiable at each point in (a, b), then there is a point c with $a < c < b$ so that $f(b) - f(a) = f'(c)(b - a)$.* †

Recall the definition of an increasing function, Definition 8.29.

Corollary 10.14. *Suppose $f : (a, b) \to \mathbb{R}$ is differentiable. The function f is increasing if and only if $f' \geq 0$ (meaning $f'(x) \geq 0$ for all $x \in (a, b)$).*

Further, if $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing. †

The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^3$ is strictly increasing, but $f'(0) = 0$.

Do Problem 10.1.

Proof. If f is increasing, then, for a fixed point $a < p < b$ and any $q \in (a, b)$ with $q \neq p$,

$$\frac{f(p) - f(q)}{p - q} \geq 0.$$

It follows from this inequality that $f'(p) \geq 0$.

For the converse, given $a < x < y < b$, by the MVT there is a $x < c < y$ such that

$$f(y) - f(x) = f'(c)(y - x) \geq 0,$$

where the inequality follows from the assumption that f' takes nonnegative values and $y - x > 0$. Thus f is increasing. If f' takes only positive values, then f is strictly increasing. □

Proposition 10.15 (A version of L'hopitals rule)**.** *Let $I = (a, b)$ and $f, g : I \to \mathbb{R}$ and suppose*

(i) both f and g are differentiable;

(ii)

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x); \text{ and;}$$

(iii) both g and g' are never 0.

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

†

The article http://en.wikipedia.org/wiki/Johann_Bernoulli#L.27H.C3.B4pital_controversy gives an amusing account of the (mis)naming of L'hopital's rule.

Proof. The functions f and g extend to be continuous on $[a,b)$ by defining $f(a)=g(a)=0$.

Let

$$L=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Given $\epsilon>0$ there is a $\delta>0$ such that if $a<y<a+\delta$, then

$$|L-\frac{f'(y)}{g'(y)}|<\epsilon.$$

From the Cauchy mean value theorem and hypothesis (iii), given $a<x<a+\delta$ there is a $a<c<x$ such that

$$\frac{f'(c)}{g'(c)}=\frac{f(x)-f(a)}{g(x)-g(a)}=\frac{f(x)}{g(x)}.$$

Thus, if $a<x<a+\delta$, then,

$$|L-\frac{f(x)}{g(x)}|=|L-\frac{f'(c)}{g'(c)}|<\epsilon.$$

□

10.4. **Taylor's Theorem.**

Theorem 10.16. *Let $I=(u,v)\subseteq\mathbb{R}$ be an open interval, $n\in\mathbb{N}$, and suppose $f:I\to\mathbb{R}$ is $(n+1)$ times differentiable. If $u<a<b<v$, then there is a c such that $a<c<b$ and*

$$f(b)=\sum_{j=0}^{n}\frac{f^{(j)}(a)(b-a)^j}{j!}+\frac{f^{(n+1)}(c)(b-a)^{n+1}}{(n+1)!}.$$

The result remains true with $b<a$. See Exercise 10.3.

Proof. Define $R_n:I\to\mathbb{R}$ by

$$R_n(x)=f(b)-\sum_{j=0}^{n}\frac{f^{(j)}(x)(b-x)^j}{j!}.$$

There is a K so that $R_n(a)=K\frac{(b-a)^{n+1}}{(n+1)!}$ and the goal is to prove there is a $a<c<b$ such that $K=f^{(n+1)}(c)$.

Let

$$\varphi(x)=R_n(x)-K\frac{(b-x)^{n+1}}{(n+1)!}.$$

Note that $\varphi:[a,b]\to\mathbb{R}$ is continuous and differentiable at each point in (a,b). Moreover, $\varphi(a)=0=\varphi(b)$. Thus, by the MVT, there is a $a<c<b$ such that $\varphi'(c)=0$. Since,

$$\varphi'(x)=-f^{(n+1)}(x)\frac{(b-x)^n}{n!}+K\frac{(b-x)^n}{n!},$$

it follows that

$$0=(-f^{(n+1)}(c)+K)\frac{(b-c)^n}{n!}.$$

The conclusion of the theorem follows. □

10.5. **Vector-Valued Functions.** Given an open set $U \subset \mathbb{R}$, the derivative of a function $f : U \to \mathbb{R}^k$ can be defined coordinate-wise. An alternate, but equivalent, definition appears later. In any case, if $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}^k$ is differentiable at $p \in D$, then, for each vector $v \in \mathbb{R}^k$, the function $g : D \to \mathbb{R}$ given by $g(x) = \langle f(x), v\rangle$ is differentiable at p and $g'(p) = \langle f'(p), v\rangle$.

Problem 10.6 explores the extent to which the MVT extends to vector-valued functions (of a real variable). You may first wish to do Exercise 10.2.

10.6. **Exercises.**

Exercise 10.1. Determine which of the following functions $f : \mathbb{R} \to \mathbb{R}$ are differentiable at 0.

(i) $f(x) = x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$;
(ii) $f(x) = x \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$;
(iii) $f(x) = x^2$ for $x \leq 0$ and $f(x) = x^3$ for $x > 0$;
(iv) $f(x) = |x|$.

Exercise 10.2. Compute the derivative of $f : \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(x) = \begin{pmatrix} x^2 \\ x^3 \end{pmatrix}.$$

Can you find a $0 < c < 1$ such that

$$f'(c) = f(1) - f(0)?$$

Exercise 10.3. Show Taylor's Theorem remains true if $b < a$ by applying the Theorem to $g(x) = f(-x)$ and the points $-a < -b$.

Exercise 10.4. Show, if $f : (a, b) \to \mathbb{R}$ is differentiable and $f' = 0$, then f is constant.

Show, if $f, g : (a, b) \to \mathbb{R}$ are differentiable, $f' = g'$ and there is a point $a < y < b$ such that $f(y) = g(y)$, then $f = g$.

10.7. **Problems.**

Problem 10.1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show, f is differentiable and $f'(0) > 0$, yet there is no open interval containing 0 on which f is increasing. [You may assume the usual rules of calculus in which case differentiability of f away from 0 is automatic. Thus, you need to show that f is differentiable at 0 and the derivative at 0 is positive. Using Corollary 10.14, to finish the problem it is enough to show, f' is not nonnegative on any open interval containing 0.]

Problem 10.2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Use the MVT to show, if f' is bounded, then f is uniformly continuous.

Problem 10.3. A function $f : (a, b) \to \mathbb{R}$ has limit ∞ at a if for every $C > 0$ there is a $\delta > 0$ such that if $a < x < a + \delta$, then $f(x) > C$.

Prove that condition (ii) in Proposition 10.15 can be replaced by

$$(ii') \quad \lim_{x\to a} g(x) = \infty = \lim_{x\to a} f(x).$$

Suggestion: Fix a $a < y < b$. Given $x < y$, by the Cauchy MVT, there is a t (depending on this choice of x and y) such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)}.$$

Deduce that

$$\frac{f(x)}{g(x)} = \frac{g(x) - g(y)}{g(x)} \frac{f'(t)}{g'(t)} + \frac{f(y)}{g(x)}.$$

Proceed.

Problem 10.4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show, if $|f'(t)| < 1$ for all t, then f has at most one fixed point (a point y such that $f(y) = y$). Show, if there is an $0 \le A < 1$ such that $|f'(t)| \le A$ for all t, then f has exactly one fixed point. [As a suggestion for the second part, choose any point x_1, let $x_{n+1} = f(x_n)$ and use Proposition 5.12.]

Note that the function any function $f : \mathbb{R} \to \mathbb{R}$ of the form $f = t + g$ where g takes only positive values, is differentiable and $|1 + g'(x)| < 1$ for all x has no fixed points, but does satisfy $|f'(x)| < 1$ for all x.

Problem 10.5. Let $f(x) = \sin(x)$ and let p_n denote the n-th Taylor polynomial for f; i.e.,

$$p_n(x) = \sum_{j=0}^{n} a_j x^j,$$

where

$$a_j = \begin{cases} 0 & j = 0 \mod 2; \\ \frac{1}{j!} & j = 1 \mod 4 \\ -\frac{1}{j!} & j = 3 \mod 4. \end{cases}$$

Use Taylor's Theorem to show that $p_n(x)$ converges to $\sin(x)$ uniformly on the interval $[-1, 1]$. (Later we will see that, for any given C, the sequence $(\frac{C^n}{n!})$ converges to 0 from which it follows that the sequence (p_n) converges uniformly to $\sin(x)$ on the interval $[-C, C]$.)

Problem 10.6. Give an example, if possible, of a function $f : [a, b] \to \mathbb{R}^2$ such that f' is continuous, but for each $t \in [a, b]$,

$$f(b) - f(a) \neq f'(t)(b - a).$$

Prove, if $f : [a, b] \to \mathbb{R}^k$ is continuous and is differentiable at each point in (a, b), then there is an $a < c < b$ such that

$$\|f(b) - f(a)\| \le \|f'(c)\|(b - a).$$

(Suggestion: Let u be a unit vector in the direction of $f(b) - f(a)$ and apply the usual Mean Value Theorem to $g(x) = \langle f(x), u \rangle$.)

Problem 10.7. Prove the following variant of the second part of Corollary 10.14. Namely, if $h : [a, b] \to \mathbb{R}$ is continuous and differentiable at each point in (a, b) and if $f'(x) > 0$ at each such point, then f is strictly increasing.

Now assume that $f, g : [a, b) \to \mathbb{R}$ are continuous, and are differentiable at each point in (a, b). Assume also that $f(a) = g(a)$ and that $f'(x) > g'(x)$ for all $x \in (a, b)$. Prove that $f(x) > g(x)$ for all $x \in (a, b)$.

Problem 10.8. Show, if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then between any two zeros of f there is a zero of f'.

Using induction, prove that a (real) polynomial of degree n can have at most n distinct real roots. (Don't use the Fundamental Theorem of Algebra to do this problem. The Fundamental Theorem of Algebra says that over the *complex* numbers, any polynomial can be factored essentially uniquely into linear terms. It's much deeper and much harder to prove than the result of this problem.)

Problem 10.9. Let X and Y be metric spaces. A function $f : X \to Y$ is called *Lipschitz continuous* $\underline{at\ p_0}$ $\in X$ if there exist $K, \delta > 0$ such that

$$d_Y(f(p), f(p_0)) \leq K d_X(p, p_0) \tag{4}$$

for all $p \in N_\delta(p_0)$. We call f *Lipschitz continuous* (or just *Lipschitz*)—with no "at p_0"—if there exists $K > 0$ such that

$$d_Y(f(p), f(q)) \leq K d_X(p, q) \tag{5}$$

for all $p, q \in X$. We call f *locally Lipschitz* if for all $p_0 \in X$, there exists $\delta > 0$ such that the restriction of f to $N_\delta(p_0)$ is Lipschitz continuous.

(Note that "locally Lipschitz" is stronger than "Lipschitz continuous at every point;" for the latter, there would be a $K(q)$ that works in (5) for each $q \in X$ and all p sufficiently close to q, but there might not be a single K that works simultaneously for all p, q sufficiently close to a given p_0. Somewhat more logical terminology for "locally Lipschitz" might be "locally uniformly Lipschitz", and a similar comment applies to "Lipschitz function" [with no "locally"]. Some mathematicians do insert the word "uniformly" in these cases, but most do not.)

(a) Prove that if $f : X \to Y$ is Lipschitz continuous at $p_0 \in X$, then f is continuous at p_0.
   (Note: the converse is false. For example, the function $[0, \infty) \to \mathbb{R}$ defined by $x \mapsto \sqrt{x}$ is not Lipschitz continuous at 0.]

For the remainder of this problem, let $U \subset \mathbb{R}$ be an open interval, and $f : U \to \mathbb{R}$ a function.

(b) Let $x_0 \in U$. Prove that if f is differentiable at x_0, then f is Lipschitz continuous at x_0.

(c) Prove that if f is differentiable, and the function $f' : U \to \mathbb{R}$ is bounded, then f is Lipschitz continuous.
(d) Prove that if f is differentiable, and the function $f' : U \to \mathbb{R}$ is continuous, then f is locally Lipschitz.

Problem 10.10. Show that the conclusion of Proposition 10.8 holds assuming only that

(i) $D, E \subseteq \mathbb{R}$;
(ii) $c \in D$ is a limit point of D,
(iii) $f : D \to E$ has an inverse g; and
(iv) g is continuous at $d = f(c)$.

Problem 10.11. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and f' is differentiable at 0. Show

$$f''(0) = \lim_{h \to 0} \frac{f(h) + f(-h) - 2f(0)}{h^2}.$$

11. Riemann Integration

This chapter develops the theory of the Riemann integral of a bounded real-valued function f on an interval $[a, b] \subset \mathbb{R}$. The approach[2] used here, approximating from above and below, is very efficient and intuitive, though a bit limited because it relies on the order structure of $\mathbb{R}$.

11.1. Definition of the Integral.

Definition 11.1. A *partition* P of the interval $[a, b] \subset \mathbb{R}$ consists of a finite set of points $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$.

Given the partition P, let $\Delta_j = x_j - x_{j-1}$. Given a bounded function $f : [a, b] \to \mathbb{R}$, let

$$\begin{aligned} m_j &= \inf\{f(x) : x_{j-1} \le x \le x_j\} \\ M_j &= \sup\{f(x) : x_{j-1} \le x \le x_j\}; \end{aligned}$$

define the *lower and upper sums* of f with respect to P by

$$\begin{aligned} L(P, f) &= \sum_{j=1}^{n} m_j \Delta_j \\ U(P, f) &= \sum_{j=1}^{n} M_j \Delta_j. \end{aligned}$$

◁

Remark 11.2. It is evident, in the context of Definition 11.1, that

$$m^* \,(b - a) \le L(P, f) \le U(P, f) \le M^* \,(b - a),$$

where

$$m^* = \inf\{f(x) : x \in [a, b]\}, \quad M^* = \sup\{f(x) : x \in [a, b]\}.$$

In particular, the set $\{L(P, f) : P\}$ is bounded above and the set $\{U(P, f) : P\}$ is bounded below. ◇

Definition 11.3. Continuing with Definition 11.1, the *lower and upper Riemann integrals* of f (on $[a, b]$) are defined by

$$\underline{\int_a^b} f\,dx = \sup\{L(P, f) : P\}, \qquad \overline{\int_a^b} f\,dx = \inf\{U(P, f) : P\}.$$

◁

Example 11.4. For the function $f : [0, 1] \to [0, 1]$ defined by $f(x) = 1$ it is evident that $U(P, f) = 1 = L(P, f)$ for every P. Hence

$$\underline{\int_0^1} 1\,dx = 1 = \overline{\int_0^1} 1\,dx.$$

△

[2]The Darboux approach.

Example 11.5. Let $f : [0,1] \to \mathbb{R}$ denote the indicator function of $[0,1] \cap \mathbb{Q}$. Thus $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Verify, for any partition P of $[0,1]$, that $L(P,f) = 0$ and $U(P,f) = 1$. Thus

$$\underline{\int_0^1} f\,dx = 0 < 1 = \overline{\int_0^1} f\,dx.$$

△

Definition 11.6. A bounded function $f : [a,b] \to \mathbb{R}$ is *Riemann integrable* on $[a,b]$ if the upper and lower integrals agree. In this case, this common value is the *Riemann integral* of f on $[a,b]$, denoted

$$\int_a^b f\,dx.$$

The set of Riemann integrable functions on $[a,b]$ is denoted by $\mathcal{RI}([a,b])$. ◁

Example 11.7. Continuing with Example 11.4, the function $f : [0,1] \to [0,1]$ defined by $f(x) = 1$ is Riemann integrable on $[0,1]$ and

$$\int_0^1 f\,dx = 1.$$

△

Example 11.8. Continuing with Example 11.5, the indicator function $f : [0,1] \to \mathbb{R}$ of $[0,1] \cap \mathbb{Q}$ is not Riemann integrable on $[0,1]$. △

Example 11.9. Let $g : [0,1] \to [0,1]$ denote the identity function, $g(x) = x$. Given a partition P as in Definition 11.1, observe,

$$\begin{aligned} U(P,g) =& \sum_{j=1}^n x_j(x_j - x_{j-1}) \\ \geq& \sum_{j=1}^n \frac{1}{2}(x_j + x_{j-1})(x_j - x_{j-1}) \\ =& \frac{1}{2}\sum_{j=1}^n [x_j^2 - x_{j-1}^2] = \frac{1}{2}. \end{aligned}$$

Hence the upper integral of g is at least $\frac{1}{2}$. A similar argument shows the lower integral is also at most $\frac{1}{2}$.

Given a positive integer n, let P_n denote the partition of $[a,b]$ given by

$$P_n = \{x_j = \frac{j}{n} : j = 0, \dots, n\}.$$

Using the well known fact that $1 + 2 + 3 + \cdots + m = m(m+1)/2$, the corresponding upper and lower sums for g are easily seen to be

$$U(P_n, g) = \sum_{j=1}^n \frac{j}{n}\frac{1}{n} = \frac{n+1}{2n}$$

and

$$L(P_n, g) = \sum_{j=0}^{n-1} \frac{j}{n}\frac{1}{n} = \frac{n-1}{2n}.$$

It follows that

$$\overline{\int_0^1} x\,dx \le \frac{1}{2}$$

and

$$\underline{\int_0^1} x\,dx \ge \frac{1}{2}.$$

Thus the upper and lower integrals are both $\frac{1}{2}$. Consequently g is integrable and its integral is $\frac{1}{2}$. △

Do Problem 11.1.

Definition 11.10. Let P and Q denote partitions of $[a, b]$. We say Q is a *refinement* of P if $P \subseteq Q$. The *common refinement* of the partitions P and Q is $P \cup Q$. ◁

Lemma 11.11. *Suppose $f : [a, b] \to \mathbb{R}$ is bounded and P and Q are partitions of $[a, b]$.*

(i) If Q is a refinement of P, then

$$L(P, f) \le L(Q, f) \le U(Q, f) \le U(P, f).$$

(ii) If P and Q are any partitions of $[a, b]$, then

$$L(P, f) \le U(Q, f).$$

(iii) In particular,

$$\underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx.$$

†

Sketch of proof. The middle inequality in item (i) is evident from the definitions (as already noted). The first and third inequalities in item (i) can be reduced to the following situation: $P = \{a < b\}$ and $Q = \{a < t < b\}$ where the result is evident.

To prove item (ii), let R denote the common refinement of P and Q and apply (i) (twice) to obtain

$$L(P, f) \le L(R, f) \le U(R, f) \le U(Q, f).$$

To prove (iii), fix a partition Q. Since $L(P, f) \le U(Q, f)$, for all partitions P, it follows that

$$\underline{\int_a^b} f\,dx \le U(Q, f).$$

Since this inequality holds for all Q, the result follows. □

Example 11.12. Returning to Example 11.9, observe that, an application of Lemma 11.11 avoids the need to first show that that the upper and lower integrals are bounded below and above respectively by $\frac{1}{2}$. △

Do Problem 11.2.

For applications, the following reformulation of integrability is often used.

Proposition 11.13. *If $f : [a,b] \to \mathbb{R}$ and f is bounded, then $f \in \mathcal{RI}([a,b])$ if and only if for each $\epsilon > 0$ there is a partition P of $[a,b]$ such that*

$$U(P,f) - L(P,f) < \epsilon.$$

†

Proof. First suppose $f \in \mathcal{RI}([a,b])$ and let $\epsilon > 0$ be given. There exist partitions Q, S such that

$$\begin{aligned} \underline{\int_a^b} f\,dx &< L(Q,f) + \epsilon \text{ and} \\ \overline{\int_a^b} f\,dx &> U(S,f) - \epsilon. \end{aligned}$$

Since the upper and lower integrals are equal, it follows that $L(Q,f) + \epsilon > U(S,f) - \epsilon$ and hence,

$$U(S,f) - L(Q,f) < 2\epsilon.$$

Choosing P equal to the common refinement of Q and S and applying Lemma 11.11(ii) twice gives

$$0 \leq U(P,f) - L(P,f) < 2\epsilon,$$

and the proof of one direction of the proposition is complete.

The estimate

$$L(P,f) \leq \underline{\int_a^b} f\,dx \leq \overline{\int_a^b} f\,dx \leq U(P,f)$$

proves the converse. □

Corollary 11.14. *If $f : [a,b] \to \mathbb{R}$ is bounded, then $f \in \mathcal{RI}([a,b])$ if and only if there is an $I \in \mathbb{R}$ such that for each $\epsilon > 0$, there exists a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that for any $x_{j-1} \leq s_j \leq x_j$,*

$$\left| I - \sum_{j=1}^{n} f(s_j)\Delta_j \right| < \epsilon.$$

In this case, I is the integral of f. †

The proof of Corollary 11.14 is left to the reader as Problem 11.4. As a remark, if only the existence of I is assumed (and not that f is bounded), it follows that f is bounded.

11.1.1. *Applying Proposition 11.13.* In using Proposition 11.13 to show a bounded function $f : [a, b] \to \mathbb{R}$ is integrable, one estimates $U(P, f) - L(P, f)$. Using the notations in Definition 11.1,

$$U(P, f) - L(P, f) = \sum_{j=1}^{n} (M_j - m_j)\Delta_j.$$

As an example, if f is increasing and $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$, then $M_j = f(x_j)$ and $m_j = f(x_{j-1})$ and thus,

$$U(P, f) - L(P, f) = \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))\ \Delta_j.$$

If P is a partition of $[a, b]$ into intervals of equal length, so $\Delta_j = (b - a)/n$ for some positive integer n, then using the telescoping nature of the sum in the middle we see that

$$U(P, f) - L(P, f) = \left(\sum_{j=1}^{n} (f(x_j) - f(x_{j-1}))\right) \frac{b - a}{n} = (f(b) - f(a))\ \frac{b - a}{n}.$$

It now follows easily from Proposition 11.13 that $f \in \mathcal{RI}([a, b])$.

Proposition 11.15. *If $f : [a, b] \to \mathbb{R}$ is increasing, then $f \in \mathcal{RI}([a, b])$.* †

The proof of Proposition 11.15 is an existence proof. In particular it does not identify the integral of f. For instance, it says that the function of example Example 11.9 is integrable, but it does not tell us the integral is $1/2$. The following theorem is a comforting wrap-up to this introductory subsection on the Riemann integral.

Theorem 11.16. *If f is continuous on $[a, b]$, then $f \in \mathcal{RI}([a, b])$.*

Proof. Let $\epsilon > 0$ be given. Since f is continuous on the compact set $[a, b]$, f is uniformly continuous. Hence there is a $\delta > 0$ so that if $a \le s, t \le b$ and $|s - t| < \delta$ then $|f(s) - f(t)| < \epsilon$.

Choose a partition P of $[a, b]$ of width less than δ; i.e., $a = x_0 < \cdots < x_n = b$ with $\Delta_j < \delta$ for all j. It follows that $M_j - m_j < \epsilon$. Hence

$$U(P, f) - L(P, f) < \epsilon(b - a).$$

An appeal to Proposition 11.13 completes the proof. □

Do Problem 11.5.

11.2. **Properties of the Integral.**

This section lists the expected and easily established properties of the Riemann integral. The proofs are mostly left to the reader.

Proposition 11.17. *If $f_1, f_2 \in \mathcal{RI}([a, b])$ and c_1, c_2 are real, then $c_1 f_1 + c_2 f_2 \in \mathcal{RI}([a, b])$ and*

$$\int_a^b (c_1 f_1 + c_2 f_2)\, dx = c_1 \int_a^b f_1\, dx + c_2 \int_a^b f_2\, dx.$$

†

Remark 11.18. The proposition says $\mathcal{RI}([a,b])$ is a (real) vector space and the mapping $I : \mathcal{RI}([a,b]) \to \mathbb{R}$ determined by the integral is linear. ◇

Do Problem 11.10.

Proposition 11.19. *If $f_1, f_2 \in \mathcal{RI}([a,b])$ and $f_1 \le f_2$, then*

$$\int_a^b f_1\,dx \le \int_a^b f_2\,dx.$$

In fact, if $f_1, f_2 : [a,b] \to \mathbb{R}$ are bounded and $f_1 \le f_2$, then

$$\underline{\int_a^b} f_1\,dx \le \underline{\int_a^b} f_2\,dx$$

and

$$\overline{\int_a^b} f_1\,dx \le \overline{\int_a^b} f_2\,dx.$$

†

It turns out that if f_1 and f_2 are integrable and $f_1 < f_2$ (meaning $f_1(x) < f_2(x)$ for all x in the interval $[a,b]$), then in fact

$$\int_a^b f_1\,dx < \int_a^b f_2\,dx,$$

though the proof is considerably more involved than that of Proposition 11.19. See Problem 11.12.

Corollary 11.20. *If $f \in \mathcal{RI}([a,b])$, then*

$$\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx.$$

†

Proof. Use $|f| \ge \pm f$ and Proposition 11.19 twice. □

Do Problem 11.13.

Proposition 11.21. *If $f \in \mathcal{RI}([a,b])$ and $a < c < b$, then $f|_{[a,c]} \in \mathcal{RI}([a,c])$ and*

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx.$$

In fact, if $g : [a,b] \to \mathbb{R}$ is bounded and $a < c < b$, then

$$\overline{\int_a^b} g\,dx = \overline{\int_a^c} g\,dx + \overline{\int_c^b} g\,dx.$$

Conversely, if $g : [a,b] \to \mathbb{R}$ is bounded and both $g|_{[a,c]}$ and $g|_{[c,d]}$ are integrable, then g is integrable. †

11.3. **The Fundamental Theorems.**

Theorem 11.22 (Second Fundamental Theorem of Calculus)**.** *If $f \in \mathcal{RI}([a,b])$, then the function $F : [a,b] \to \mathbb{R}$ defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is continuous. Further, if f is continuous at $a < y < b$, then F is differentiable at y and $F'(y) = f(y)$.

Proof. Let $M = \sup\{|f(x)| : a \le x \le b\}$. Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M+1}$. For $a \le x < z \le b$ and $z - x < \delta$ we have,

$$\begin{aligned} |F(z) - F(x)| &= \left|\int_a^z f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &= \left|\int_x^z f(t)\,dt\right| \\ &\le \int_x^z |f(t)|\,dt \\ &\le M(z-x) \\ &< \epsilon. \end{aligned}$$

Thus F is (uniformly) continuous.

Next, suppose f is continuous at y. Given $\epsilon > 0$, there is a $\delta > 0$ so that if $a \le t \le b$ and $|t - y| < \delta$, then $|f(t) - f(y)| < \epsilon$. Thus, if $a \le y < z \le b$ and $z - y < \delta$, then

$$\begin{aligned} \left|\frac{F(z) - F(y)}{z - y} - f(y)\right| &= \left|\frac{1}{z-y}\int_y^z f(t)\,dt - f(y)\right| \\ &= \left|\frac{1}{z-y}\left(\int_y^z (f(t) - f(y))\,dt\right)\right| \\ &\le \frac{1}{z-y}\int_y^z \epsilon\,dt \\ &\le \epsilon. \end{aligned}$$

A similar argument prevails for $a \le z < y \le b$ and the conclusion follows. □

Corollary 11.23. *If $f : [a,b] \to \mathbb{R}$ is continuous, then there is a continuous function $F : [a,b] \to \mathbb{R}$ such that F is differentiable on (a,b) and $F' = f$ on (a,b).* †

Example 11.24. Consider $f : (0,\infty) \to \mathbb{R}$ defined by $f(t) = 1/t$. The *logarithm* or *natural log*, denoted $\log : (0,\infty) \to \mathbb{R}$ is defined by

$$F(x) = \log(x) = \int_1^x \frac{1}{t}\,dt.$$

By Theorem 11.22, $F'(x) = 1/x$, from which the familiar properties of the log follow. (See Problems 11.16 and 11.15.) In particular $\log(1/x) = -\log(x)$.

Note that, by considering appropriate lower sums,

$$\log(n+1) \geq \sum_{j=2}^{n+1} \frac{1}{j}.$$

Since the harmonic series diverges and the log is continuous, it follows that the range of the log contains $[0,\infty)$. Using $\log(1/x) = -\log(x)$ the range of log also contains $(-\infty,0]$. Hence the range of log is all of $\mathbb{R}$.

Since the derivative of log is strictly positive, log is strictly increasing and in particular one-one. Thus log has an inverse, which is called the *exponential* function $\exp : \mathbb{R} \to (0,\infty)$.

The usual properties of exp now follow from those of log. (See Problem 11.17.)

Recall, to this point, for positive real numbers x, the power x^a has only been defined for a a rational number. In Problem 11.15 the reader is asked to show $\log(x^a) = a\log(x)$ for $x > 0$ and $a \in \mathbb{Q}$. In view of this fact, we now define, for $x > 0$ and a any real number,

$$x^a = \exp(a\log(x)).$$

In particular,

$$\exp(1)^a = \exp(a).$$

Hence, setting $e = \exp(1)$ gives $e^a = \exp(a)$ and it is customary to denote the exponential function by e^x. △

Theorem 11.25 (First Fundamental Theorem of Calculus)**.** *If $F : [a,b] \to \mathbb{R}$ is differentiable, and F' is bounded, then, for all partitions $\mathcal{P}$ of $[a,b]$,*

$$L(\mathcal{P},F') \leq F(b) - F(a) \leq U(\mathcal{P},F').$$

In particular, if $F' \in \mathcal{RI}([a,b])$, then

$$F(b) - F(a) = \int_a^b F' dx.$$

Proof. For notational ease, let $f = F'$.

Let $\mathcal{P} = \{x_0 < x_1 < \cdots < x_n = b\}$ denote a given partition of $[a,b]$. For each j there exists, by the mean value theorem (Corollary 10.14), a $x_{j-1} < t_j < x_j$ such that

$$F(x_j) - F(x_{j-1}) = f(t_j)(x_j - x_{j-1}). \tag{6}$$

Summing (6) over j and using the telescoping nature of the sum on the left hand side gives,

$$F(b) - F(a) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}). \tag{7}$$

Further, by Exercise 11.1,

$$L(\mathcal{P},f) \leq \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \leq U(\mathcal{P},f). \tag{8}$$

Combining (7) and (8) gives

$$L(\mathcal{P}, f) \leq F(b) - F(a) \leq U(\mathcal{P}, f).$$

□

Do Problem 11.18.

Corollary 11.26. *Suppose $F, G : [a,b] \to \mathbb{R}$ are differentiable. If F', G' are Riemann integrable on $[a,b]$, then FG' and GF' are Riemann integrable on $[a,b]$ and*

$$\int_a^b FG' dx = F(b)G(b) - F(a)G(a) - \int_a^b F'G dx.$$

†

Proof. The hypotheses imply the function $H = FG$ is differentiable and its derivative is Riemann integrable. Hence, by the product rule and the first FTC,

$$H(b) - H(a) = \int_a^b H' dx = \int_a^b FG' dx + \int_a^b G'F dx.$$

Rearranging gives the result. □

11.4. **Products of integrable functions.** Conspicuously missing from the elementary properties of the integral from Subection 11.2 is the fact that the product of two integrable functions is integrable.

Definition 11.27. A function $f : [a,b] \to \mathbb{R}$ is *Lipschitz continuous* if there is a constant C so that for all $x, y \in [a,b]$,

$$|f(x) - f(y)| \leq C|x - y|. \tag{9}$$

◁

Remark 11.28. Regardless of the domain, a Lipschitz continuous function is automatically uniformly continuous. On the other hand, the function $f(x) = \sqrt{x}$ on the interval $[0,1]$ is uniformly continuous but is not Lipschitz continuous. To see this note that if f were Lipschitz continuous then there would be a constant C so that

$$|\sqrt{x} - \sqrt{0}| \leq C|x - 0|$$

for all $x \in [0,1]$. However, this simplifies to $C \geq x^{-1/2}$, which is unbounded on the interval $[0,1]$.

Given a subset D of $\mathbb{R}$, a function $f : D \to \mathbb{R}$ is *continuously differentiable* if it is differentiable and $f' : D \to \mathbb{R}$ is continuous. If f is continuously differentiable on a closed bounded interval $[a,b]$, then f' is bounded on $[a,b]$, so suppose $|f'(x)| \leq M$ for all $x \in [a,b]$. By the Mean Value Theorem, it follows that for all $x, y \in [a,b]$, there is a point c between x and y so that

$$f(x) - f(y) = f'(c)(x - y),$$

and this implies that $|f(x) - f(y)| \leq M|x - y|$. (Problem 10.2 represents a slight generalization of this fact.) ◇

It is rather easy to prove the following special case of Theorem 11.34 below (Problem 11.8). It provides a powerful way to build new integrable functions.

Theorem 11.29. *If g is Lipschitz continuous and f is integrable then $h = g \circ f$ is integrable.*

Corollary 11.30. *If $f \in \mathcal{RI}([a,b])$, then so are*

(i) $|f|^p$ *for* $p \geq 1$;
(ii) $f_+ = \max\{f, 0\}$; *and*
(iii) $f_- = \min\{f, 0\}$.

†

Proof. For $p \geq 1$, the function $g(t) = |t|^p$, restricted to any bounded interval, is Lipschitz continuous. Hence Theorem 11.29 applies.

To prove (ii), consider $g : \mathbb{R} \to \mathbb{R}$ given by $g(t) = \max\{t, 0\} = (|t| + t)/2$.

To prove (iii), consider $g(t) = -(|t| - t)/2$. □

Remark 11.31. Using Theorem 11.34 below, Corollary 11.30(i) extends to $p \geq 0$. For instance if $f \in \mathcal{RI}([a,b])$, then so is $|f|^{\frac{1}{2}}$. ⋄

Corollary 11.32. *If $f, g \in \mathcal{RI}([a,b])$, then so is fg.* †

Proof. By Proposition 11.17, $f + g \in \mathcal{RI}([a,b])$. By three applications of Corollary 11.30(i) with $p = 2$ and several more applications of Proposition 11.17, it then follows that $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2] \in \mathcal{RI}([a,b])$. □

11.5. Further sufficient conditions for integrability.

11.5.1. *Limits of integrable functions.*

Proposition 11.33. *If $f_n : [a,b] \to \mathbb{R}$ is a sequence of Riemann integrable functions that converges uniformly to a function $f : [a,b] \to \mathbb{R}$, then $f \in \mathcal{RI}([a,b])$ and*

$$\lim_{n\to\infty} \int_a^b f_n \, dx = \int_a^b f \, dx.$$

†

Proof. Let $\epsilon > 0$ be given. Since (f_n) converges uniformly to f, there is an N such that if $n \geq N$, then $|f(x) - f_n(x)| < \epsilon$ for all $a \leq x \leq b$. In particular, $f_N(x) - \epsilon < f(x) < f_N(x) + \epsilon$ for all $a \leq x \leq b$. It follows that, if Q is a partition of $[a,b]$, then

$$L(Q, f_N) - \epsilon(b-a) \leq L(Q, f) \leq U(Q, f) \leq U(Q, f_N) + \epsilon(b-a).$$

By Proposition 11.13 there is a partition P such that $U(P, f_N) - L(P, f_N) < \epsilon$. Hence,

$$U(P, f) - L(P, f) < [U(P, f_N) + \epsilon(b-a)] - [L(P, f_N) - \epsilon(b-a)] \leq \epsilon(1 + 2(b-a)).$$

An application of (the other direction of) Proposition 11.13 shows $f \in \mathcal{RI}([a,b])$. The rest of the proof is left as an exercise—see Problem 11.14. □

11.5.2. *A generalization of Theorem 11.29.* Theorem 11.34 below generalizes Theorem 11.29 by relaxing the Lipshcitz continuity hypothesis to simple continuity.

Theorem 11.34. *Suppose $f \in \mathcal{RI}([a,b])$ and $f : [a,b] \to [c,d]$. If $g : [c,d] \to \mathbb{R}$ is continuous, then $h = g \circ f \in \mathcal{RI}([a,b])$.*

The proof of Theorem 11.34 is broken down into a series of lemmas.

We (provisionally) say that the function $g : [a,b] \to \mathbb{R}$ is *piecewise Lipschitz continuous* if there exists $a = x_0 < x_1 < \cdots < x_n = b$ such that $g|_{[x_j,x_{j+1}]}$ is Lipshcitz continuous for each j.

Lemma 11.35. *If $g : [a,b] \to \mathbb{R}$ is piecewise Lipschitz continuous then g is Liptschitz continuous.* †

Proof. By hypothesis there is a partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$ such that, for each k, the restriction of g to $[x_k, x_{k+1}]$ is Lipschitz continuous; that is, there exists C_k such that if $u, v \in [x_k, x_{k+1}]$, then

$$|g(u) - g(v)| \le C_k |u - v|.$$

Setting $C = \max\{C_1, \dots, C_n\}$ we will show if $a \le y < z \le b$, then

$$|g(z) - g(y)| \le C(z - y),$$

There exist indices k and m such that $x_{k-1} \le y < x_k < \cdots < x_{m-1} < z \le x_m$. Thus,

$$\begin{aligned} |g(z) - g(y)| &= \left| g(x_k) - g(y) + \left(\sum_{j=k+1}^{m-1} g(x_j) - g(x_{j-1}) \right) + g(z) - g(x_{m-1}) \right| \\ &\le |g(x_k) - g(y)| + \left(\sum_{j=k+1}^{m-1} |g(x_j) - g(x_{j-1})| \right) + |g(z) - g(x_{m-1})| \\ &\le C_k(x_k - y) + \left(\sum_{j=k+1}^{m-1} C_{j-1}(x_j - x_{j-1}) \right) + C_{m-1}(z - x_{m-1}) \\ &\le C(z - y), \end{aligned}$$

completing the proof. □

Lemma 11.36. *If $g : [a,b] \to \mathbb{R}$ is continuous, then for each $\epsilon > 0$ there is a Lipschitz continuous function $h : [a,b] \to \mathbb{R}$ such that $|g(x) - h(x)| < \epsilon$ for all $a \le x \le b$. In other words, g is uniformly approximable by Lipschitz continuous functions.* †

Proof. Let $\epsilon > 0$ be given. Since $g : [a,b] \to \mathbb{R}$ is continuous, it is uniformly continuous. Hence there is a positive integer n such that if $|x - y| < (b-a)/n$ then $|g(x) - g(y)| < \epsilon$. Let P denote the partition

$$\{x_j = a + \frac{j(b-a)}{n} : 0 \le j \le n\}$$

of $[a,b]$ and define $h : [a,b] \to \mathbb{R}$ to be the piecewise linear function whose graph connects the points $(x_j, g(x_j))$. Thus $|h(x) - g(x)| < \epsilon$ for all $x \in [a,b]$. Moreover, on each interval

$[x_k, x_{k+1}]$, h is linear and thus Lipschitz continuous. It follows that h is piece Lipschitz continuous, and thus by our previous lemma, it is Lipschitz continuous. □

Proof of Theorem 11.34. Using Lemma 11.36 that we just proved, there is a sequence $g_n : [a,b] \to \mathbb{R}$ of Lipschitz continuous functions such that (g_n) converges uniformly to g. The functions $h_n = g_n \circ f$ are Riemann integrable by Theorem 11.29. Furthermore, (h_n) converges uniformly to $g \circ f$. An application of Proposition 11.33 shows $g \circ f$ is Riemann integrable. □

11.5.3. *Almost continuous functions.* Here we show that functions that are "almost continuous" are still integrable:

Proposition 11.37. *Suppose $f : [a,b] \to \mathbb{R}$ is bounded. If f is continuous except at finitely many points, then $f \in \mathcal{RI}([a,b])$.* †

To prove Proposition 11.37 we require the following lemma.

Lemma 11.38. *If $f : [a,b] \to \mathbb{R}$ is bounded and continuous on (a,b), then $f \in \mathcal{RI}([a,b])$ and*

$$\int_a^b f\,dx = \lim_{c \to a^+} \lim_{d \to b^-} \int_c^d f\,dx.$$

†

Proof. Since f is bounded, there exist m and M such that $m \leq f(x) \leq M$ for all $x \in [a,b]$. Given $\epsilon > 0$ choose $0 < \delta \leq \min\{\frac{b-a}{2}, \frac{\epsilon}{2(M-m)}\}$. Let $c = a + \delta$ and $d = b - \delta$. Since $g = f|_{[c,d]} : [c,d] \to \mathbb{R}$ is continuous, it is integrable by Theorem 11.16. Thus, by Proposition 11.21,

$$\begin{aligned}\overline{\int_a^b} f\,dx =& \overline{\int_a^c} f\,dx + \overline{\int_c^d} f\,dx + \overline{\int_d^b} f\,dx \\ \leq& (c-a)M + \int_c^d f\,dx + (d-b)M.\end{aligned}$$

Similarly,

$$\underline{\int_a^b} f\,dx \geq (c-a)m + \int_c^d f\,dx + (b-d)m.$$

Putting the last two inequalities together,

$$\overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \leq (M-m)[(c-a)+(b-d)] < \epsilon.$$

Hence f is integrable. The remainder of the proof is left as an exercise. See Problem 11.3. □

Proof of Proposition 11.37. Let $a \leq x_1 < x_2 \cdots < x_k \leq b$ denote the points where f is not necessarily continuous. Let $x_0 = a$ and $x_{k+1} = b$. From Lemma 11.38, for each $0 \leq j \leq k$ the function $f_j = f|_{[x_j, x_{j+1}]}$ is integrable. Hence, by Proposition 11.21, f is integrable. □

Do Problem 11.9.

11.6. **The L^p norms.** Fix $a < b$ and $1 \le p < \infty$. For $f \in C([a,b])$, the function $|f|^p$ is integrable by Corollary 11.30 and, by Problem 11.5,

$$\int_a^b |f|^p \, dx = 0$$

if and only if $f = 0$ (is the zero function). It turns out that the function $\|\cdot\|_p : C([a,b]) \to [0,\infty)$ defined by

$$\|f\|_p = \left(\int_a^b |f|^p \, dx\right)^{1/p}$$

is a norm on the vector space $C([a,b])$. For $p = 1$ it is easy to show $\|\cdot\|_1$ is a norm.

It is also easy to see that the mapping $\langle \cdot, \cdot \rangle : C([a,b]) \times C([a,b]) \to \mathbb{R}$ defined by

$$\langle f, g \rangle = \int_a^b fg \, dx$$

is an inner product on $C([a,b])$ and hence,

$$\|f\|_2 = \sqrt{\langle f, f \rangle}$$

is a norm on $C([a,b])$ by Proposition 3.11.

The norms $\|\cdot\|_p$ give rise, in the usual way, to metrics on $C([a,b])$,

$$d_p(f,g) = \|f - g\|_p.$$

The limiting case (as p tends to infinity) gives the uniform metric,

$$\|f\|_\infty = \max\{|f(t)| : a \le t \le b\}.$$

(See Problem 11.6.)

For $1 \le p < \infty$, the metric space $(C([a,b], d_p)$ is not complete. Constructing the completion of $(C([a,b], d_1)$ naturally leads to the notion of *Lebesgue measure* and the *Lebesgue integral*, topics for a more advanced course in analysis.

11.7. **Integration of vector valued functions.** [3]

Definition 11.39. Suppose $f : [a,b] \to \mathbb{R}^k$ is bounded. Writing $f = (f_1, \dots, f_k)$, the function f is *Riemann integrable*, denoted $f \in \mathcal{RI}([a,b])$ if each $f_j \in \mathcal{RI}([a,b])$. In this case the *Riemann integral of* f is

$$\int_a^b f \, dx = \left(\int_a^b f_1 \, dx, \dots, \int_a^b f_k \, dx\right) \in \mathbb{R}^k.$$

Thus the integral of a $\mathbb{R}^k$-valued function is defined entry-wise and is a vector in (element of) $\mathbb{R}^k$. ◁

[3] This Subsection depends on Subsubsection 11.5.2.

Proposition 11.40. *Suppose $f : [a, b] \to \mathbb{R}^k$ and $f \in \mathcal{RI}([a, b])$. If $\gamma \in \mathbb{R}^k$, then the function*

$$f_\gamma(x) = \langle f(x), \gamma \rangle$$

is in $\mathcal{RI}([a, b])$ and

$$\int_a^b f_\gamma \, dx = \left\langle \int_a^b f \, dx, \gamma \right\rangle.$$

†

The proof is simply a matter of writing everything out in terms of the standard basis for $\mathbb{R}^k$ and using properties of the integral. The details are left to the reader. The proposition provides a coordinate-free way to define the integral of a vector valued function. Namely, the integral, if it exists, is the unique vector $I \in \mathbb{R}^k$ such that for each $\gamma \in \mathbb{R}^k$,

$$\langle I, \gamma \rangle = \int_a^b f_\gamma \, dx.$$

Proposition 11.41. *Suppose $F : [a, b] \to \mathbb{R}^k$ is differentiable. If $F' \in \mathcal{RI}([a, b])$, then*

$$F(b) - F(a) = \int_a^b F' dt.$$

†

The result follows immediately from applying the first fundamental theorem of calculus coordinate-wise.

Proposition 11.42. *Suppose $f : [a, b] \to \mathbb{R}^k$. If $f \in \mathcal{RI}([a, b])$, then $\|f\|_2 \in \mathcal{RI}([a, b])$, and*

$$\left\| \int_a^b f \, dx \right\|_2 \leq \int_a^b \|f\|_2 \, dx.$$

†

Proof. By hypothesis, each $f_j \in \mathcal{RI}([a, b])$. Thus each $|f_j|^2 \in \mathcal{RI}([a, b])$ by Corollary 11.30 part (i) (with $p = 2$). Since the sum of integrable functions is integrable, $\sum |f_j|^2 \in \mathcal{RI}([a, b])$. Finally, by Remark 11.31,

$$\|f\|_2 = \left(\sum |f_j|^2 \right)^{1/2} \in \mathcal{RI}([a, b]).$$

Assuming it is not 0, let u denote a unit vector in the direction of the integral of f and estimate, using Proposition 11.40 and the Cauchy–Schwarz inequality (Proposition 3.10),

$$\begin{aligned}\left\|\int_a^b f\,dx\right\|_2 &= \left|\left\langle\int_a^b f\,dx, u\right\rangle\right|\\ &= \left|\int_a^b \langle f,u\rangle\,dx\right|\\ &\le \int_a^b |\langle f,u\rangle|\,dx\\ &\le \int_a^b \|f\|_2\|u\|_2\,dx.\end{aligned}$$

Since $\|u\|_2 = 1$ the desired inequality follows. □

11.8. **Differentiability of a limit.**

Theorem 11.43. *Suppose $f_n : (a,b) \to \mathbb{R}$ is a sequence of continuously differentiable functions that converges pointwise to $f : (a,b) \to \mathbb{R}$. If the sequence of functions $f_n' : (a,b) \to \mathbb{R}$ converges uniformly to $g : (a,b) \to \mathbb{R}$, then f is differentiable and $f' = g$.*

Proof. Fix a point $a < c < b$. From the first fundamental theorem of calculus, for $a < x < b$,

$$f_n(x) - f_n(c) = \int_c^x f_n'(t)\,dt.$$

The uniform limit g of f_n' is continuous and thus integrable on closed subintervals of (a,b) and moreover, by Proposition 11.33,

$$\int_c^x f_n'(t)\,dt \to \int_c^x g(t)\,dt$$

for $a < x < b$. Since $f_n(x)$ and $f_n(c)$ converge to $f(x)$ and $f(c)$ respectively,

$$f(x) = f(c) + \int_c^x g(t)\,dt.$$

From the second fundamental theorem of calculus and using the fact that g is continuous, $f'(x) = g(x)$. □

11.9. **Exercises.**

Exercise 11.1. Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function and $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition. Show, if $x_{j-1} \le t_j \le x_j$, then

$$L(P,f) \le \sum_{j=1}^n f(t_j)(x_j - x_{j-1}) \le U(P,f).$$

The sum above is a *Riemann sum*.

Exercise 11.2. Give an example of a sequence (f_n) of Riemann integrable functions $f : [a,b] \to \mathbb{R}$ that converge pointwise to a bounded function f that is not Riemann integrable.

Exercise 11.3. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = n$ if $0 < x \le 1/n$ and $f_n(x) = 0$ otherwise. Explain why each f_n is Riemann integrable, the sequence (f_n) converges pointwise to a Riemann integrable function, but

$$\lim_{n\to\infty} \int_0^1 f_n\,dx \ne \int_0^1 f\,dx,$$

even though the limit on the left hand side exists. Compare with Proposition 11.33.

Exercise 11.4. Let $f : [-1,1] \to \mathbb{R}$ denote the function with $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. Show $f \in \mathcal{RI}([-1,1])$ and

$$\int_{-1}^1 f\,dx = 0.$$

Compare with Problem 11.5 and 11.12. See also Problem 11.9.

11.10. **Problems.**

Problem 11.1. Let $f : [-1,1] \to \mathbb{R}$ denote the function $f(x) = 1$ if $0 \le x \le 1$ and $f(x) = 0$ otherwise. Prove, directly from the definitions, that $f \in \mathcal{RI}([-1,1])$ and

$$\int_{-1}^1 f\,dx = 1.$$

Problem 11.2. Recall that

$$\sum_{j=1}^n j^2 = \frac{1}{6}n(n+1)(2n+1).$$

Use this formula, the definition and Lemma 11.11 to show $h : [0,1] \to \mathbb{R}$ defined by $h(x) = x^2$ is Riemann integrable.

Problem 11.3. Suppose $f : [a,b] \to \mathbb{R}$ is Riemann integrable. Prove,

$$\lim_{c\to b,\, c<b} \int_a^c f\,dx = \int_a^b f\,dx.$$

Problem 11.4. Prove Corollary 11.14. [Suggestion: See Exercise 11.1.]

Problem 11.5. Suppose $f : [-1,1] \to \mathbb{R}$ takes nonnegative values. Show, if f is integrable, continuous at 0 and if $f(0) > 0$, then

$$\int_{-1}^1 f\,dx > 0.$$

Problem 11.6. Prove,

$$\lim_{n\to\infty} \|f\|_n = \|f\|_\infty.$$

Here the limit is taken through $n \in \mathbb{N}^+$ (so is the limit of a sequence).

Problem 11.7. Show, for $1 \le p < \infty$, the metric spaces $(C([a,b],d_p)$ are not complete.

Problem 11.8. Prove Theorem 11.29. 2.9.

Problem 11.9. Suppose $f,g : [a,b] \to \mathbb{R}$ and f and g are equal except possibly at a point c with $a < c < b$. Show that if f is Riemann integrable then so is g and moreover,

$$\int_a^b f\,dx = \int_a^b g\,dx.$$

Note that, by induction, the result holds if f and g agree except possibly at finitely many points. Compare with Exercise 11.4 and Proposition 11.37.

Problem 11.10. Prove Proposition 11.17. [Suggestion: Use Corollary 11.14.]

Problem 11.11. Show, if $f : [0,1] \to \mathbb{R}$ is bounded and the lower integral of f is positive, then there is an open interval on which $f > 0$. (Compare with problems 11.5 above and 11.12 below.)

Problem 11.12. (a) Returning to Problem 11.5, give an example where the conclusion fails if f is not assumed continuous at 0 (but still assumed integrable).

(b) Consider the following variant of the function from Example (8.2) (v) Define $f : [0,1] \to \mathbb{R}$ by $f(x) = 1$ if $x \notin \mathbb{Q}$ and $f(x) = \frac{1}{q}$, where $x = \frac{p}{q}$, $p \in \mathbb{N}$, $q \in \mathbb{N}^+$, and $\gcd(p,q) = 1$. Show that f takes only positive values, yet the lower integral of f is 0. Show f is not Riemann integrable.

(c) Suppose now that $f : [a,b] \to \mathbb{R}$ is bounded. Show, if $f \ge 0$ and the upper integral of f is 0, then f is zero on a dense subset of $[a,b]$ by completing the following outline.

(i) Show, if $I = [\alpha,\beta] \subset [a,b]$ is any nontrivial (meaning $\alpha < \beta$) closed subinterval, and $\epsilon > 0$, then there is a further nontrivial closed subinterval J of I on which $0 \le f \le \epsilon$.

(ii) Starting with $I = I_0$, construct a nested sequence I_n of nontrivial closed subintervals, $I_0 \supset I_1 \supset I_2 \supset \dots$ such that $0 \le f(x) \le \frac{1}{n}$ for $x \in I_n$.

(iii) Conclude that there is a point $y \in \cap_{n=0}^{\infty} I_n$ and moreover, that $f(y) = 0$.

(iv) Conclude that every open interval in $[a,b]$ contains a point y such that $f(y) = 0$.

(v) Conclude that the set $Z(f) = \{y \in [a,b] : f(y) = 0\}$ is dense in $[a,b]$.

Finally, conclude, if $f : [a,b] \to \mathbb{R}$ is Riemann integrable, $f \ge 0$, and $f > 0$ on an open set, then the integral of f is positive.

Problem 11.13. Prove Proposition 11.19.

Problem 11.14. Suppose $f_n : [a,b] \to \mathbb{R}$ is a sequence of Riemann integrable functions that converges uniformly to the Riemann integrable function $f : [a,b] \to \mathbb{R}$. Prove, and

$$\lim_{n\to\infty} \int_a^b f_n\,dx = \int_a^b f\,dx.$$

[Suggestion: First observe that f is bounded and you may wish to apply Proposition 11.19]

Problem 11.15. Prove for $a \in \mathbb{Q}$ and $x \in \mathbb{R}^+$ (meaning x is a positive real number), that $\log(x^a) = a\log(x)$. Suggestion, consider $g(x) = \log(x^a)$ and compute $g'(x)$.

It now makes sense to define $x^r = \exp(r\log(x))$ for $r \in \mathbb{R}$.

Problem 11.16. Let $f(x) = \log(x)$. Given $a > 0$, let $g(x) = f(ax)$. Prove, $g'(x) = f'(x)$ and thus there exists a c so that $g(x) = f(x) + c$. Prove, $c = \log(a)$ and thus $\log(ax) = \log(a) + \log(x)$. (See Exercise 10.4.)

Problem 11.17. Prove $\exp(a + b) = \exp(a)\exp(b)$ and $\exp(ab) = \exp(a)^b$.

Problem 11.18. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and $\varphi : [\alpha, \beta] \to [a, b]$ is strictly increasing and continuously differentiable. Show,

$$\int_{\varphi(A)}^{\varphi(B)} f\, dx = \int_A^B f(\varphi(t))\varphi'(t)\, dt.$$

Problem 11.19. Suppose (s_n) is uniformly distributed in $[0, 1]$, meaning, for each $0 \leq a < b \leq 1$,

$$\lim_{n\to\infty} \frac{|\{k \leq n : s_k \in [a, b]\}|}{n} = b - a.$$

Show, if $f : [0, 1] \to \mathbb{R}$ is Riemann integrable, then

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} f(s_j)}{n} = \int_0^1 f\, dx.$$

12. Series

Numerical series and power series are the subjects of this section. While it is possible to work over the complex numbers $\mathbb{C}$ or even in a normed vector space, the exposition here focuses on real-valued sequences and series. In particular, throughout this section a sequence (a_n) is a numerical sequence; i.e., $a_n \in \mathbb{R}$.

Much of the theory depends on the following elementary identity for $r \in \mathbb{R}$. Namely,

$$(1-r)\sum_{j=0}^{n} r^j = 1 - r^{n+1}. \tag{10}$$

Use will also be made of the inequalities

$$\begin{aligned} \sum_{j=1}^{2^n-1} \frac{1}{j^p} &\le \sum_{m=0}^{n-1} \left(\frac{1}{2^{p-1}}\right)^m \\ \frac{1}{2^p}\sum_{m=0}^{n-1} \left(\frac{1}{2^{p-1}}\right)^m &\le \sum_{j=2}^{2^n} \frac{1}{j^p} \end{aligned} \tag{11}$$

valid for natural numbers n and positive real numbers p. Both inequalities are obtained by grouping terms as follows:

$$\sum_{j=1}^{2^n-1} \frac{1}{j^p} = \frac{1}{1} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{2^{2p}} + \cdots + \frac{1}{7^p}\right) + \cdots + \left(\frac{1}{2^{(n-1)p}} + \cdots + \frac{1}{(2^n-1)^p}\right),$$

and

$$\sum_{j=2}^{2^n} \frac{1}{j^p} = \frac{1}{2^p} + \left(\frac{1}{3^p} + \frac{1}{2^{2p}}\right) + \left(\frac{1}{5^p} + \cdots + \frac{1}{2^{3p}}\right) + \cdots + \left(\frac{1}{(2^{(n-1)}+1)^p} + \cdots + \frac{1}{2^{np}}\right).$$

Do Problem 12.1.

12.1. Some Numerical Sequences.

Recall that the sequence (a_n) from $\mathbb{R}$ *converges* if there is an $A \in \mathbb{R}$ such that for every $\epsilon > 0$ there is an N such that if $n \ge N$, then $|a_n - A| < \epsilon$. In this case the sequence is said to *converge to* A and A is the *limit of the sequence*, written

$$\lim_{n\to\infty} a_n = \lim a_n = A.$$

Recall also, that if a sequence converges, then its limit is unique.

It is also convenient to introduce the following notion. A sequence (a_n) from $\mathbb{R}$ *diverges to* ∞, denoted

$$\lim_{n\to\infty} a_n = \infty$$

if, for every $C > 0$ there exists an N such that $a_n > C$ for all $n \ge N$.

Proposition 12.1. *Suppose $r, \rho \in \mathbb{R}$ and $0 \le r < 1$ and $0 < \rho$.*

(i) The sequence (r^n) converges to 0;
(ii) For each $k \in \mathbb{N}$, the sequence $(n^k r^n)$ converges to 0;

(iii) The sequence $(\rho^{\frac{1}{n}})$ *converges to* 1*;*
(iv) The sequence $(\frac{\log(n)}{n})$ *converges to* 0*;*
(v) For p *real, the sequence* $(n^{\frac{p}{n}})$ *converges to* 1*.*

†

The proofs use Theorem 4.12 (a bounded increasing sequence of real numbers converges) in several parts.

Proof. Item (i) and item (ii) in the case that $k = 1$ is the content of Proposition 4.21. Problem 4.9 handles the case $k > 1$ in item (ii).

Item (iii) and item (iv) in the case $p = 1$ is Proposition 4.6. Using the fact that log is continuous, it follows that

$$0 = \log(1) = \lim_{n\to\infty} \log(n^{\frac{1}{n}}) = \lim_{n\to\infty} \frac{\log(n)}{n}.$$

In particular, for $p \in \mathbb{R}$,

$$0 = \lim_{n\to\infty} \frac{p\log(n)}{n} = \lim_{n\to\infty} \log(n^{\frac{p}{n}}).$$

Using the fact that exp is continuous, it now follows that

$$1 = \exp(0) = \lim_{n\to\infty} n^{\frac{p}{n}}.$$

□

12.2. **Numerical Series.** Given a sequence $(a_n)_{n=k}^{\infty}$, the *series*

$$\sum_{j=k}^{\infty} a_j = \sum a_j, \tag{12}$$

is the sequence (s_n) of *partial sums*,

$$s_n = \sum_{j=k}^{n} a_j. \tag{13}$$

The *series converges* if the sequence (s_n) converges and in this case we write

$$\sum_{j=k}^{\infty} a_j = \lim s_n.$$

Thus we have used the same symbol to denote the sequence (s_n) and, if it converges, its limit. Otherwise the series *does not converge*.

If (s_n) diverges to ∞, the the series *diverges to infinity*, written

$$\sum_{j=k}^{\infty} a_j = \infty.$$

Example 12.2 (The Geometric Series)**.** The series $\sum r^j$ is known as the *geometric series.* Using (10), it is easy to show, if $|r| < 1$, then

$$\sum_{j=0}^{\infty} r^j = \frac{1}{1-r};$$

if $r \geq 1$, then

$$\sum r^j = \infty;$$

and if $r \leq -1$, then the series $\sum r^j$ does not converge. △

Do Problem 12.3.

Proposition 12.3. *Consider the series* $\sum a_j = \sum_{j=k}^{\infty} a_j$ *and its partial sums* $s_n = \sum_{j=k}^{n} a_j$.

(i) The series $\sum a_j$ *converges if and only if, for each* $\ell \geq k$, *the series* $\sum_{j=\ell}^{\infty} a_j$ *converges;*

(ii) If $a_j \geq 0$ *for all* j, *then the series* $\sum a_j$ *converges if and only if the partial sums* (s_n) *form a bounded sequence; i.e., if and only if there exists a constant* M *such that* $s_n \leq M$ *for all* n*;*

(iii) If there exists an $\ell \geq k$ *and a sequence* (b_j) *such that* $b_j \geq a_j \geq 0$ *for all* $j \geq \ell$ *and*

$$\sum_{j=\ell}^{\infty} b_j$$

converges, then the series $\sum a_j$ *converges;*

(iv) If there exists an $\ell \geq k$ *and a sequence* (b_j) *such that* $a_j \geq b_j \geq 0$ *for all* $j \geq \ell$ *and*

$$\sum_{j=\ell}^{\infty} b_j$$

diverges to infinity, then the series $\sum a_j$ *diverges to infinity;*

(v) The series $\sum a_j$ *converges if and only if for every* $\epsilon > 0$ *there is an* N *such that for all* $n > m \geq N$,

$$\left| \sum_{j=m+1}^{n} a_j \right| < \epsilon.$$

(vi) If the series

$$\sum_{j=k}^{\infty} |a_j|$$

converges, then so does the series $\sum a_j$ *and moreover,*

$$\left| \sum_{j=k}^{\infty} a_j \right| \leq \sum_{j=k}^{\infty} |a_j|;$$

(vii) If the series $\sum a_j$ *converges, then* (a_n) *converges to* 0.

†

Remark 12.4. In this case (where $a_j \geq 0$), if the sequence is not bounded, then the series diverges to ∞.

Items (iii) and (iv) together are the *comparison test.*

Item (v) is the Cauchy criteria (which applies since $\mathbb{R}$ is complete).

If $\sum |a_j|$ converges, then $\sum a_j$ *converges absolutely.* Thus, item (vi) says absolute convergence implies convergence.

If the series $\sum a_j$ converges, but the series $\sum |a_j| = \infty$, then $\sum a_j$ is said to *converge conditionally* . An example of a conditionally convergent series is the alternating harmonic series, see Example 12.12 below. ◇

Proof. To prove (i), note that the sequence of partial sums

$$t_n = \sum_{j=\ell}^{n} a_j$$

for the series $\sum_{j=\ell}^{\infty} a_j$ are related to the partial sums s_n for the original series by,

$$t_n = s_n - c,$$

where $c = \sum_{j=k}^{\ell-1} a_j$. Hence (t_n) converges if and only if (s_n) converges.

For item (ii), if $a_j \geq 0$ for all j, then (s_n) is an increasing sequence. Thus (s_n) converges if and only if it is a bounded sequence.

To prove item (iii), define s'_n by

$$s'_n = \sum_{j=\ell}^{\infty} a_j.$$

Since the corresponding series with terms b_j converges, its partial sums are bounded by some positive number M. Hence,

$$s'_n = \sum_{j=\ell}^{n} a_j \leq \sum_{j=\ell}^{n} b_j \leq M.$$

Thus (s'_n) is bounded and by part (ii) converges. Hence the original series converges by item (i).

Item (iv) is essentially the contrapositive of item (iii). The details of the proof are left to the reader.

Item (v) is just a restatement of the Cauchy criteria.

To prove item (vi), let t_n denote the partial sums

$$t_n = \sum_{j=k}^{n} |a_j|.$$

Observe that, for $n > m$,

$$\begin{aligned} |s_n - s_m| &= |\sum_{j=m+1}^{n} a_j| \\ &\leq \sum_{j=m+1}^{n} |a_j| \\ &= |t_n - t_m|. \end{aligned}$$

Since (t_n) converges, it is a Cauchy sequence. It follows from the inequality above that (s_n) is Cauchy. Hence (s_n) converges and item (vi) is proved.

If (s_n) converges, then it is Cauchy. Hence, given $\epsilon > 0$ there is an N such that if $n > m \geq N$, then $|s_n - s_m| < \epsilon$. In particular, if $n > N$ and $m = n - 1$, then $|a_n| < \epsilon$. This shows (a_n) converges to 0. □

Example 12.5 (The Harmonic Series)**.** The series

$$\sum_{j=1}^{\infty} \frac{1}{j}$$

is the *harmonic series*.

Since its sequence (s_n) of partial sums is increasing and, from the second inequality in equation (11), the subsequence (s_{2^n}) is unbounded, the harmonic series diverges to infinity. △

Example 12.6 (p-series)**.** More generally, for $0 < p$, the series

$$\sum_{j=1}^{\infty} \frac{1}{j^p}$$

is a p-series.

As we have already seen, the series diverges to infinity for $p = 1$ and thus diverges to infinity for $p < 1$ by the comparison test, Proposition 12.3 part (v).

For $p > 1$, the first estimate of equation (11) shows that the partial sums are bounded above by

$$\sum_{m=0}^{\infty} (\frac{1}{2^{p-1}})^m = \frac{2^{p-1}}{2^{p-1} - 1}.$$

Thus, by Proposition 12.3 part (ii), the series converges. △

Do Problems 12.2, 12.4, 12.5, and 12.6.

12.3. **The Root Test.** In preparation for the proof of the root test below recall the notion of the *limit superior* (lim sup) of a sequence of non-negative real numbers (a_n).

Definition 12.7. Let (a_n) be a non-negative sequence of real numbers. If (a_n) is unbounded, then $\limsup a_n = \infty$. If (a_n) is bounded, let

$$\alpha_n = \sup\{a_m : m \ge n\}.$$

Thus, (α_n) is a decreasing sequence which is bounded below by 0 and hence converges to some L. Set $\limsup a_n = L$. ◁

Theorem 12.8 (Root Test)**.** *Consider the series* $\sum a_j$ *and let*

$$L = \limsup |a_n|^{1/n}.$$

(i) If $L < 1$, then the series converges;
(ii) If $L > 1$, then (a_n) is unbounded (so doesn't converge to 0) and thus the series does not converge;
(iii) If $L = 1$, then the test is inconclusive.

Proof. Suppose $L < 1$. Choose $L < \rho < 1$. From Proposition 4.32, there is an N so that if $n \ge N$, then

$$|a_n|^{1/n} < \rho.$$

Thus, $|a_n| < \rho^n$ for $n \ge N$. Hence the series converges by comparison to the geometric series $\sum \rho^j$.

Suppose $L > 1$. Choose $L > \rho > 1$. For each n there is an $m \ge n$ such that $|a_m| > \rho^m$. It follows that (a_n) is not bounded above. Hence, by Proposition 12.3(vi), the series diverges.

The sequence $((1/n^p)^{1/n}) = (n^{-p/n})$ converges to 1 by Proposition 12.1(v). It follows that hypothesis of part (iii) prevails for all p-series; however some p series converge (when $p > 1$), while others diverge to infinity (when $0 < p \le 1$). Hence, if $L = 1$, the root test is inconclusive. □

12.4. **Series Squibs.** A *squib* (among other meanings) refers to a short, sometimes humorous piece in a newspaper or magazine, usually used as a filler. It also can mean a firecracker which burns out without exploding (*a dud*).

Theorem 12.9 (Ratio Test)**.** *Let (c_n) be a sequence of positive real numbers and let*

$$a_n = \frac{c_{n+1}}{c_n}.$$

(i) If $\limsup a_n < 1$, then the series $\sum c_n$ converges; and
(ii) If $\liminf a_n > 1$, then the sequence (c_n) tends to infinity (does not converge to 0) and hence the series $\sum c_n$ diverges to infinity.

Note the asymmetry between the hypotheses of the root and ratio tests. Problem 12.13 shows that the root test is a stronger result than the ratio test, though of course when it does apply, often the ratio test is easier to use. In the case that the sequence $(\frac{c_{n+1}}{c_n})$ converges, say to L, then the series converges if $L < 1$ and diverges if $L > 1$. In the case that $L = 1$, the test fails.

Proof. Suppose $L = \limsup a_n < 1$. Choose $L < \rho < 1$. By Proposition 4.32, there is an N so that if $n \geq N$, then $a_n \leq \rho$. It follows that

$$c_n \leq \rho c_{n-1}.$$

Iterating this inequality and writing $n = N + m$ give

$$c_n \leq \rho^m c_N = \rho^n(\rho^{-N} c_N)$$

Thus $\sum c_n$ converges by comparison to the geometric series $\sum \rho^n$.

Now suppose $L = \liminf a_n > 1$. Choose $L > \rho > 1$. There is an N so that

$$\inf\{a_n : n \geq N\} \geq \rho.$$

Arguing as before,

$$c_{n+N} \geq \rho^n c_N,$$

which tends to infinity. □

Remark 12.10. The root and ratio tests are rather crude tests for divergence. Indeed, the sufficient condition in each case implies the terms of the series do not converge to 0. On the other hand, the root and ratio test can be used to determine whether a sequence (a_n) of positive terms converges to 0. Indeed, if either $\limsup |a_n|^{\frac{1}{n}}$ or $\limsup \frac{a_{n+1}}{a_n}$ is (strictly) less than 1, then $\lim a_n = 0$. (See Problem 4.9.)

As an example, for $r > 0$ and fixed,

$$\lim_{n\to\infty} \frac{r^n}{n!} = 0.$$

◇

Theorem 12.11 (Alternating Series)**.** *If a_n is a decreasing sequence of positive numbers which converges to 0, then the alternating series*

$$\sum_{j=k}^{\infty} (-1)^j a_j$$

converges.

Proof. Let s_n denote the partial sums. For a natural numbers m, k with k even,

$$\begin{aligned}|s_{m+k} - s_m| = &\left|\sum_{j=m+1}^{m+k} (-1)^j a_j\right| \\ =&|(a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \cdots + (a_{m+k-1} - a_{m+k})| \\ =&(a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \cdots + (a_{m+k-1} - a_{m+k}) \\ =&a_{m+1} - (a_{m+2} - a_{m+3}) - \cdots - (a_{m+k-2} - a_{m+k-1}) - a_{m+k} \\ \le&a_{m+1},\end{aligned}$$

where the decreasing hypothesis is used in the third equality and the inequality.

For k odd,

$$\begin{aligned}|s_{m+k} - s_m| = &\left|\sum_{j=m+1}^{m+k} (-1)^j a_j\right| \\ =&|(a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \cdots + (a_{m+k-2} - a_{m+k-1}) + a_{m+k}| \\ =&(a_{m+1} - a_{m+2}) + (a_{m+3} - a_{m+4}) + \cdots + (a_{m+k-2} - a_{m+k-1}) + a_{m+k} \\ =&a_{m+1} - (a_{m+2} - a_{m+3}) - \cdots - (a_{m+k-1} - a_{m+k}) \le a_{m+1}.\end{aligned}$$

Since a_n converges to 0, the sequence (s_n) is Cauchy and thus converges. □

Example 12.12. The alternating harmonic series

$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

converges, but not absolutely. Thus it is a conditionally convergent series. △

12.5. **Power Series.**

Definition 12.13. Let (a_n) be a sequence of real numbers. The expression

$$\sum_{j=0}^{\infty} a_j x^j \tag{14}$$

is a *power series.* ◁

Remark 12.14. Let D denote those real numbers x for which the series (14) converges. The power series (14) determines a function $s : D \to \mathbb{R}$ defined by

$$s(x) = \sum_{j=0}^{\infty} a_j x^j. \tag{15}$$

Let

$$s_n = \sum_{j=0}^{n} a_j x^j. \tag{16}$$

denote the *partial sums* of the power series. The s_n can be thought of as either functions on D or on all of $\mathbb{R}$, as dictated by context.

The following theorem says that D is not too complicated. $\diamond$

Theorem 12.15. *Given the power series* (14)*, let* $L = \limsup |a_n|^{\frac{1}{n}}$ *and let* $R = \frac{1}{L}$ *(interpreted as* 0 *if* $L = \infty$*, and* ∞ *if* $L = 0$*).*

(i) The series converges absolutely for $|x| < R$ *and diverges for* $|x| > R$*;*
(ii) $(-R, R) \subseteq D \subseteq [-R, R]$ *and thus* D *is an interval; and*
(iii) If $R' \in [0, \infty]$ *and the series converges absolutely for* $|x| < R'$ *and diverges for* $|x| > R'$*, then* $R = R'$*.*

Definition 12.16. The number R is the *radius of convergence* and the set D is the *interval of convergence.* $\triangleleft$

Problem 12.9 provides examples showing there is no more general statement possible about the interval of convergence (domain) of a power series. Do Problem 12.8.

Proof. Let the real number x be given. Let $c_n = a_n x^n$ and note that an application of Lemma 4.31 gives,

$$\limsup |c_n|^{\frac{1}{n}} = |x| \limsup |a_n|^{\frac{1}{n}} = |x|L.$$

By the root test if $|x|L < 1$, then the series converges absolutely, and if $|x|L > 1$, then the series does not converge. $\square$

Lemma 12.17. *If the power series* s *has radius of convergence* $R > 0$ *and* $0 < u < R$*, then the sequence* (s_n) *converges uniformly on* $[-u, u]$*. In particular the limit* s *is continuous on* $|x| < R$*.* $\dagger$

Remark 12.18. It is natural to ask, as Abel did, if say the interval of convergence is $I = (-R, R]$, is then the function $f : I \to \mathbb{R}$ of Remark 12.14 continuous; i.e., continuous at R. The answer is yes. See Problem 12.11 $\diamond$

Proof. Since u is within the radius of convergence of the series, the sequence

$$t_n = \sum_{j=0}^{n} |a_j u^j|$$

converges and thus satisfies the Cauchy condition of Proposition 12.3(v). In particular, given $\epsilon > 0$ there is an N so that if $n > m \geq N$, then

$$\epsilon > |t_n - t_m| = \sum_{j=m+1}^{n} |a_j| u^j.$$

If $|x| \le u$ and $n > m \ge N$, then

$$\begin{aligned} |s_n(x) - s_m(x)| &= |\sum_{j=m+1}^{n} a_j x^j| \\ &\le \sum_{j=m+1}^{n} |a_j||x|^j \\ &\le \sum_{j=m+1}^{n} |a_j| u^j \\ &< \epsilon. \end{aligned}$$

Thus (s_n) converges uniformly on $[-u, u]$ and thus converges uniformly to its pointwise limit s on this interval. Since each s_n is continuous, so is the limit on the interval $[-u, u]$. Thus the limit s is continuous on $(-R, R)$. □

Lemma 12.19. *If the series* (14) *has radius of convergence R, then both of the series*

(i) $\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ *and;*
(ii) $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^n$

have radius of convergence R too. †

Proof. To prove the lemma, note that, by Lemma 4.31 and by Proposition 12.1(d), that

$$\limsup |na_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} = \limsup |\frac{a_n}{n+1}|^{\frac{1}{n}}.$$

Hence all three (the original plus those in (i) and (ii)) have the same radius of convergence. □

Remark 12.20. It can of course happen that the interval of convergence of the these series is different than that for the original series. See Exercise 12.3. It turns out that, compared to the interval of convergence for the original series, the interval in (i) could only possibly lose endpoints; and that in (ii) could only possibly gain endpoints. Summation by parts can be used to prove this assertion. See Lemma 13.15 and Problem 13.5. ⋄

Proposition 12.21. *Suppose the power series* (14) *has radius of convergence $R > 0$ and let s denote the sum of the series. If $0 \le u < R$, then s is integrable on $[-u, u]$ and*

$$\int_0^u s\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} u^{n+1}.$$

The function s is differentiable on $|x| < R$ and

$$s'(x) = \sum_{j=0}^{\infty} (j+1) a_{j+1} x^j$$

†

Proof. On the interval $[-u,u]$ the sequence (s_n) converges uniformly to s. Thus, by Problem 11.33,

$$\begin{aligned}\int_0^u s\,dx &= \lim_{n\to\infty}\int_0^u s_n\,dx\\ &= \lim_{n\to\infty}\sum_{j=0}^n \int_0^u a_j x^j\,dx\\ &= \sum_{j=0}^\infty \frac{a_j u^{j+1}}{j+1}.\end{aligned}$$

Let (t_n) denote the partial sums of the series

$$t(x)=\sum_{j=0}^\infty (j+1)a_{j+1}x^j.$$

By Lemma 12.19, this series has radius of convergences R and by Lemma 12.17, the sequence (t_n) converges uniformly to t on each interval $(-u,u)$, for $0<u<R$. Since also $s_n' = t_{n-1}$, Theorem 11.43 applies with the conclusion that s is differentiable and $s'=t$ on $(-u,u)$ and therefore on $(-R,R)$. □

Remark 12.22. Thus power series can be integrated and differentiated *term by term*. In particular, a power series (14) is infinitely differentiable within its radius of convergence. Moreover,

$$s^{(m)}(0)=a_m m!$$

◇

12.6. **Functions as Power Series.** From the geometric series

$$\frac{1}{1-x}=\sum_{j=0}^\infty x^j, \quad |x|<1, \tag{17}$$

it is possible to derive a number of other series representations.

Differentiating (17) term by term gives

$$\frac{1}{(1-x)^2}=\sum_{j=0}^\infty (j+1)x^j, \quad |x|<1.$$

Replacing x by $-t^2$ in (17) gives,

$$\frac{1}{1+t^2}=\sum_{j=0}^\infty (-1)^j t^{2j}, \quad |t|<1.$$

Integrating this last series term by term gives

$$\arctan(x)=\sum_{j=0}^\infty (-1)^j \frac{x^{2j+1}}{2j+1}, \quad |x|<1.$$

Likewise integrating (17) term by term gives

$$-\log(1-x) = \sum_{j=0}^{\infty} \frac{x^{j+1}}{j+1}, \quad |x| < 1.$$

12.7. **Taylor Series.** The following consequence of Taylor's Theorem is suitable for establishing power series representations for the exponential, sine and cosine functions.

Theorem 12.23. *Suppose $f : \mathbb{R} \to \mathbb{R}$. If f is infinitely differentiable and for each x there is a C_x such that*

$$|f^{(j)}(y)| \le C_x$$

for all $j \in \mathbb{N}$ and $|y| \le |x|$, then the power series

$$s(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j$$

has infinite radius of convergence and $f(x) = s(x)$. Moreover, this series for f converges uniformly on bounded sets (to f).

Proof. Given $n \in \mathbb{N}$ and $x \in \mathbb{R}$, from Taylor's Theorem there is a c between 0 and x such that

$$f(x) = s_n(x) + \frac{f^{(n+1)}(c) x^{n+1}}{(n+1)!}$$

Thus,

$$|f(x) - s_n(x)| \le C_x \frac{|x|^{n+1}}{(n+1)!},$$

where C_x depends only upon x (and not on n). Thus it suffices to show that, for a given x, the right hand side converges to 0 (see Remark 12.10). Hence, for each x, the sequence $(s_n(x))$ converges to $f(x)$. Consequently s has infinite radius of convergence and thus, by Lemma 12.17, (s_n) converges to f uniformly on every bounded interval (and hence uniformly on every bounded set). $\square$

Example 12.24. Let $f(x) = \exp(x)$ and note $f^{(n)}(x) = \exp(x)$. Since if $|y| \le |x|$, then $\exp(y) \le \exp(|x|)$ and hence $|f^{(n)}(y)| \le \exp(|x|)$. Theorem 12.23 implies

$$\exp(x) = \sum_{j=0}^{\infty} \frac{x^n}{n!}, \quad x \in \mathbb{R}.$$

Let $g(x) = \sin(x)$. Then $|g^{(n)}(x)| \le 1$ for all n and x. It follows that

$$\sin(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{2j+1}}{(2j+1)!}.$$

$\triangle$

Remark 12.25. If $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and the series

$$s(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

has infinite radius of convergence, it is natural to ask if $f = s$. The answer is, without hypotheses such as those in Theorem 12.23, no as can be seen from the following example (found in nearly every Calculus text). For the function f given by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(http://en.wikipedia.org/wiki/Non-analytic_smooth_function) it can be shown that $f^{(j)}(0) = 0$ for all j. Thus s has infinite radius of convergence, but $s = 0 \neq f$.

◇

12.8. **Exercises.**

Exercise 12.1. Show if (s_n) is an increasing sequence of real numbers and if (s_n) has a bounded subsequence, then (s_n) converges. Interpret this result in terms of series.

Exercise 12.2. Test the following series for convergence.

(i)

$$\sum_{n=1}^{\infty} (\frac{n}{n+2})^{n^2}.$$

(ii)

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdots 2n}.$$

(iii)

$$\sum_{n=1}^{\infty} \frac{1 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdots (2n)^2}.$$

Exercise 12.3. For the power series,

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n},$$

let A, B, C denote the interval of convergence of f and the series (i) and (ii) respectively in Lemma 12.19. Verify, $B \subsetneq A \subsetneq C$.

Exercise 12.4. Find a power series representation for $\cos(x)$ and verify that it converges uniformly to $\cos(x)$ on every bounded interval.

Exercise 12.5. Find a power series representation for the *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt.$$

`http://en.wikipedia.org/wiki/Error_function`

Exercise 12.6. Show, if (a_n) is a sequence of nonnegative numbers and the series

$$\sum_{j=0}^{\infty} a_j$$

converges, then for every $\epsilon > 0$ there is an M so that if $M \le N$, then

$$\sum_{j=N}^{\infty} a_j < \epsilon.$$

Exercise 12.7. Given a sequence $(b_n)_{n=0}^{\infty}$, let $a_n = b_{n+1} - b_n$. The series,

$$\sum_{j=0}^{\infty} a_j$$

is a *telescoping series*. What are its partial sums? When does the series converge.

Exercise 12.8. Show, if (a_n) is a bounded sequence, then the radius of convergence of the series (14) has radius of convergence at least 1.

Exercise 12.9. Explain why the series,

$$\sum_{n=1}^{\infty} (-1)^n n^{-s}$$

converges for $s > 0$ and diverges for $s \le 0$. It thus determines a function $\eta(s)$ with domain $(0, \infty)$. Explain why series converges absolutely for $s > 1$ and conditionally for $0 < s \le 1$ and compare with the situation for a power series, where there are at most two points where the series converges conditionally. The series here is a *Dirichlet Series* (which are more naturally thought of as a function of a complex variable s). For more on Dirichlet Series see Subsection 13.3.

12.9. **Problems.**

Problem 12.1. Suppose (a_n) a decreasing sequence of positive numbers. Show, for positive integers n,

$$\sum_{j=1}^{2^n-1} a_j \le \sum_{k=0}^{n-1} 2^k a_{2^k}$$

and likewise,

$$\sum_{j=2}^{2^n} a_j \ge \frac{1}{2} \sum_{k=1}^{n} 2^k a_{2^k}.$$

Verify the inequalities in Equation (11) are special cases.

Problem 12.2. Suppose (a_n) a decreasing sequence of positive numbers. Use Problem 12.1 to show the series $\sum_{j=0}^{\infty} a_j$ converges if and only if the series

$$\sum_{j=0}^{\infty} 2^j a_{2^j}$$

converges. (You might find Exercise 12.1 useful.)

Determine the convergence of the series

(i)

$$\sum_{n=2}^{\infty} \frac{1}{n\log(n)};$$

(ii)

$$\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^2}.$$

Problem 12.3. Show, if (a_n) is a decreasing sequence of positive real numbers and

$$\sum_{j=0}^{\infty} a_j$$

converges, then $\lim na_n = 0$. [Suggestion: Observe, $\lim_{n\to\infty} a_n = 0$, for each N and each $n > N$,

$$\sum_{j=N}^{n} a_j \geq (n-N)a_n,$$

and use Exercise 12.6.]

Problem 12.4. Suppose (a_j) and (b_j) are sequences of real numbers. Show, if both

$$\sum a_j^2, \quad \text{and} \quad \sum b_j^2$$

converge, then so does

$$\sum a_j b_j.$$

[Suggestion: Use the inequality $2|ab| \leq a^2 + b^2$, or apply the Cauchy–Schwarz inequality at the level of partial sums.]

Problem 12.5. Show, if (a_n) is a sequence of non-negative numbers and the series $\sum a_j$ converges, then so does the series $\sum a_j^2$.

Problem 12.6 (Integral Test)**.** Suppose $f : [0,\infty) \to [0,\infty)$ is continuous and decreasing. Show,

$$\sum_{j=0}^{\infty} f(j)$$

converges if and only if

$$F(x) = \int_0^x f\, dt$$

is bounded above independent of x (meaning there is an M such that $F(x) \le M$ for all $x \ge 0$).

Problem 12.7. Suppose (a_j) is a sequence from $\mathbb{R}^k$. Show, if

$$\sum \|a_j\|$$

converges, then

$$\sum a_j$$

converges in $\mathbb{R}^k$.

Problem 12.8. Show, if (a_n) is a sequence of non-zero real numbers and

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$$

converges with limit L, then the radius of convergence of the power series $\sum a_j x^j$ is $R = \frac{1}{L}$ (properly interpreted).

Problem 12.9. Find the interval of convergence for the following power series.

(i) $\sum n^{-n} x^n$;
(ii) $\sum \frac{x^n}{n^2}$;
(iii) $\sum \frac{x^n}{n}$;
(iv) $\sum n^n x^n$.

Problem 12.10. Suppose (a_n) is a sequence from a metric space X. Show that the conclusion of Proposition 5.12 holds if the series $\sum d(a_{n+1}, a_n)$ converges.

Problem 12.11 (Abel's Theorem)**.** Suppose the series s of (14) has radius of convergence 1 and suppose

$$\sum_{j=0}^{\infty} a_j$$

converges. Prove

$$\lim_{r\to 1^-} \sum_{j=0}^{\infty} a_j r^j = \sum_{j=0}^{\infty} a_j.$$

Thus, the (function defined by the) series s is continuous at 1.

Here is an outline to follow if you like.

(i) It can be assumed that $\sum_{j=0}^{\infty} a_j = 0$.
(ii) With $t_n = \sum_{j=0}^{n} a_j$, show if N is a positive integer and $|t_j| \le C$ for all $j \ge N$, then for $n \ge N$,

$$|\sum_{j=N+1}^{n} t_j x^j| \le C \frac{x^N}{1-x}.$$

In particular, the series

$$g(x) = \sum_{n=0}^{\infty} t_n x^n$$

has radius of convergence at least one.

(iii) Show
$$s(x) = (1-x)g(x), \quad |x| < 1.$$

(iv) Given $\epsilon > 0$, choose N such $|t_j| < \epsilon$ for $j \geq N$ and use, for $0 < x < 1$,
$$(1-x)g_n(x) = (1-x)[\sum_{j=0}^{N-1} t_j x^j + \sum_{j=N}^{n} t_j x^j] \leq (1-x)[CN + \frac{\epsilon}{1-x}],$$
where g_n are the partial sums of the series g and C is an bound on the $\{|t_j| : j \in \mathbb{N}\}$, to complete the proof.

Problem 12.12. Use Abel's theorem and the power series representation for $\arctan(x)$ to show
$$\frac{\pi}{4} = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}.$$

Problem 12.13. Let (a_n) be a sequence of positive reals. Show,
$$\limsup \frac{a_{n+1}}{a_n} \geq \limsup |a_n|^{\frac{1}{n}}$$
and also
$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf |a_n|^{\frac{1}{n}}.$$
[Suggestion: Follow the proof of the ratio test.]

Problem 12.14. Prove, Bernoulli's inequality,
$$(1+x)^n \geq 1 + nx,$$
for positive integers n and $x \geq -1$. [Suggestion: Induct.]

Prove, the sequence $(e_n = (1+\frac{1}{n})^n)$ is increasing. [Suggestion: Observe
$$\left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^{n+1} = (1 - \frac{1}{(n+1)^2})^{n+1}$$
and apply Bernoulli's inequality.]

Prove the sequence $(f_n = (1+\frac{1}{n})^{n+x})$ is decreasing. [Suggestion: Simplify
$$\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)^{n+2}$$
and apply Bernoulli's inequality.

Show,
$$e_n \leq \sum_{j=0}^{n} \frac{1}{n!} \leq f_n.$$
Conclude (e_n) and (f_n) converge to $\exp(1)$. Thus,
$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n.$$

Problem 12.15. Prove, for x real and $x \geq -2$ and positive integers n that

$$(1+x)^n \geq 1+nx.$$

[Suggestion: Note that the result is true for $n=1,2$ by direct verification. Now suppose, arguing by induction, that $x \geq -2$ and that $(1+x)^m \geq 1+mx$ for all $m \leq n$. Write

$$(1+x)^{n+1} = (1+x)^{n-1}(1+x)^2$$

and apply the induction hypothesis the first term in the product on the right hand side.]

Problem 12.16. Let $r_n = \sum_{j=1}^n \frac{1}{j} - \log(n)$. Show that the sequence is bounded below (by 0) and decreasing. It thus has a limit known as the Euler-Mascheroni constant (http://en.wikipedia.org/wiki/Euler%E2%80%93Mascheroni_constant).

Problem 12.17. Suppose X is a metric space and $(f_j)_{j=k}^\infty$ is a sequence of functions $f_n : X \to \mathbb{R}$. If there exists a sequence (of nonnegative numbers) (M_j) such that, for each n and $x \in X$,

$$|f_n(x)| \leq M_j$$

and the series

$$\sum_{j=k}^\infty M_j$$

converges, then the series of functions,

$$F_n(x) = \sum_{j=k}^n f_j(x)$$

converges uniformly. This result is the *Weierstrass M test*.

Problem 12.18. Suppose $f : (-R,R) \to \mathbb{R}$ is infinitely differentiable and for each j there is an M_j such that $|f^{(j)}(t)| \leq M_j$ for each j and $t \in (-R,R)$. If

$$\sum_{j=0}^\infty M_j \frac{R^j}{j!}$$

converges, then the Taylor series for f converges to f uniformly on $(-R,R)$.

13. Complex Numbers and Series*

The $*$ here indicates that this section is optional. Subsection 13.1 is needed for a later section on Fourier Series. Let $V = (V, \|\cdot\|)$ be a complete normed vector space (a *Banach space*). Thus, V is a vector space, $\|\cdot\|$ is a norm on V and the metric space $V = (V, d)$ with $d(x, y) = \|x - y\|$ is complete. An example is of course $\mathbb{R}^k$ with the usual Euclidean norm. To a sequence (a_n) from V there is the naturally associated series (s_n) with

$$s_n = \sum_{j=0}^{n} a_j.$$

As before, let

$$\sum_{j=0}^{\infty} a_j \tag{18}$$

denote both the sequence of partial sums and its limit, should it exist.

Definition 13.1. The series of Equation (18) *converges absolutely* if the numerical series

$$\sum_{j=0}^{\infty} \|a_j\|$$

converges. ◁

Most of the facts about numerical series carry over to series from V. The following proposition gives a sampling of such results.

Proposition 13.2. *For the series S from Equation (18) the results of Proposition 12.3 hold, with the obvious exceptions. In particular, if S converges absolutely, then S converges.*

For $V = \mathbb{R}^k$, write $a_j = (a_j(1), \dots, a_j(k))$, where $a_j(\ell)$ denotes the ℓ-th entry of a_j. In this case the series S converges if and only if for each ℓ the series $\sum a_j(\ell)$ converges. Moreover, in this case,

$$\sum a_j = \left(\sum a_j(1) \quad \dots \quad \sum a_j(k)\right).$$

†

13.1. **Complex Numbers.** This section contains a review of the basic facts about the field of complex numbers. Recall that $\mathbb{C}$, the complex plane, is, as a (real) vector space, $\mathbb{R}^2$. A point $(a, b) \in \mathbb{R}^2$ is identified with the complex number $a + \imath b \in \mathbb{C}$ and the product of $z = a + \imath b$ and $w = u + \imath v$ is

$$zw = (au - bv) + \imath(av + bu).$$

Thus, $\imath^2 = -1$.

Given $z = a+\imath b \neq 0$, it is natural to write $z = r(\frac{a}{r}+\imath\frac{b}{r})$, where $r = \sqrt{a^2+b^2}$. The point $(\frac{a}{r},\frac{b}{r})$ lies on the unit circle and thus has the form $(\cos(t),\sin(t))$. Hence, the complex number z can be written as

$$z = r(\cos(t) + \imath\sin(t)). \tag{19}$$

The number $r = \sqrt{a^2+b^2}$ is the *modulus* of z and is denoted $|z|$; the number t is the *argument* of z; and the representation (19) is the *polar decomposition* of z.

If $z = a + \imath b$ is a complex number, we sometimes write $a = \Re z$ and $b = \Im z$; these are called the *real part* and *imaginary part* of z, respectively. The *complex conjugate* of $z = a + ib$ is defined to be $\overline{z} := a - \imath b$. Notice that $\Re z$ and $\Im z$ can be recovered from z and $\overline{z}$ by the formulas

$$\Re z = \frac{z+\overline{z}}{2}, \qquad \Im z = \frac{z-\overline{z}}{2\imath}.$$

The useful formula

$$z\overline{z} = |z|^2$$

is readily verified.

Given two complex numbers $z = |z|(\cos(t) + i\sin(t))$ and $w = |w|(\cos(s) + i\sin(s))$ a routine calculation using angle sum formulas for sine and cosine shows,

$$zw = |z|\,|w|(\cos(s+t) + \imath\sin(s+t)).$$

Thus complex multiplication can be interpreted geometrically in terms of the product of the moduli and sum of the arguments.

A function $f : X \to \mathbb{C}$ can be expressed in terms of its real and imaginary parts as $f = u + \imath v$, where $u, v : X \to \mathbb{R}$. The pointwise complex conjugate of f, denoted $\overline{f}$, is given by $\overline{f} = u - \imath v$.

13.2. **Power Series and Complex Numbers.** Since, as a real vector space, $\mathbb{C}$ it is nothing more than $\mathbb{R}^2$, the discussion at the outset of this section applies.

Let (a_n) be a sequence of complex numbers. The expression, for complex numbers z,

$$\sum_{j=0}^{\infty} a_j z^j, \tag{20}$$

is the complex version of a power series.

Theorem 13.3. *Either the series of Equation (20) converges absolutely for every complex number z or there is a real number R such that if $|z| < R$ then the series converges and if $|z| > R$, then the series diverges.*

In the case that the series converges for all z, its *radius of convergence* is ∞ and otherwise R of the theorem is the radius of convergence.

Example 13.4. The power series

$$\sum_{j=0}^{\infty} z^n$$

has radius of convergence 1 and for $|z| < 1$ converges to $\frac{1}{1-z}$. For $|z| \geq 1$ the series diverges.

The power series

$$\sum_{j=0}^{\infty} \frac{z^n}{n}$$

also has radius of convergence 1 and evidently diverges if $z = 1$. In particular, the series does not converge absolutely for $|z| \geq 1$. On the other hand, a generalization of the alternating series test (which will not be discussed) can be used to show that if $|z| = 1$, but $z \neq 1$, then the series converges (conditionally).

The power series,

$$E(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

converges for all z and thus has an infinite radius of convergence. It defines a function $E : \mathbb{C} \to \mathbb{C}$. Note that, in view of Example 12.24, $E(x) = \exp(x)$ for $x \in \mathbb{R}$. Accordingly E is the *complex exponential* and will be denoted by exp or e^z. Thus,

$$E(z) = \exp(z) = e^z.$$

△

Proposition 13.5. *$E(z + w) = E(z)E(w)$ for $z, w \in \mathbb{C}$.* †

The proof of the proposition uses the following lemma which is of independent interest.

Lemma 13.6. *Suppose $\sum a_j$ and $\sum b_j$ converge to A and B respectively and let*

$$c_m = \sum_{j=0}^{m} a_j b_{m-j}.$$

(i) If $\sum a_j$ converge absolutely, then $\sum c_m$ converges to AB.
(ii) If both $\sum a_j$ and $\sum b_j$ converge absolutely, then so does $\sum c_m$.

†

Proof. Let t_k denote the partial sums of the series $\sum_{j=0}^{\infty} b_j$ and observe,

$$\sum_{m=0}^{n} c_m = \sum_{j=0}^{n} a_j t_{n-j} = B \sum_{j=0}^{n} a_j + \sum_{j=0}^{n} a_j (t_{n-j} - B).$$

To complete the proof, it suffices to show that the last term on the right hand side above converges to 0. To this end, let $\epsilon > 0$ be given. There is an N so that $|B - t_k| < \epsilon$ for $k \geq N$ because (t_k) converges to B and at the same time

$$\sum_{j=N}^{\infty} |a_j| \leq \epsilon,$$

since the series $\sum a_j$ is assumed to converge absolutely. Since $|B - t_j|$ converges, there is an M such that $|B - t_j| \leq M$ for all j. For $n \geq 2N$,

$$\begin{aligned}
|\sum_{j=0}^{n} a_{n-j}(B - t_j)| \leq & |\sum_{j=0}^{n-N} a_{n-j}(B - t_j)| + |\sum_{j=n-N+1}^{n} a_{n-j}(B - t_j)| \\
\leq & \sum_{j=N}^{n} M|a_j| + (\sum_{j=0}^{\infty} |a_j|)\epsilon \\
\leq & \epsilon(M + \sum |a_j|).
\end{aligned}$$

□

Proof of Proposition 13.5. From the Lemma,

$$\begin{aligned}
E(z)E(w) &= \sum_{m=0}^{\infty} (\sum_{j=0}^{m} \frac{z^j w^{m-j}}{j!(m-j)!}) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} z^j w^{m-j} \\
&= \sum_{m=0}^{\infty} \frac{(z+w)^m}{m!} = E(z+w).
\end{aligned}$$

□

Remark 13.7. The polar decomposition of Equation (19) can be expressed in terms of the exponential function using the formula,

$$e^{it} = \exp(it) = \cos(t) + \imath \sin(t),$$

valid for t real. See Problem 13.3. ◇

13.3. **Dirichlet Series.** For positive integers n and complex numbers s, let

$$n^{-s} = \exp(-s \log(n)).$$

For n fixed, n^{-s} is thus defined for all $s \in \mathbb{C}$. Note that

$$|n^{-s}| = n^{-\Re s}.$$

Given a sequence $a = (a_n)_{n=1}^{\infty}$ of complex numbers, the expression

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{21}$$

is a *Dirichlet series*. Of course, this series determines a function with domain equal D_a, the set of those $s \in \mathbb{C}$ for which the series converges. Writing $s = \sigma + \imath t$ in terms of its real and imaginary parts, the series converges absolutely if and only if,

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma}$$

converges.

The following proposition collects some elementary facts about convergence of Dirichlet series. The proof are left as an Exercise for the gentle reader. See Problem 13.4.

Proposition 13.8. *Suppose $C > 0$ and σ_0 is real. If $|a_n| n^{-\sigma_0} \le C$ for all n, then the Dirichlet series $f(s)$ of Equation (21) converges absolutely for $\sigma = \Re s > \sigma_0 + 1$.*

If the series $f(s)$ converges absolutely at $s_0 = \sigma_0 + \imath t_0$, then the series converges absolutely for every s with $\Re s \ge \sigma_0$. Further, in this case, the series converges uniformly on $\{s \in \mathbb{C} : \Re s \ge \sigma_0\}$.

Either the series $f(s)$ converges absolutely for all s; fails to converge for all s; or there is a real number σ_a such that the series converges absolutely for $\Re s > \sigma_a$ and does not converge absolutely for $\Re s < \sigma_a$. †

Definition 13.9. Interpreting σ_a as either $\pm\infty$ if needed, the number σ_a is the *abscissa of absolute convergence* of the Dirichlet series $f(s)$. ◁

Example 13.10. For $a_n = 1$, the resulting series is known as the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

It converges absolutely for $\sigma = \Re s > 1$. It turns out that it diverges if $\sigma \le 1$. For $\sigma < 1$ the divergence follows from Theorem 13.12. The case $\sigma = 1$ will not be dealt with here. △

Example 13.11. The Dirichlet series

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^n n^{-s}$$

converges absolutely for $\sigma > 1$, conditionally for $0 < \sigma \le 1$ and diverges otherwise. Here, as usual, $s = \sigma + \imath t$. Thus $\eta(s)$ determines a function, known as the Dirichlet η (or alternating ζ) function with domain $\{s \in \mathbb{C} : \Re s > 0\}$. △

Theorem 13.12. *The Dirichlet series of Equation (21) either converges for all s; converges for no s; or there is a real number σ_c such that the series converges for $\Re s > \sigma_c$ and diverges for $\Re s < \sigma_c$. Moreover, $\sigma_c + 1 \ge \sigma_a$.*

Definition 13.13. Again, allowing $\sigma_c = \pm\infty$ if needed, σ_c is the *abscissa of (simple) convergence* of the Dirichlet series $f(s)$. ◁

Remark 13.14. Evidently $\sigma_c \le \sigma_a$ The inequality $\sigma_a \le \sigma_c + 1$ follows immediately from Proposition 13.8. The examples of the zeta and eta functions show that the inequalities are the best possible. ◇

The proof of Theorem 13.12 requires a couple of Lemmas, the first of which is known, for obvious reasons, as summation by parts.

Lemma 13.15 (Summation by Parts)**.** *Given complex numbers* $a_1, \dots, a_n; b_1, \dots, b_n$*, let* $B_k = \sum_{j=1}^{k} b_j$*. For* $m \geq 2$*,*

$$\sum_{j=m}^{n} a_j b_j = a_n B_n - a_m B_{m-1} - \sum_{j=m}^{n-1} B_j(a_{j+1} - a_j).$$

†

Proof. Observe,

$$\begin{aligned}\sum_{j=m}^{n} a_j b_j =& a_m b_m + \sum_{j=m+1}^{n} a_j(B_j - B_{j-1}) \\ =& a_m b_m + \sum_{j=m+1}^{n} a_j B_j - \sum_{j=m}^{n-1} a_{j+1} B_j \\ =& a_m b_m + a_n B_n - a_{m+1} B_m - \sum_{j=m+1}^{n-1} (a_{j+1} - a_j) B_j \\ =& a_n B_n - a_m B_{m-1} - (a_{m+1} - a_m) B_m - \sum_{j=m+1}^{n-1} (a_{j+1} - a_j) B_j \\ =& a_n B_n - a_m B_{m-1} - \sum_{j=m}^{n-1} (a_{j+1} - a_j) B_j\end{aligned}$$

□

Lemma 13.16. *Suppose* $s_0 = \sigma_0 + \imath t_0$ *and there exists a* C *such that for all* $N \in \mathbb{N}^+$*,*

$$|\sum_{n=1}^{N} a_n n^{-s_0}| \leq C.$$

If $s = \sigma + \imath t$ *and* $\sigma > \sigma_0$*, then for all* m *and* N*,*

$$|\sum_{n=m}^{N} a_n n^{-s_0}| \leq 4Cm^{\sigma_0 - \sigma}.$$

†

Proof. Applying Lemma 13.15 (summation by parts) to $n^{\sigma_0 - \sigma}$ and $a_n n^{\sigma_0}$ gives,

$$\sum_{n=m}^{N} a_n n^{\sigma} n^{\sigma_0 - \sigma} = N^{\sigma_0 - \sigma} B_N - m^{\sigma_0 - \sigma} B_{m-1} - \sum_{n=m}^{N-1} ((n+1)^{\sigma_0 - \sigma} - n^{\sigma_0 - \sigma}) B_n,$$

where

$$B_n = \sum_{j=1}^{n} a_n n^{\sigma_0}.$$

From the hypothesis, (B_n) is bounded by C. Hence,

$$|\sum_{n=m}^{N} a_n n^{\sigma} n^{\sigma_0-\sigma}| \leq C[N^{\sigma_0-\sigma} + m^{\sigma_0-\sigma} + \sum_{n=m}^{N-1} |(n+1)^{\sigma_0-\sigma} - n^{\sigma_0-\sigma}|].$$

Observe by its telescoping nature,

$$\begin{aligned}\sum_{n=m}^{N-1} |(n+1)^{\sigma_0-\sigma} - n^{\sigma_0-\sigma}| =& m^{\sigma_0-\sigma} - N^{\sigma_0-\sigma} \\ =& m^{\sigma_0-\sigma}|(\frac{N}{m})^{\sigma_0-\sigma} - 1| \leq 2m^{\sigma_0-\sigma}.\end{aligned}$$

Since also $N^{\sigma_0-\sigma} \leq m^{\sigma_0-\sigma}$, the conclusion of the lemma follows. □

Lemma 13.17. *Suppose the series $f(s)$ of (21) converges at $s_0 = \sigma_0 + \imath t_0$. If $s = \sigma + \imath t$ and $\sigma > \sigma_0$, then the series converges at s.* †

Proof. Here is a sketch of the proof. Let $S_k(s)$ denote the partial sums of $f(s)$. If the series converges at s_0 the partials sums $(S_k(s_0))$ are bounded, say by C. The sequence $(m^{\sigma_0-\sigma})_m$ converges to 0 as $\sigma_0 - \sigma < 0$. It follows that the partial sums of $f(s)$ are Cauchy by Lemma 13.16. Thus the series converges at s. □

Proof of Theorem 13.12. Suppose $f(s)$ converges for some, but not all s. Let

$$\tau = \inf\{\sigma \in \mathbb{R} : \text{ there exists a } t \text{ such that } f(s = \sigma + \imath t) \text{ converges}\}.$$

Given $\sigma > \tau$, there exists a $\sigma > \sigma_0 > \tau$, by the definition of τ as an infimum. There exists a t_0 such that f converges at $s_0 = \sigma_0 + \imath t_0$. By Lemma 13.17, the series converges for every s with $\Re s > \sigma_0$ and for $\sigma + \imath t$.

On the other hand, if $\sigma < \tau$ and $t \in \mathbb{R}$, then $f(\sigma + \imath t)$ doesn't converge by the choice of τ and the proof is complete. □

Returning to complete the Example 13.10 of the zeta function $\zeta(s)$, note that the series diverges for $s = \sigma < 1$, but converges absolutely for $s = \sigma + \imath t$ when $\sigma > 1$. Hence, its abscissa of convergence is $\sigma_c = 1 = \sigma_a$.

13.4. **Problems.**

Problem 13.1. Determine the radius of convergence and the set of $z \in \mathbb{C}$ for which the power series,

$$\sum_{j=0}^{\infty} \frac{z^j}{j^2+1}$$

converges.

Problem 13.2. Fix a positive integer m. Show that the expression, for $z \in \mathbb{C}$,

$$\sum_{j=0}^{\infty} z^{mj}$$

is a power series. Using the result stated (without proof) in Example 13.4, determine the set of z for which this series converges.

Problem 13.3. Prove, for $x \in \mathbb{R}$,

$$\exp(\imath x) + \exp(-\imath x) = 2\cos(x).$$

Find a similar formula for $\sin(x)$.

Conclude,

$$\exp(it) = \cos(t) + \imath \sin(t)$$

and thus the representation of Equation (19) can be written as

$$z = re^{it}.$$

Problem 13.4. Prove Proposition 13.8.

Problem 13.5. Prove the assertion in Remark 12.20.

14. Linear Algebra Review

This section reviews linear algebra in the Euclidean spaces $\mathbb{R}^n$ in preparation for studying the derivative of mappings from one Euclidean space to another. It is assumed that the reader has had a course in linear algebra and is conversant with matrix computations.

14.1. Matrices and Linear Maps Between Euclidean Spaces.

Definition 14.1. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if, for all $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

(i) $T(x + y) = T(x) + T(y)$; and
(ii) $T(cx) = cT(x)$.

In this case it is customary to write Tx instead of $T(x)$. ◁

Example 14.2. Verify that the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x_1, x_2) = (2x_1, x_1 + x_2)$$

is linear.

Verify that the functions $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\begin{aligned} f(x_1, x_2) =& (x_1 x_2, x_1 + x_2) \text{ and;} \\ g(x_1, x_2) =& (2x_1, x_1 + x_2 + 1) \end{aligned}$$

are not linear. △

Definition 14.3. Let $a_1, \dots, a_n \in \mathbb{R}^m$ denote the columns of the $m \times n$ matrix A so that

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

If $x \in \mathbb{R}^n$, then the product of A and x is

$$Ax = \sum x_j a_j \in \mathbb{R}^m.$$

In particular, $a_j = Ae_j$, where $e_j \in \mathbb{R}^n$ is the j-th *standard basis vector*, namely the vector with a 1 in the j-th position and 0 elsewhere.

Given an $m \times n$ matrix A, let $\mathfrak{T}_A$ denote the mapping $\mathfrak{T}_A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$\mathfrak{T}_A x = Ax.$$

◁

Lemma 14.4. If A is an $m \times n$ matrix, then the mapping $\mathfrak{T}_A : \mathbb{R}^n \to \mathbb{R}^m$ is linear. †

Definition 14.5. Given a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$, let $\mathfrak{A}_T$ denote the matrix

$$\mathfrak{A}_T = \begin{pmatrix} Te_1 & Te_2 & \dots Te_n \end{pmatrix}.$$

Thus $\mathfrak{A}_T$ is the $m \times n$ matrix with j-th column Te_j and is called the *matrix representation* of T. ◁

Example 14.6. Compute the matrix representation $\mathfrak{A}_T$ for the linear transformation in Example 14.2. △

The following proposition justifies identifying $m \times n$ matrices with linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proposition 14.7. *(i) If A is an $m \times n$ matrix, then $\mathfrak{A}_{\mathfrak{T}_A} = A$.*
(ii) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $\mathfrak{T}_{\mathfrak{A}_T} = T$.

†

If $S, T : \mathbb{R}^n \to \mathbb{R}^m$ are linear maps and $c \in \mathbb{R}$, then $cS + T : \mathbb{R}^n \to \mathbb{R}^m$ is naturally defined by

$$(cS + T)x = cSx + Tx.$$

If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are both linear, then the composition $S \circ T$ is written ST, notation justified by the following proposition.

Proposition 14.8. *The correspondence between matrices and linear maps enjoys the following properties.*

(i) If A and B are $m \times n$ matrices and $c \in \mathbb{R}$, then

$$\mathfrak{T}_{cA+B} = c\mathfrak{T}_A + \mathfrak{T}_B.$$

(ii) If $S, T : \mathbb{R}^n \to \mathbb{R}^m$ are linear and $c \in \mathbb{R}$, then $cS + T$ is linear and moreover,

$$\mathfrak{A}_{cS+T} = c\mathfrak{A}_S + \mathfrak{A}_T.$$

(iii) If A and B are $m \times n$ and $p \times m$ matrices respectively, then

$$\mathfrak{T}_{BA} = \mathfrak{T}_B\mathfrak{T}_A.$$

(iv) If $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ are linear, then so is ST. Moreover,

$$\mathfrak{A}_{ST} = \mathfrak{A}_S\mathfrak{A}_T.$$

(v) If $m = n$ and T is invertible, then its inverse, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is also linear, the matrix $\mathfrak{A}_T$ is invertible, and

$$\mathfrak{A}_T^{-1} = \mathfrak{A}_{T^{-1}}.$$

(vi) Likewise, if A is an invertible $n \times n$ matrix, then $\mathfrak{T}_A$ is invertible and

$$\mathfrak{T}_A^{-1} = \mathfrak{T}_{A^{-1}}.$$

†

Proof. The first two items are left to the gentle reader as Problem 14.1. The arguments are similar to those of item (iii) and (iv) to follow.

To prove item (iii), let A and B as in the statement of the proposition be given and note that, for $x \in \mathbb{R}^n$,

$$\mathfrak{T}_{BA}x = BAx = B\mathfrak{T}_Ax = \mathfrak{T}_B(\mathfrak{T}_Ax) = \mathfrak{T}_B\mathfrak{T}_Ax.$$

For item (iv), given S, T, observe by item (iii) and Proposition 14.7(ii),

$$\mathfrak{T}_{\mathfrak{A}_S\mathfrak{A}_T} = \mathfrak{T}_{\mathfrak{A}_S}\mathfrak{T}_{\mathfrak{A}_T} = ST.$$

In particular, ST is linear. Moreover, applying Proposition 14.7(i),

$$\mathfrak{A}_S\mathfrak{A}_T = \mathfrak{A}_{\mathfrak{T}_{\mathfrak{A}_S\mathfrak{A}_T}} = \mathfrak{A}_{ST}$$

The remainder of the proof - items (v) and (vi) - is left to the gentle reader as Problem 14.2. □

Now that matrices and linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ have been identified, often speak of a matrix as a linear map and conversely. Let I_n denote the $n \times n$ identity matrix. The linear map it induces is of course the identity mapping $\mathfrak{A}_{I_n}x = x$. Occasionally, we will write I in place of I_n when the size n is apparent from the context.

Proposition 14.9. *Suppose A is an $m \times n$ matrix. If $m > n$, then A is not onto. If $m < n$, then A is not one-one.*

The matrix A is one-one if and only if $Ax = 0$ implies $x = 0$.

For an $n \times n$ matrix A, the following are equivalent.

(i) A is invertible;
(ii) A is one-one;
(iii) A is onto;
(iv) there exists an $n \times n$ matrix B such that $BA = I_n$ (and in this case $B = A^{-1}$);
(v) there exists an $n \times n$ matrix C such that $AC = I_n$ (and in this case $C = A^{-1}$);
(vi) $\det(A) \neq 0$;

†

Example 14.10. Let T denote the linear transformation from Example 14.2. Verify that T is one-one and hence invertible. Find T^{-1}. △

14.2. **Norms on $\mathbb{R}^n$.** Let $\|\cdot\|_2$ denote the usual Euclidean norm on $\mathbb{R}^n$. Thus, for $x = (x_1, \dots, x_n)$,

$$\|x\|_2^2 = \sum_{j=1}^n x_j^2.$$

In the usual way $(\mathbb{R}^n, d_2)$ is a metric space, where $d_2(x, y) = \|x - y\|_2$. Let $\{e_1, \dots, e_n\}$ denote the *standard basis* for $\mathbb{R}^n$; i.e., e_j has a 1 in the j-th entry and zeros elsewhere and

$$x = \sum_{j=1}^n x_j e_j.$$

Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$, the *unit sphere* in $\mathbb{R}^n$. Note that it is a compact set (in $(\mathbb{R}^n, d_2)$).

Recall too the ℓ_1 norm on $\mathbb{R}^n$ defined by

$$\|x\|_1 = \sum |x_j|,$$

where $x = \sum_{j=1}^n x_j e_j$ and $\{e_1, \dots, e_n\}$ is the standard orthonormal basis for $\mathbb{R}^n$. Let d_1 denote the resulting metric. In particular we have,

$$\|x\|_1 \leq \sqrt{n}\|x\|_2.$$

Definition 14.11. *Two norms $\|\cdot\|$ and $\|\cdot\|_*$ on $\mathbb{R}^n$ are equivalent norms if there exists $0 < c < C$ such that*

$$c\|x\| \leq \|x\|_* \leq C\|x\|$$

for all $x \in \mathbb{R}^n$. ◁

Remark 14.12. The metric properties of equivalent norms are the same; i.e., notions of convergence and continuity are the same. Accordingly, we can freely move between equivalent norms for many purposes of analysis. ◇

If $\|\cdot\|$ is a norm on $\mathbb{R}^n$, let $(\mathbb{R}^n, \|\cdot\|)$ denote both this normed vector space and the associated metric space (with metric $d(x, y) = \|x - y\|$).

Theorem 14.13. *All norms on $\mathbb{R}^n$ are equivalent. In particular, if $\|\cdot\|$ is a norm on $\mathbb{R}^n$, then closed bounded sets in the metric space $(\mathbb{R}^n, \|\cdot\|)$ are compact.*

This theorem depends upon the fact that $\mathbb{R}^n$ is a finite dimensional vector space. There are examples of inequivalent norms on infinite dimensional vector spaces (see Problem 14.6).

Proof. The strategy is to show that every norm $\|\cdot\|$ on $\mathbb{R}^n$ is equivalent to $\|\cdot\|_2$. To this end, first observe that, writing $x = \sum_{j=1}^n x_j e_j$,

$$\|x\| \leq \sum_{j=1}^n |x_j|\,\|e_j\| \leq M\|x\|_1 \leq \sqrt{n}M\|x\|_2,$$

where $M = \max\{\|e_1\|, \dots, \|e_n\|\}$. It follows that, with d denoting the metric determined by $\|\cdot\|$,

$$d(x, y) \leq \sqrt{n}M d_2(x, y)$$

for all $x, y \in \mathbb{R}^n$. Hence the function $F : \mathbb{R}^n \mapsto \mathbb{R}$ defined by $F(x) = \|x\|$ is (uniformly) continuous with respect to the Euclidean distance function $d_2(x, y) = \|x - y\|_2$ on $\mathbb{R}^n$. Since $G = F|_{S^{n-1}}$ is a continuous function on a compact set, it achieves its minimum c. This minimum is not 0 since G does not take the value 0 (norms are positive definite and $0 \notin S^{n-1}$). Now, for $x \in S^{n-1}$,

$$c \leq \|x\| = F(x).$$

If $y \neq 0$, then $x = \frac{y}{\|y\|_2} \in S^{n-1}$ and properties of norms,

$$c\,\|y\|_2 \leq \|y\|$$

and the proof is complete. □

Proposition 14.14. *Suppose $\|\cdot\|$ and $\|\cdot\|_*$ are norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then* *(i) there is a $C > 0$ such that*

$$\|Tx\|_* \le C\|x\|; \quad \textit{and}$$

(ii) T is uniformly continuous as a mapping from $(\mathbb{R}^n, \|\cdot\|) \to (\mathbb{R}^m, \|\cdot\|_*)$;

Finally,

$$\inf\{C' : \|Tx\|_* \le C'\|x\| \text{ for all } x \in \mathbb{R}^n\} = \sup\{\|Tx\|_* : x \in \mathbb{R}^n, \quad \|x\| = 1\}$$

and both the infimum and supremum are attained. †

Note that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible, then T^{-1} is also linear and hence continuous (with respect to any two norms on $\mathbb{R}^n$).

Proof. Let $\|\cdot\|_1$ denote the ℓ_1 norm on $\mathbb{R}^n$. It is equivalent to the norm $\|\cdot\|$. Hence there exists a K such that, for all $x \in \mathbb{R}^n$,

$$\|x\|_1 \le K\,\|x\|$$

Let $L = \max\{\|Te_1\|_*, \dots, \|Te_n\|_* : 1 \le j \le n\}$. Writing $x = \sum_{j=1}^n x_j e_j$ in the usual way, estimate,

$$\|Tx\|_* = \|\sum_{j=1}^n x_j T(e_j)\|_* \le \sum_{j=1}^n |x_j|\,\|Te_j\| \le L\|x\|_1 \le LK\,\|x\|,$$

proving (i) with $C = LK$.

Item (ii) follows immediately from item (i).

To prove the final item, let $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$. From Theorem 14.13, S is compact. The mapping $G : S \to \mathbb{R}$ defined by $G(x) = \|Tx\|_*$ is continuous (as a mapping from $(\mathbb{R}^n, \|\cdot\|)$ to $\mathbb{R}$) by what has already been proved. Hence G attains its supremum, say M; i.e., there is a point $x_0 \in S$ such that $\|Tx_0\|_* = M$ and if $x \in S$, then $\|Tx\|_* \le M$ and, by properties of norms it readily follows that $\|Tx\|_* \le M\,\|x\|$ for all $x \in \mathbb{R}^n$. Thus the infimum in the statement of the proposition is at most M which is also the supremum in the statement. On the other hand, if $\|Tx\|_* \le C\|x\|$ for all x, then $C \ge M = \|Tx_0\|_*$ and the proof is complete. □

14.3. **The Vector Space of $m \times n$ Matrices.** Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Propositions 14.7 and 14.8 describe the canonical identification of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ with the set $M_{m,n}$ of $m \times n$ matrices. Both $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $M_{m,n}$ are vector spaces and, as vector spaces, can identified with R^{mn}. In particular, $M_{m,n}$ can be given the Euclidean norm by defining, for $X = (x_{j,k}) \in M_{m,n}$,

$$\|X\|_2^2 = \sum_{j,k} |x_{j,k}|^2,$$

which is often called the *Frobenius norm*. However, there is another norm on $M_{m,n}$ (of course equivalent to the Frobenius norm by Theorem 14.13) which, for many purposes, is more natural - and easier to work with. Given an $m \times n$ matrix T, (equivalently a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$) by Proposition 14.14, there is a C such that

$$\|Tx\|_2 \le C\|x\|_2$$

for all $x \in \mathbb{R}^n$.

Proposition 14.15. *The mapping $\|\cdot\| : M_{m,n} \to \mathbb{R}$ defined by*

$$\|T\| = \inf\{C : \|Tx\|_2 \le C\|x\|_2 \text{ for all } x \in \mathbb{R}^n\}$$

is a norm on $M_{m,n}$. †

Note that by Proposition 14.14, $\|T\| = \sup\{\|Tx\|_2 : x \in \mathbb{R}^n,\ \|x\|_2 = 1\}$. This norm is called the *operator norm* (or sometimes the *matrix norm*) and is also denoted by $\|T\|_{\text{op}}$. That $\|T\|_{\text{op}}$ defines defines a norm is left to the gentle reader as Problem 14.5. The following Proposition collects some immediate properties of the operator norm.

Proposition 14.16. *Let T be an $m \times n$ matrix.*

(i) The norm of an $m \times n$ matrix T is also given by

$$\|T\| = \max\{\|Tx\|_2 : \|x\|_2 = 1\}.$$

(ii) If $y \in \mathbb{R}^n$, then,

$$\|Ty\|_2 \le \|T\|\,\|y\|_2.$$

(iii) If $C > 0$ and

$$\|Ty\|_2 \le C\|y\|_2$$

for all $y \in \mathbb{R}^n$, then $\|T\| \le C$.

(iv) If S is an $n \times p$ matrix, then $\|TS\| \le \|T\|\,\|S\|$.

†

14.4. **The Set of Invertible Matrices.** This section closes by reviewing some basic facts about inverse of matrices (equivalently linear transformations on Euclidean space). Throughout, unless otherwise indicated, $\|\cdot\|$ stands for the Euclidean norm $\|\cdot\|_2$ or the operator norm $\|\cdot\|_{\text{op}}$ as context dictates.

Given $n \in \mathbb{N}^+$, recall $I = I_n$ denotes the $n \times n$ identity matrix and $\mathfrak{T}_I$ is the identity mapping, $\mathfrak{T}_I x = x$.

Lemma 14.17. *If A is an $n \times n$ matrix and $\|A\| < 1$, then $I - A$ is invertible. Moreover,*

$$\|(I-A)^{-1}\| \le \frac{1}{1 - \|A\|}.$$

†

Proof. Observe, for $x \in \mathbb{R}^n$, that

$$\|(I-A)x\| \ge \|x\| - \|Ax\| \ge \|x\| - \|A\|\|x\| = \|x\|(1 - \|A\|).$$

In particular, if $x \ne 0$, then $(I-A)x \ne 0$ and thus, by Proposition 14.9 (the equivalence of items (i) and (ii)), $I - A$ is invertible.

Given $x \in \mathbb{R}^n$, note that

$$\|x\| = \|(I-A)(I-A)^{-1}x\| \geq \|(I-A)^{-1}x\|(1-\|A\|).$$

Thus,

$$\|(I-A)^{-1}x\| \leq \frac{1}{1-\|A\|}\|x\|.$$

Hence $\|(I-A)^{-1}\| \leq (1-\|A\|)^{-1}$ by Proposition 14.16(iii). □

Proposition 14.18. *Fix n. The set $\mathcal{I}_n$ of invertible $n \times n$ matrices is an open subset of M_n and the mapping $F : \mathcal{I}_n \to \mathcal{I}_n$*

$$F(A) = A^{-1}$$

is continuous. †

Proof. Fix $A \in \mathcal{I}_n$. Choose $\eta = \frac{1}{2\|A^{-1}\|}$ and suppose $\|H\| < \eta$. In this case,

$$\|-A^{-1}H\| \leq \|A^{-1}\|\,\|H\| < \frac{1}{2}$$

and hence $I + A^{-1}H$ is invertible by Lemma 14.17. Consequently,

$$A + H = A(I + A^{-1}H)$$

is invertible, proving that the η-neighborhood of A lies in $\mathcal{I}_n$ (since if B is in this η neighborhood, then $H = B - A$ has (operator) norm at most η). Lemma 14.17 also gives

$$\|(A+H)^{-1}\| \leq \|A^{-1}\|\frac{1}{1-\|A^{-1}H\|} \leq 2\|A^{-1}\|.$$

To see that F is continuous, again suppose $\|H\| < \eta$ and note

$$\begin{aligned}\|F(A+H) - F(A)\| =&\|(A+H)^{-1}[A-(A+H)]A^{-1}\| \\ \leq&\|A+H\|^{-1}\,\|H\|\,\|A^{-1}\| \\ \leq&2\|A^{-1}\|^2\,\|H\|.\end{aligned}$$

To complete the proof, given $\epsilon > 0$, choose $0 < \delta \leq \eta$ and such that $\delta < \frac{\epsilon}{2\|A^{-1}\|}$. □

14.5. **Exercises.**

Exercise 14.1. Show, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $T0 = 0$. (Here the first 0 is the zero vector in $\mathbb{R}^n$ and the second is the zero vector in $\mathbb{R}^m$.)

Exercise 14.2. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x_1, x_2, x_3) = (2x_1 - x_2 + 3x_3, x_3 - x_1).$$

Find a matrix A such that $T = \mathfrak{T}_A$ and explain how doing so shows that T is linear and $A = \mathfrak{A}_T$.

Exercise 14.3. Prove that the relation of equivalence of norms is an equivalence relation (symmetric, reflexive, and transitive).

Exercise 14.4. Suppose D is an $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$. Show $\|D\| = \max\{|\lambda_j| : 1 \le j \le n\}$ (the operator norm).

Show D is invertible if and only if all the diagonal entries are different from 0.

Exercise 14.5. Given n and $y_1, \dots, y_n$, the (row) $1 \times n$ matrix

$$Y = \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}$$

is identified with the corresponding linear map $Y : \mathbb{R}^n \to \mathbb{R}$. Find $\|Y\|$ (the operator norm).

Exercise 14.6. Fix $c > 0$ and let

$$S = c \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Show $\|S\| = c$. Show also that $I - S$ is invertible (even if $c \ge 1$) and compare with Lemma 14.17

14.6. **Problems.**

Problem 14.1. If $T, S : \mathbb{R}^n \to \mathbb{R}^m$, then $T + S : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $(T + S)x = Tx + Sx$ as one would expect. Show, if S and T are linear, then $T + S$ is linear and

$$\mathfrak{A}_{T+S} = \mathfrak{A}_T + \mathfrak{A}_S.$$

Show further, if $c \in \mathbb{R}$, then $cT : \mathbb{R}^n \to \mathbb{R}^m$ defined by $cT(x) = c(Tx)$ (and denoted simply cTx) is a linear map and further

$$\mathfrak{A}_{cT} = c\mathfrak{A}_T.$$

In the other direction, show, if A and B are $m \times n$ matrices, then

$$\mathfrak{T}_{A+B} = \mathfrak{T}_A + \mathfrak{T}_B$$

and also

$$\mathfrak{T}_{cA} = c\mathfrak{T}_A.$$

Problem 14.2. Complete the proof of Proposition 14.8.

Problem 14.3. Show if $\|\cdot\|$ is a norm on $\mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and one-one, then the function $\|\cdot\|_* : \mathbb{R}^n \to \mathbb{R}$ given by

$$\|x\|_* = \|Tx\|$$

is a norm on $\mathbb{R}^n$.

Problem 14.4. Suppose A and B are $n \times n$ and $m \times m$ matrices respectively and that C is an $n \times m$ matrix. Let

$$X = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

Prove X is invertible if and only if both A and B are.

Problem 14.5. Prove Proposition 14.15.

Problem 14.6. Consider the vector space $C([0,1])$ with the norms,

$$\|f\|_2^2 = \int_0^1 |f|^2 dt$$

and

$$\|f\|_\infty = \max\{|f(t)| : 0 \le t \le 1\}.$$

Let $f_n(t) = t^n$ (defined on $[0,1]$) and show that

$$\lim_{n\to\infty} \|f_n\|_2 = 0,$$

whereas $\|f_n\|_\infty = 1$ for all n. Conclude the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are not equivalent.

Problem 14.7. Complete the outline of the following alternate proof of Lemma 14.17. Suppose A is an $n \times n$ matrix and $\|A\| < 1$. Show that the series,

$$\sum_{j=0}^{\infty} A^j$$

is Cauchy in $M_{n,n}$ and hence converges to some T in $M_{n,n}$ and moreover,

$$\|T\| \le \sum_{j=0}^{\infty} \|A\|^j = \frac{1}{1 - \|A\|}.$$

Show,

$$(I - A)\sum_{j=0}^{n} A^j = I - A^{n+1}$$

and the right hand side converges to 0 with n. Conclude $AT = I$.

15. Derivatives of Mappings Between Euclidean Spaces

The calculus of functions $f : U \to \mathbb{R}^m$, where U is an open set in $\mathbb{R}^n$, is the topic of this section. Here $\mathbb{R}^n$ denotes n-dimensional Euclidean space. Thus $\mathbb{R}^n$ is the space of (column) n-vectors, $x = (x_1, \dots, x_n)^T$, (here ^{T} denotes *transpose*) with real entries, the usual pointwise operations and the standard Euclidean norm,

$$\|x\|^2 = \sum_{j=1}^{n} x_j^2.$$

(Note this notation differs slightly from that in some earlier sections where $\|\cdot\|_2$ was used for the Euclidean norm.) Of course, by Theorem 14.13 we could work with any pair of norms on $\mathbb{R}^n$ and $\mathbb{R}^m$.

The derivative is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ and Subsection 14.1 reviewed the connection between matrices and linear maps between Euclidean spaces. The definition and basic examples of derivatives are given in Subsection 15.1. Properties of the derivative appear in Subsection 15.2. Directional derivatives and the connections between partial derivatives and the derivative are detailed in Subsections 15.3 and 15.4 respectively.

15.1. The Derivative: Definition and Examples.

A linear map $T : \mathbb{R} \to \mathbb{R}$ can be identified with the real number $t = T1$ and conversely. Indeed, $Th = hT1 = th$; i.e., T corresponds to the 1×1 matrix $[T1]$.

By definition, $f : (a, b) \to \mathbb{R}$ is differentiable at $a < c < b$ if there is a number t so that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = t.$$

Rewriting, f is differentiable at c if and only if there is a t such that

$$\lim_{h \to 0} \frac{|f(c+h) - f(c) - th|}{|h|} = 0.$$

Note that $U \subseteq \mathbb{R}^n$ is open if and only if for each $c \in U$ there is an $\eta > 0$ such that if $h \in \mathbb{R}^n$ and $\|h\| < \eta$, then $c + h \in U$. In particular,

$$N_\eta(c) = N_\eta(0) + c := \{h + c : \|h\| < \eta\}.$$

Definition 15.1. Suppose U is an open subset of $\mathbb{R}^n$, $c \in U$ and $f : U \to \mathbb{R}^m$. The function f is *differentiable at* c if there is a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{\|f(c+h) - f(c) - Th\|}{\|h\|} = 0. \tag{22}$$

If f is differentiable at each $c \in U$, then f is *differentiable* ◁

Proposition 15.2. *Suppose U is an open subset of $\mathbb{R}^n$, $c \in U$, and $f : U \to \mathbb{R}^m$. If $S, T : \mathbb{R}^n \to \mathbb{R}^m$ are linear maps and both*

$$\lim_{h \to 0} \frac{\|f(c+h) - f(c) - Th\|}{\|h\|} = 0 = \lim_{h \to 0} \frac{\|f(c+h) - f(c) - Sh\|}{\|h\|},$$

then $S = T$. †

Proof. Given $\epsilon > 0$ there is a δ such that if $0 < \|h\| < \delta$, then $c + h \in U$ and both

$$\begin{aligned}\|f(c+h) - f(c) - Th\| &< \epsilon\|h\| \\ \|f(c+h) - f(c) - Sh\| &< \epsilon\|h\|.\end{aligned}$$

Hence,

$$\|Th - Sh\| \leq \|f(c+h) - f(c) - Sh\| + \|f(c+h) - f(c) - Th\| < 2\epsilon\|h\|.$$

Now suppose $k \in \mathbb{R}^n$ is given. For t real and $|t| < \frac{\delta}{\|k\|+1}$, the vector $h = tk$ satisfies $\|h\| < \delta$ and thus,

$$\|T(tk) - S(tk)\| = |t|\,\|Tk - Sk\| < 2\epsilon|t|\|k\|.$$

Since $\epsilon > 0$ is arbitrary, it follows that $Tk = Sk$. □

Definition 15.3. If f is differentiable at c, then the unique linear map T satisfying (22) is the *derivative of f at c*, written

$$f'(c) = Df(c) = T.$$

◁

Example 15.4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (x_1^2, x_1 x_2)$. With $c = (x_1, x_2)$ fixed, let X denote the matrix,

$$X = \begin{pmatrix} 2x_1 & 0 \\ x_2 & x_1 \end{pmatrix}.$$

Given a point $c = (c_1, c_2) \in \mathbb{R}^2$ (the domain of f) and a vector $h = (h_1, h_2) \in \mathbb{R}^2$,

$$\|f(c_1 + h_1, c_2 + h_2) - f(c_1, c_2) - Xh\| = \|(h_1^2, h_1 h_2)\| \leq |h_1|\|h\|.$$

It follows that f is differentiable at (c_1, c_2) and X (really the linear map $\mathfrak{T}_X$ it determines) is $Df(c_1, c_2)$. △

Do Exercise 15.1.

15.2. Properties of the Derivative.

Proposition 15.5. *Suppose $U \subseteq \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^m$. If f is differentiable at c, then f is continuous at c.* †

The proof of this proposition is left to the reader as Problem 15.1.

Proposition 15.6 (Chain Rule)**.** *Suppose $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open and that $f : U \to V$ and $g : V \to \mathbb{R}^k$. If f is differentiable at $c \in U$ and g is differentiable at $d = f(c)$, then $\psi = g \circ f$ is differentiable at c and*

$$D\psi(c) = Dg(f(c))Df(c).$$

†

Proof. For notational ease, let $S = Df(c)$ and $T = Dg(f(c))$. Also, for $h \in \mathbb{R}^n$ and $k \in \mathbb{R}^m$ such that $c + h \in U$ and $d + k \in V$, let

$$\begin{aligned}\gamma(k) =& g(d+k) - g(d) - Tk\\ \eta(h) =& f(c+h) - f(c) - Sh\\ \Gamma(h) =& \eta(h) + Sh = f(c+h) - f(c).\end{aligned}$$

With these notations,

$$\begin{aligned}\psi(c+h) - \psi(c) - TSh =& g(f(c+h)) - g(f(c)) - TSh\\ =& g(f(c) + \Gamma(h)) - g(f(c)) - T(\Gamma(h) - \eta(h))\\ =& \gamma(\Gamma(h)) + T\eta(h).\end{aligned}$$

Now let $1 \geq \epsilon > 0$ be given. Since g is differentiable at d, there exists a $\delta > 0$ such that if $\|k\| < \delta$, then $\|\gamma(k)\| < \epsilon\|k\|$. Since f is differentiable at c, there exists a $\mu > 0$ such that if $\|h\| < \mu$, then $\|\eta(h)\| < \epsilon\|h\| \leq \|h\|$ and at the same time

$$\begin{aligned}\|\Gamma(h)\| \leq& \|\eta(h)\| + \|S\|\|h\|\\ \leq& (1 + \|S\|)\|h\| < \delta.\end{aligned}$$

Thus for $\|h\| < \mu$,

$$\begin{aligned}\|\gamma(\Gamma(h)) + T\eta(h)\| \leq& \epsilon\|\Gamma(h)\| + \|T\|\,\|\eta(h)\|\\ \leq& \epsilon(1 + \|S\|)\|h\| + \|T\|\epsilon\|h\|\\ \leq& \epsilon(1 + \|S\| + \|T\|)\|h\|.\end{aligned}$$

Thus,

$$\lim_{h \to 0} \frac{\|\psi(c+h) - \psi(c) - TSh\|}{\|h\|} = 0$$

and the proof is complete. □

As is easy to prove, the derivative is also linear in the sense that if U is an open set in $\mathbb{R}^n$, the point c lies in U, the functions $f, g : U \to \mathbb{R}^m$ are differentiable at c and $r \in \mathbb{R}$, then $rf + g$ is differentiable at c and

$$D(rf + g)(c) = rDf(c) + Dg(c).$$

15.3. **Directional Derivatives.** A subset C of $\mathbb{R}^n$ is *convex* if $a, b \in C$ and $0 \leq t \leq 1$ implies $(1-t)a + tb \in C$. For $c \in \mathbb{R}^n$ and $r > 0$, the neighborhood $N_r(c)$ is evidently convex. In fact, because neighborhoods are open sets, given $a, b \in N_r(c)$, there exists a $\delta > 0$ such that $(1-t)a + tb \in N_r(c)$ for $-\delta < t < 1 + \delta$. Thus, given an open subset U of $\mathbb{R}^n$ a function $f : U \to \mathbb{R}^m$ and a point $c \in U$ and an $r > 0$ such that $N_r(c) \subseteq U$, if $a, b \in N_r(c)$ then, for some $\delta > 0$, we can consider the function $h : (-\delta, 1+\delta) \to U$ defined by $h(t) = (1-t)a + tb = a + t(b-a)$ and thus the composition $g : (-\delta, 1+\delta) \to \mathbb{R}^m$,

$$g(t) = f(h(t)) = f(a + t(b-a)).$$

Definition 15.7. Suppose

(i) U is an open subset of $\mathbb{R}^n$;

(ii) c is in U;
(iii) $u \in \mathbb{R}^n$ is a unit vector;
(iv) $f : U \to \mathbb{R}^m$.

In this case there is a $\delta > 0$ such that $c + tu \in U$ whenever $t \in \mathbb{R}$ and $|t| < \delta$. The *directional derivative of f in the direction u* is the derivative of $g : (-\delta, \delta) \to \mathbb{R}^m$ defined by $g(t) = f(c + tu)$ at 0, if it exists. It is denoted $D_u f(c)$. Thus,

$$D_u f(c) = \lim_{t\to 0} \frac{f(c+tu) - f(c)}{t},$$

if this limit exists. ◁

Remark 15.8. If it exists, $D_u f(c) \in \mathbb{R}^m$. (It is a vector.) ◇

The following simple corollary of the chain rule relates the derivative to directional derivatives.

Corollary 15.9. *Suppose $U \subseteq \mathbb{R}^n$ is open, $c \in U$, $f : U \to \mathbb{R}^m$. For each $h \in \mathbb{R}^n$, there is a $\delta > 0$ such that $c + th \in U$ for $|t| < \delta$. If f is differentiable at c, then the function $g : (-\delta, \delta) \to \mathbb{R}^m$ defined by $g(t) = f(c + th)$ is differentiable at 0 and*

$$g'(0) = Df(c)h.$$

Further, if f is differentiable, then so is g and

$$g'(t) = Df(c + th)h.$$

In particular, if f is differentiable at c and $u \in \mathbb{R}^n$ is a unit vector, then

$$D_u f(c) = Df(c)u \in \mathbb{R}^m.$$

†

Proof. Since U is open and $c \in U$, there is a $\delta > 0$ such that $c + tu \in U$ for $|t| < \delta$. Define $h : (-\delta, \delta) \to \mathbb{R}^n$ by $h(t) = c + tu$. Then h is differentiable at 0 and $h'(0) = u$. Thus, by the chain rule, $g = f \circ h$ is differentiable at 0 and $g'(0) = f'(h(0))h'(0)$. □

It can happen that all directional derivatives exist, even though f is not differentiable at c as the following example shows.

Example 15.10. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0, 0) = 0$ and otherwise

$$f(x, y) = \frac{x^3}{x^2 + y^2}.$$

Then all the directional derivatives of f exist; however, f is not differentiable at 0. To prove this last assertion, suppose $T : \mathbb{R}^2 \to \mathbb{R}$ is a linear and both

$$\lim_{t\to 0} \frac{|f(te_1) - f(0) - T(te_1)|}{|t|} = 0 = \lim_{s\to 0} \frac{|f(se_2) - f(0) - T(se_2)|}{|s|}.$$

In this case $Te_1 = 1$ and $Te_2 = 0$. In particular, as a matrix,

$$T = \begin{pmatrix} 1 & 0 \end{pmatrix} = e_1^T.$$

But then, with $h = (e_1 + e_2)$,

$$\lim_{t\to 0} \frac{|f(th) - f(0) - T(th)|}{\|th\|} = \frac{1}{2\sqrt{2}}.$$

It follows that

$$\lim_{h\to 0} \frac{|f(h) - f(0) - Th|}{\|h\|} \neq 0.$$

Hence f is not differentiable at 0. □

In Problem 15.4 you will show that in fact the composition of f with any differentiable curve, not just curves of the form $g(t) = th$ (for a fixed vector h), is differentiable at 0. △

15.4. **Partial Derivatives and the Derivative.** This subsection begins with the familiar definition of the partial derivative.

Definition 15.11. Suppose U is an open subset of $\mathbb{R}^n$, $c \in U$ and $f : U \to \mathbb{R}^m$. The *partial derivative of* f with respect to x_j at c is the directional derivative of f in the direction e_j at c (where $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{R}^n$) and is denoted by $D_j f(c)$.

In the case that $f : \mathbb{R}^n \to \mathbb{R}$ ($m = 1$), it is customary to write

$$\frac{\partial f}{\partial x_j}$$

instead of $D_j f$. ◁

Proposition 15.12. *If f, as in the definition, is differentiable at c, then the matrix representation of $Df(c)$, in terms of its columns, is*

$$\begin{pmatrix} D_1 f(c) & \cdots & D_n f(c) \end{pmatrix}$$

and moreover,

$$Df(c)_{i,j} = \langle Df(c)e_j, e_i\rangle = \langle D_j f(c), e_i\rangle.$$

Thus, the matrix representation for $Df(c)$ is

$$Df(c) = \left(\tfrac{\partial f_i}{\partial x_j}(c)\right)_{i,j=1}^{m,n}.$$

†

Proof. Given $U \subseteq \mathbb{R}^n$ open and $f : U \to \mathbb{R}^m$, for $1 \le i \le m$, let $f_i : U \to \mathbb{R}$ denote the function $f_i(x) = \langle f(x), e_i\rangle$; i.e.,

$$f(x) = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}.$$

Assuming f is differentiable at $c \in U$, each f_i is differentiable at c since f_i is the composition of the differentiable mapping $E_i : \mathbb{R}^m \to \mathbb{R}$ defined by

$$E_i(y) = \langle y, e_i\rangle = e_i^T y$$

with the mapping f which is differentiable at c. Using $f_i = E_i \circ f$ and the fact that $DE_i(c)h = E_i(h)$ (see Exercise 15.8), the chain rule (Proposition 15.6) and Corollary 15.9 imply, for $1 \le j \le n$,

$$\frac{\partial f_i}{\partial x_j}(c) = D_j f_i(c) = Df_i(c)e_j = E_i(f(c))Df(c)e_j = \langle Df(c)e_j, e_i\rangle.$$

□

Example 15.13. Returning to Example 15.4, $f_1(x_1, x_2) = x_1^2$ and $f_2(x_1, x_2) = x_1 x_2$. Thus,

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a,b) & \frac{\partial f_1}{\partial x_2}(a,b) \\ \frac{\partial f_2}{\partial x_1}(a,b) & \frac{\partial f_2}{\partial x_2}(a,b) \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ b & a \end{pmatrix}.$$

△

Remark 15.14. If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ has partial derivatives at at $c \in U$, then the *gradient* of f is the $1 \times n$ matrix

$$\nabla f(c) = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

In particular, if f is differentiable at c, then the matrix representation of f at c is $\nabla f(c)$. ⋄

Proposition 15.15. *Suppose $U \subseteq \mathbb{R}^n$ is open, $c \in U$ and $f : U \to \mathbb{R}$. If $\eta > 0$, if the neighborhood $N_\eta(c) = \{x \in \mathbb{R}^n : \|x - c\| < \eta\}$ lies in U and if each $\frac{\partial f}{\partial x_j}$ exists on $N_\eta(c)$ and is continuous at c, then f is differentiable at c.* †

Proof. The case $n = 2$ is proved. The details for the general case can easily be filled in by the gentle reader. Since the partials of f exist at c, $\nabla f(c) = \begin{pmatrix} D_1 f(c) & D_2 f(c) \end{pmatrix}$ exists as a vector. From Proposition 15.12, what needs to be proved is that

$$\lim_{h \to 0} \frac{\|f(c+h) - f(c) - \nabla f(c)h\|}{\|h\|} = 0.$$

Let $\epsilon > 0$ be given. By continuity of the partial derivatives of f at c, there is a δ such that $\eta > \delta > 0$ and if $\|k\| < \delta$, then

$$|D_j f(c+k) - D_j f(c)| < \epsilon.$$

Write $c = c_1 e_1 + c_2 e_2$. Let $h = h_1 e_1 + h_2 e_2$ with $\|h\| < \eta$ be given. Observe, that $c + h_1 e_1 \in U$ and

$$f(c+h) - f(c) = f(c+h) - f(c + h_1 e_1) + f(c + h_1 e_1) - f(c). \tag{23}$$

Next, note that for $0 \le s \le 1$ that $c + s h_1 e_1 \in N_\delta(c)$ and $c + h_1 e_1 + s h_2 e_2 \in N_\delta(c)$. Accordingly define $g_j : [0,1] \to \mathbb{R}$ by $g_1(s) = f(c + s h_1 e_1)$ and $g_2(s) = f(c + h_1 e_1 + s h_2 e_2)$. The assumption that the partial derivatives of f exist on $N_\eta(c)$ imply that both g_1 and g_2 are continuous on $[0,1]$ and differentiable on $(0,1)$. Hence, by the mean value theorem there exists points $t_j \in (0,1)$ such that,

$$\begin{aligned} f(c+h) - f(c + h_1 e_1) &= h_2 D_2 f(c + h_1 e_1 + t_2 h_2 e_2) \\ f(c + h_1 e_1) - f(c) &= h_1 D_1 f(c + t_1 h_1 e_1). \end{aligned} \tag{24}$$

Thus, if $\|h\| < \delta$, then, combining Equations (23) and (24),

$$\begin{aligned}
&|f(c+h) - f(c) - \nabla f(c)h| \\
&\quad \leq |f(c+h) - f(c+h_1e_1) - h_2D_2f(c+h_1e_1+t_2e_2)| \\
&\quad + |h_2D_2f(c+h_1e_1+t_2e_2) - h_2D_2f(c)| \\
&\quad + |f(c+h_1e_1) - f(c) - h_1D_1f(c+t_1e_1)| \\
&\quad + |h_1D_1f(c+t_1e_1) - h_1D_1f(c)| \\
&\quad \leq \epsilon(|h_2| + |h_1|) \\
&\quad \leq 2\epsilon\|h\|.
\end{aligned}$$

This proves that f is differentiable at c (and of course $Df(c) = \nabla f(c)$). □

Proposition 15.16. *Suppose U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$. If all the partials D_jf_i exist on U and are continuous at c, then f is differentiable at c.* †

Proof. From the preceding proposition, each f_i, defined by $f_i(x) = \langle f(x), e_i\rangle$ is differentiable and $Df_i = \nabla f_i$.

Fix $c \in U$ and let T denote the matrix with (i,j) entry $D_jf_i(c)$. For $\|h\|$ small enough that $c + h \in U$, sufficiently small,

$$\begin{aligned}
\frac{\|f(c+h) - f(c) - Th\|}{\|h\|} &= \frac{\|\sum\langle f(c+h) - f(c) - Th, e_i\rangle e_i\|}{\|h\|} \\
&\leq \sum \frac{|\langle f(c+h) - f(c) - Th, e_i\rangle|}{\|h\|} \\
&\leq \sum \frac{|f_i(c+h) - f_i(c) - Df_ih|}{\|h\|}.
\end{aligned}$$

Each term on the right hand side converges to 0 with h by Proposition 15.15. □

Remark 15.17. Given $U \subset \mathbb{R}^n$ an open set and $f : U \to \mathbb{R}$, if f is differentiable and $Df : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is continuous, then f is said to be *continuously differentiable*. Note that f is continuously differentiable if and only if all its partials are.

Summarizing, if all the partials of f exists in a neighborhood of a point c and are continuous at c, then f is differentiable and its derivative is identified with its matrix of partial derivatives. If further, all the partials are continuous, then f is continuously differentiable. ◇

15.5. **Exercises.**

Exercise 15.1. Show, directly from the definition, that $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x_1, x_2) = (x_1^2 - x_2^2, 2x_1x_2)$$

is differentiable and compute its derivative (at each point).

At which points c does the derivative $Df(c)$ fail to be invertible?

Exercise 15.2. Show that $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(r,t) = (r\cos(t), r\sin(t))$$

is differentiable and compute its derivative using Propositions 15.12 and 15.16 (and the well known rules of calculus for sin and cos).

At which points c does $Df(c)$ fail to be invertible?

Exercise 15.3. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. For $c \in \mathbb{R}^n$, find u which maximizes $D_u f(c)$ (over unit vectors u). This direction is the *direction of maximum increase of f at c*. Compare with Exercise 14.5.

Exercise 15.4. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ have continuous (partial) derivatives. Write out the chain rule for $h = g \circ f$ explicitly in terms of these (partial) derivatives.

Exercise 15.5. Same as Exercise 15.4, but with $f, g : \mathbb{R}^2 \to \mathbb{R}^2$.

Exercise 15.6. Define $F : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$F(x,y,u,v) = (x^2 - y^2 + u^2 - v^2 - 1, xy + uv).$$

Compute the derivative of F.

Determine the *rank* of $DF(c)$ at a point c where $F(c) = 0$. (There are numerous equivalent definitions of the rank of a matrix. Feel free to use any that you are comfortable with. If you don't know a definition, you might choose to use the rank is the largest k such a (principal) $k \times k$ submatrix has non-zero determinant. In particular, the rank of $DF(c)$ is at most two.)

Exercise 15.7. Verify the chain rule for $f \circ F$, where F is defined in Exercise 15.6 and f is given in Example 15.4.

Exercise 15.8. Suppose A is an $m \times n$ matrix and a is a vector in $\mathbb{R}^m$. Show that $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$f(x) = Ax + a$$

is differentiable and $Df(c) = A$ (for all c).

Exercise 15.9. Suppose $f : \mathbb{R}^3 \to \mathbb{R}^2$ is differentiable at 0 and

$$Df(0) = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

Find the directional derivative in the direction

$$v = \begin{pmatrix} 3 & 1 & 5 \end{pmatrix}^T.$$

15.6. **Problems.**

Problem 15.1. Prove Proposition 15.5.

Problem 15.2. Suppose $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}$. Define *f has a local minimum at c local minimum* and prove, if f has a local minimum at c and f is differentiable at c, then $Df(c) = 0$.

Problem 15.3. Suppose $U \subsetneq \mathbb{R}^n$ is open and connected. Show, if $f : U \to \mathbb{R}$ is differentiable and $Df = 0$ ($Df(x) = 0$ for all $x \in U$), then f is constant. You may wish to use the following outline.

(i) Suppose $a \in U$ the vector $h \in \mathbb{R}^n$ and there is a $\delta > 0$ such that $a + th \in U$ for $-\delta < t < 1 + \delta$. Show $f(a) = f(a + h)$.
(ii) Show, if $r > 0$ and $N_r(a) \subseteq U$, then $f(x) = f(a)$ for all $x \in N_r(a)$.
(iii) Apply Problem 8.3.

Show the same result holds with the codomain of f replaced by $\mathbb{R}^m$.

Problem 15.4. Show, in Example 15.10, if $\gamma : (-\delta, \delta) \to \mathbb{R}^2$ is differentiable, $\gamma(0) = 0$, and $\gamma'(0) \neq 0$, then $f \circ \gamma$ is differentiable at 0.

Problem 15.5. Define $f(x, y) = \frac{xy}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Show the partial derivatives $D_j f(0, 0)$ exist, even though f is not continuous at 0.

Problem 15.6. Suppose $U \subset \mathbb{R}^2$ is open and $f : U \to \mathbb{R}$. Prove if the partial derivatives of f exist and are bounded, then f is continuous.

16. The Inverse and Implicit Function Theorems

In this section the Inverse and Implicit Function Theorems are established. The approach here is to prove the Inverse Function Theorem first and then use it to prove the Implicit Function Theorem. It is possible to do things the other way around.

16.1. Lipschitz Continuity.

This subsection collects a couple of facts used in the proof of the inverse function theorem.

Suppose U is an open set in $\mathbb{R}^n$. A function $f : U \to \mathbb{R}^m$ is *Lipschitz continuous* if there is an M such that

$$\|f(x) - f(y)\| \leq M\|x - y\|$$

for all $x, y \in U$.

Recall, a subset C of a vector space V is *convex* if $x, y \in C$ and $s, t \geq 0$, $s + t = 1$ implies

$$sx + ty \in C.$$

Note that, given $c \in \mathbb{R}^n$ and $\epsilon > 0$,

$$N_\epsilon(c) = \{x \in V : \|c - x\| < \epsilon\},$$

is an open convex set.

Proposition 16.1. *Suppose $U \subset \mathbb{R}^n$ is open and convex. If $f : U \to \mathbb{R}^m$ is continuously differentiable and if $Df : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is bounded, say $\|Df(x)\| \leq M$ for all $x \in U$, then f is Lipschitz continuous with constant M.*

In particular, if $O \subseteq \overline{O} \subseteq U$ and O is open, convex and bounded, then f is Lipschitz continuous on O. †

Proof. Given $x, y \in U$, since U is open and convex there is a $\delta > 0$ such that $ty + (1-t)x \in U$ for $-\delta < t < 1 + \delta$. Define $\psi : (-\delta, 1 + \delta) \to U$ by

$$\psi(t) = ty + (1 - t)x$$

and let $g(t) = f \circ \psi(t)$. Since both f and ψ are differentiable, by the chain rule g is differentiable and

$$g'(t) = Df(\psi(t))\psi'(t) = Df(ty + (1 - t)x)(y - x).$$

Hence,

$$\|g'(t)\| \leq \|Df(ty + (1 - t)x)\| \, \|y - x\| \leq M\|y - x\|.$$

From the First Fundamental Theorem of Calculus, Theorem 11.41,

$$f(y) - f(x) = g(1) - g(0) = \int_0^1 g'(t)\, dt.$$

By Proposition 11.42,

$$\|f(y) - f(x)\| \leq \int_0^1 \|g'(t)\|\, dt \leq M\|y - x\|.$$

□

Lemma 16.2. *The function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(x) = \|x\|^2 = \langle x, x\rangle$ is differentiable and $\nabla g(z) = 2z^T$.* †

Proof. Writing $g(x) = x^T x$, note that

$$g(x+h) - g(x) - 2x^T h = h^T h$$

from which it easily follows that $Dg(x)h = 2x^T h$. □

16.2. **The Inverse Function Theorem.**

Theorem 16.3 (Inverse Function)**.** *Suppose U is an open subset of $\mathbb{R}^n$, $c \in U$, and $f : U \to \mathbb{R}^n$. If f is continuously differentiable on U and $Df(c)$ is invertible, then there is an open subset $O \subseteq U$ containing c such that*

(i) $f(O)$ is open;
(ii) $\tilde{f} = f|_O : O \to f(O)$ is a bijection;
(iii) the inverse of $\tilde{f}$ is continuously differentiable; and
(iv) $D\tilde{f}^{-1}(f(c)) = Df(c)^{-1}$.

Proof. The statement is proved first under the additional assumption that $c = 0$ and $Df(0) = I$.

Consider the function $g : U \to \mathbb{R}^n$ defined by $g(x) = x - f(x)$. Since $Dg(0) = 0$, and Dg is continuous, there is a $\delta > 0$ such that $N_\delta(0) \subseteq U$ and $\|Dg(x)\| \le \frac{1}{2}$ for $\|x\| < \delta$. By Proposition 16.1, if $\|x\|, \|y\| < \delta$, then

$$\|g(y) - g(x)\| \le \frac{1}{2}\|y - x\|.$$

Hence,

$$\begin{aligned}\|y - x\| \le& \|(y-x) - (f(y) - f(x))\| + \|f(y) - f(x)\| \\ =& \|g(y) - g(x)\| + \|f(y) - f(x)\| \\ \le& \frac{1}{2}\|y - x\| + \|f(y) - f(x)\|.\end{aligned}$$

Rearranging gives

$$\|y - x\| \le 2\|f(y) - f(x)\|. \tag{25}$$

This proves that f is one-one on $N_\delta(0)$. It also shows that the inverse of the mapping $f : N_\delta(0) \to f(N_\delta(0))$ is (Lipschitz) continuous.

Note that $Dg(x) = I - Df(x)$ and since $\|Dg(x)\| \le \frac{1}{2}$ on $N_\delta(0)$, it follows from Lemma 14.17 that $Df(x)$ is invertible on $N_\delta(0)$.

Choose $0 < \eta < \delta$. The set $S_\eta = \{x : \|x\| = \eta\}$ is closed in $\mathbb{R}^n$ and hence compact. The function $S_\eta \ni x \mapsto \|f(x) - f(0)\|^2$ is continuous and, since f is one-one on $N_\delta(0)$, never 0. It follows, from the extreme value theorem, that there is a $d > 0$ such that $\|f(x) - f(0)\| \ge 2d$ for all $x \in S_\eta$.

Let $K = B_\eta(0)$, the η ball centered at 0. In particular, K is the closure of $N_\eta(0)$ and is compact since it is a closed bounded subset of $\mathbb{R}^n$. Moreover, S_η is the boundary of K. Given $y \in N_d(f(0))$, let

$$\varphi(x) = \langle f(x) - y, f(x) - y \rangle = \|f(x) - y\|^2.$$

The function φ is continuous on K and hence attains its minimum at some $z \in K$. This minimum is not on the boundary S_η since $\|y - f(0)\| < d$ and at the same time, for $x \in S_\eta$,

$$\|f(x) - y\| \geq \|f(x) - f(0)\| - \|f(0) - y\| \geq 2d - d = d.$$

Now, by Problem 15.2, $\nabla\varphi(z) = 0$ and on the other hand, by the chain rule, Proposition 15.6, and Lemma 16.2

$$0 = \nabla\varphi(z) = 2(f(z) - y)^T Df(z).$$

Since $Df(z)$ is invertible, it follows that $f(z) - y = 0$ and we conclude that $y = f(z)$ for some $z \in N_\eta(0)$. Hence, for each $y \in N_d(f(0))$, there is $z \in N_\eta(0)$ such that $f(z) = y$.

Let $O = N_\eta(0) \cap f^{-1}(N_d(f(0)))$. Then O is open, $0 \in O$, and $f : O \to N_d(f(0))$ is one-one and onto. The first two items are now proved (under some additional hypotheses).

Let ψ denote the inverse to $\tilde{f} = f|_O : O \to N_d(f(0))$. The inequality (25) says that ψ is continuous (in fact Lipschitz).

To prove (iii), let $y \in N_d(f(0))$ be given. By (ii) there is an (unique) $x \in O$ such that $f(x) = y$. There is a $\sigma > 0$ such that if $\|k\| < \sigma$, then $y + k \in N_d(f(0))$. By (i) there there is an h such that $x + h \in O$ and $f(x + h) = y + k$. In particular, $\psi(y + k) - \psi(y) = x + h - x = h$. Let $T = Df(x)$ which, by the choice of η, is invertible. We have,

$$\begin{aligned}
\psi(y + k) - \psi(y) - T^{-1}k =& h - T^{-1}k \\
=& -T^{-1}(k - Th) \\
=& -T^{-1}(f(x + h) - y - Th) \\
=& -T^{-1}(f(x + h) - f(x) - Th).
\end{aligned}$$

Thus,

$$\|\psi(y + k) - \psi(y) - T^{-1}k\| \leq \|T^{-1}\|\,\|f(x + h) - f(x) - Th\|.$$

Since also

$$\|h\| = \|x + h - x\| \leq 2\|f(x + h) - f(x)\| = 2\|y + k - y\| = 2\|k\|,$$

it follows that

$$\frac{\|\psi(y + k) - \psi(y) - T^{-1}k\|}{\|k\|} \leq 2\|T^{-1}\|\frac{\|f(x + h) - f(x) - Th\|}{\|h\|}.$$

This last estimate shows that ψ is differentiable at y and $D\psi(y) = T^{-1} = Df(\psi(y))^{-1}$. Finally, the mapping $N_d(f(0)) \ni y \mapsto Df(\psi(y))$ is continuous (as Df is assumed continuous on U and ψ has been shown to be continuous) as is the mapping taking a matrix to its inverse (see Proposition 14.18). Thus $D\psi$ is the composition of continuous maps and hence continuous. The proof of (iii) is complete (under some additional hypotheses).

Now suppose still that $c = 0$, but assume only that $Df(0)$ is invertible. Let $A = Df(0)$ and let G denote the mapping $G(x) = Ax$. Since A is invertible, G is invertible and continuous and moreover both G and G^{-1} are continuously differentiable (see Exercise 15.8). Indeed, $G^{-1}(x) = A^{-1}x$ and, for instance $DG(x)$ is constantly equal to A. Let $F = A^{-1}f = G^{-1} \circ f$. Then F is continuously differentiable on U and $DF(0) = A^{-1}Df(0) = I$. Hence, by what has already been proved, there is an open set O such that $F(O)$ is open and F restricted to O is a continuous bijection between O and $F(O)$ whose inverse is continuously differentiable. It now follows that $f = G \circ F$ maps O bijectively onto the open set $f(0) = G(F(0)) = (G^{-1})^{-1}(F(O))$ and the inverse of f restricted to $f(O)$ is the composition of continuously differentiable functions and is thus continuously differentiable.

The passage from $c = 0$ to a general c is left to the gentle reader. □

Corollary 16.4. *Suppose U is an open subset of $\mathbb{R}^n$, $f : U \to \mathbb{R}^n$ is continuously differentiable, and $Df(x)$ is invertible for each $x \in U$. If $V \subseteq U$ is open, then $f(V)$ is open.* †

Example 16.5. Consider the the mapping given by

$$(r, t) \mapsto e^r(\cos(t), \sin(t)).$$

In particular, it maps the line (r_0, t) to the circle of radius r_0 (many times over). Similarly, it maps the line (r, t_0) to the ray (emanating at, but not containing, the origin) $e^r(\cos(t_0), \sin(t_0))$.

The derivative of F is the 2×2 matrix,

$$DF(r, t) = \begin{pmatrix} e^r\cos(t) & -e^r\sin(t) \\ e^r\sin(t) & e^r\cos(t) \end{pmatrix}$$

which is easily seen to be invertible (its determinant is e^{2r}). Thus, for each point (r_0, t_0) there is open set U of (r_0, t_0) on which F is one-one and $F(U)$ is open.

From the corollary, if V is any open subset of $\mathbb{R}^2$, then $F(V)$ is open. In particular, the range of F is an open set. Of course, the range of F is $\mathbb{R}^2 \setminus \{(0, 0)\}$ which is evidently open.

As an exercise, find an open set U containing $(0, 0)$ on which F is one-one and compute the inverse of $F : U \to F(U)$; likewise find an open set V containing $(0, \frac{\pi}{2})$ on which F is one-one and determine the inverse of $F : V \to F(V)$. Ditto for $(0, \pi)$.

By comparison, consider the function f of Example 16.5. Determine the image of the lines $\{(r_0, t) : t\}$ and $\{(r, t_0) : r\}$ under f. Note that Df is invertible at (r_0, t_0) precisely when $r_0 \neq 0$. See Exercise 15.2. △

16.3. **The Implicit Function Theorem.** It will be convenient to think of the Euclidean space $\mathbb{R}^{n+m}$ as the *direct sum* $\mathbb{R}^n \oplus \mathbb{R}^m$ which is the set

$$\mathbb{R}^n \oplus \mathbb{R}^m = \{(x, y) : x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m\}$$

with coordinate-wise addition

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2),$$

scalar multiplication

$$c(x,y) = (cx, cy)$$

and (Euclidean) norm,

$$\|(x,y)\|^2 = \|x\|^2 + \|y\|^2.$$

If $L \in \mathcal{L}(\mathbb{R}^{n+m}, \mathbb{R}^m)$ and $J : \mathbb{R}^m \mapsto \mathbb{R}^{n+m}$ is the *inclusion*

$$Jy = 0 \oplus y = \begin{pmatrix} 0 \\ y \end{pmatrix}, \tag{26}$$

then $LJ \in \mathcal{L}(\mathbb{R}^m)$. Suppose $U \subseteq \mathbb{R}^n \oplus \mathbb{R}^m$ is open, $c = (a,b) \in U$ (so that $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$), and $f : U \to \mathbb{R}^m$. Writing $z \in U$ as $z = x \oplus y$, if f is differentiable at c, then the matrix representation of $Df(c)$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & \vdots & \vdots & | & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} & & \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{pmatrix}.$$

As short hand, write

$$\begin{pmatrix} \frac{\partial f}{\partial x} & | & \frac{\partial f}{\partial y} \end{pmatrix}.$$

With a similar short hand, the matrix representation for J has the form,

$$J = \begin{pmatrix} 0_{n,m} \\ - \\ I_m \end{pmatrix}$$

and thus,

$$Df(c)J = \frac{\partial f}{\partial y} := \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial y_1} & \cdots & \frac{\partial f_m}{\partial y_m} \end{pmatrix}.$$

Theorem 16.6 (Implicit Function)**.** *Suppose $U \subseteq \mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$ is open, $(a,b) \in U$, and $f : U \to \mathbb{R}^m$. If*

(a) f is continuously differentiable;
(b) $f(a,b) = 0$; and
(c) $Df(a,b)J$ is invertible,

then there is

(i) an open set $(a,b) \in O \subseteq \mathbb{R}^{n+m}$;
(ii) an open set $a \in W \subseteq \mathbb{R}^n$; and
(iii) a unique function $g : W \to \mathbb{R}^m$ such that
  (a) $G(x) = (x, g(x))$ maps W into O;
  (b)
$$\{(x,y) : f(x,y) = 0\} \cap O = \{G(x) : x \in W\}; \text{ and;}$$
  (c)
$$f(x, g(x)) = 0 = f(G(x)).$$

Moreover, W and O can be chosen so that g is continuously differentiable.

When the conclusion of the implicit function theorem holds we say that $f(x,y)=0$ defines y as a function of x *near* the point (a,b).

Proof. Define $F : U \to \mathbb{R}^{n+m}$ by

$$F(x,y) = (x, f(x,y)).$$

It is readily checked that F is continuously differentiable and further

$$DF(x,y) = \begin{pmatrix} I & 0 \\ * & Df(x,y)J \end{pmatrix}.$$

By Problem 14.4 and the hypothesis that $Df(a,b)J$ is invertible, the matrix $DF(a,b)$ is invertible.

By the Inverse Function Theorem, there is an open set O containing (a,b) such that F is one-one on O, the set $F(O)$ is open, and $F : O \to F(O)$ has a continuously differentiable inverse $H : F(O) \to O$.

Let $W = \{u \in \mathbb{R}^n : (u,0) \in F(O)\}$. Thus W is the set of those $u \in \mathbb{R}^n$ such that there exists a $v \in \mathbb{R}^m$ such that $(u,v) \in O$ and $f(u,v) = 0$. Note that W is open and $a \in W$ since $(a,b) \in O$ and $f(a,b) = 0$.

Write $H(x,y) = (h_1(x,y), h_2(x,y)) \in \mathbb{R}^n \oplus \mathbb{R}^m$. Thus,

$$(x,y) = H(F(x,y)) = H(x, f(x,y)) = (h_1(x, f(x,y)), h_2(x, f(x,y))).$$

If $x \in W$, then $(x,0) \in F(0)$. Thus, there is a y such that $(x,y) \in O$ and $f(x,y) = 0$. It follows that $h_1(x,0) = x$ and we may define $g : W \to \mathbb{R}^m$ by $g(x) = h_2(x,0)$. In this case, for such x,

$$G(x) = (x, g(x)) = (h_1(x,0), h_2(x,0)) = H(x,0) \in O$$

and thus $G : W \to O$ and condition (a) of item (iii) holds. Since g is the composition of continuously differentiable mappings (inclusion, then H, then projection onto the last m-coordinates), g is continuously differentiable. Moreover, if $x \in W$, then

$$(x,0) = F \circ H(x,0) = F(x, g(x)) = (x, f(x, g(x))).$$

Thus $f(G(x)) = 0$ for all $x \in W$ and condition (c) in item (iii) holds as does the reverse inclusion in (b). On the other hand, if $(x,y) \in O$ and $f(x,y) = 0$, then $x \in W$ and thus $F(x, g(x)) = (x,0) = F(x,y)$. Since F is one-one, it follows that $y = g(x)$. Hence, condition (b) in item (iii) holds and moreover, g is uniquely determined.

□

Example 16.7. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^2 + y^2 - 1$. The derivative (gradient) of f is then,

$$\nabla f(x,y) = 2(x \quad y).$$

In particular, $\nabla f \neq 0$ on the set $f(x,y) = 0$. If $f(a,b) = 0$ and $b \neq 0$, then there is an open set W containing a (in $\mathbb{R}$), an open set O containing (a,b) and a continuously differentiable function $g : W \to \mathbb{R}$ such that $g(a) = b$ and $f(x, g(x)) = 0$. In fact, $(x, y) \in O$ and $f(x, y) = 0$ if and only if $y = g(x)$.

Likewise, if $a \neq 0$, then there is an open set V containing b (in $\mathbb{R}$) and a continuously differentiable function $h : V \to \mathbb{R}$ such that $h(b) = a$ and $f(h(y), y) = 0$.

Of course in this example it is a simple matter to actually solve g and h. △

Example 16.8. Consider the lemniscate

$$f(x, y) = (x^2 + y^2)^2 - 2(x^2 - y^2) = 0.$$

(See `en.wikipedia.org/wiki/Lemniscate_of_Bernoulli` for a picture.) In polar coordinates it takes the form $r^2 = 2\cos(2\theta)$, from which it can be seen to look like a (horizontal) figure eight. The gradient of f is

$$\nabla f(x, y) = (4x^3 + 4xy^2 - 4x, 4y^3 + 4x^2 y + 4y)$$

which can vanish only for $y = 0$ in which case $x = 0$ or $x = \pm 1$. Since the points $(\pm 1, 0)$ are not on the lemniscate, except for the point $(0, 0)$, the Implicit Function Theorem says that the lemniscate is (*locally*) the graph of a function (either $y = g(x)$ of $x = h(y)$). The theorem is silent on whether this is possible near $(0, 0)$; however from the picture it is evident that the lemniscate is not the graph of a function in any open set containing $(0, 0)$. △

Example 16.9. Let $f(x, y) = y - x^3$. The set $f(x, y) = 0$ is just the graph of a continuously differentiable function, namely $y = x^3$. It is also the graph of $x = y^{\frac{1}{3}}$ which is not continuously differentiable at 0. Note that $\nabla f(0, 0) = (0, 1)$ so that the implicit function theorem is silent on writing x as a function of y near $(0, 0)$ △

Example 16.10. The solution set of $x^2 = y^3$ is the Neile parabola (see `en.wikipedia.org/wiki/Neile_parabola`). Let $f(x, y) = x^2 - y^3$. Then the gradient of f vanishes at $(0, 0)$. In this case the set $f(x, y) = 0$ is the graph of a function $y = x^{\frac{2}{3}} = g(x)$, but $g(x)$ is not differentiable at 0. On the other hand, this set is not the graph of a function $x = h(y)$ near $(0, 0)$. △

Example 16.11. Define $F : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$F(x, y, u, v) = ((x^2 - y^2) - (u^3 - 3uv^2), 2xy - (3u^2 v - v^3)).$$

Verify that near any point, except for 0, on the set $F(x, y, u, v) = 0$ it is possible to solve for either $(u, v) = g(x, y)$ or $(x, y) = h(u, v)$. (Note: using complex numbers, this example can be written as $z^2 = w^3$ and so is the complex version of the Neile parabola). △

The discussion of the Implicit Function Theorem continues in the following subsection.

16.4. **Immersions, Embeddings, and Surfaces*.** As indicated by the $*$ this section is optional. It contains an informal introduction to surfaces. The main technical tool is the Implicit Function Theorem.

Suppose T is an $m \times k$ matrix. Viewed as a linear mapping, T is one-one if and only if the columns of T form a linearly independent set (and of course necessarily $m \geq k$).

Likewise, T is onto if and only if the columns of T span $\mathbb{R}^m$ (and of course necessarily $k \geq m$). In particular, T is onto if and only if T has an $m \times m$ invertible submatrix.

Definition 16.12. Let $W \subset \mathbb{R}^n$ be an open set. An *immersion* $f : W \to \mathbb{R}^k$ is a continuously differentiable function such that $Df(x)$ is one-one at each point $x \in W$. Note that necessarily $k \geq n$. ◁

Example 16.13. The map $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) = (\cos(t), \sin(t))$ is an immersion. △

Definition 16.14. The immersion $f : W \to \mathbb{R}^k$ is an *embedding* if it is one-one and the inverse of $f : W \to f(W)$ is continuous. ◁

Example 16.15. Define $\psi : (-1, 2\pi) \to \mathbb{R}^2$ as follows. Let $\psi(t) = (1, t)$ for $-1 < t \leq 0$; and $\psi(t) = (\cos(t), \sin(t))$ for $0 \leq t < 2\pi$. Then ψ is a one-one immersion. However, it is not an embedding, since the inverse is not continuous at $(1, 0)$. △

Example 16.16. The function $g_1 : (-\pi, \pi) \to \mathbb{R}^2$ defined by $g_1(t) = (\cos(t), \sin(t))$ is an embedding; as is $g_2 : (0, 2\pi)$ defined by $g_2(t) = (\cos(t), \sin(t))$. Thus, the circle $x^2+y^2 = 1$ is locally the image of an embedding. △

If $W \subseteq \mathbb{R}^n$ is open and $f : W \to \mathbb{R}^m$ is continuously differentiable, then the mapping $G : W \to \mathbb{R}^{n+m}$ defined by

$$G(x) = (x, f(x))$$

is a embedding. Indeed, in block matrix form,

$$DG(x) = \begin{pmatrix} I_n \\ - \\ Df(x) \end{pmatrix}$$

and hence $DG(x)$ is one-one. Further, the projection mapping $\pi : \mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m \to \mathbb{R}^n$ defined by $\pi(x, y) = x$ is continuous and its restriction to the range of G is the inverse of G. Hence G has a continuous inverse.

Definition 16.17. An *n dimensional surface* in $\mathbb{R}^k$ is a (non-empty) set $S \subseteq \mathbb{R}^k$ such that for each point $s \in S$ there is an open set O containing s in $\mathbb{R}^k$, an open subset $W \subseteq \mathbb{R}^n$, and an embedding $G : W \to \mathbb{R}^k$ such that $G(W) = O \cap S$. ◁

Example 16.18. The discussion preceding the definition shows that the graph of a continuously differentiable function is a surface.

Example 16.15 shows that the circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is a 1-dimensional surface in $\mathbb{R}^2$. △

Theorem 16.19. *Suppose $U \subseteq \mathbb{R}^k$ is an open set, $f : U \to \mathbb{R}^m$ is continuously differentiable, and $S = \{z \in U : f(z) = 0\}$ is non-empty.*

If $Df(z)$ is onto for each $z \in S$, then S is a $k - m$ dimensional surface.

Note that the onto hypotheses implies $k \geq m$. Let $n = k - m$ so that $k = n + m$.

Proof. Let $s \in S$ be given. Since $Df(s)$ is onto, by relabeling the variables if needed, it can be assumed that $Df(s)J : \mathbb{R}^m \to \mathbb{R}^m$ is invertible, where J is define in Equation (26). By the Implicit Function Theorem (Theorem 16.6), there exists open sets $a \in W \subseteq \mathbb{R}^n$ and $(a,b) \in O \subseteq \mathbb{R}^{n+m}$ and a (unique) continuously differentiable function $g : W \to \mathbb{R}^m$ such that

$$\{z = (x,y) : f(x,y) = 0\} \cap O = \{(x, g(x)) : x \in W\}.$$

To see that $G : W \to O$ given by $G(x) = (x, g(x))$ is the desired embedding, note that its inverse (on its range) is given by $(x, g(x)) \mapsto x$ and is thus continuous (as it is the restriction of a coordinate projection to the range of G). ☐

Example 16.20. The following example shows that the converse of the preceding theorem is false. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x^2 + y^2 - 1$, let $h = f^2$. The curve $f(x,y) = 0$ is of course the unit circle as is the curve $h(x,y) = 0$. Thus $h(x,y) = 0$ defines a surface. On the other hand,

$$\nabla(h) = 4f(x,y)\nabla(f)$$

which vanishes at every point on $f(x,y) = 0$.

For a more subtle example, consider $0 = p(x,y) = y^3 + 2x^2y - x^4$. The gradient of p at $(0,0)$ is $(0,0)$. On the other hand, near $(0,0)$ this curve can also be expressed as $0 = g(x,y) = x^2 - y(1 + \sqrt{1+y})$ and $\nabla(g)(0,0) = (0\ \ 1) \neq 0$. △

Example 16.21. Define $F : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$F(x,y,u,v) = (x^2 - y^2 + u^2 - v^2 - 1, 2xy + 2uv).$$

Verify that $F(x,y,u,v) = 0$ is a 2 dimensional surface (in $\mathbb{R}^4$). (Note, this is the complex sphere $z^2 + w^2 = 1$.) △

16.5. **Exercises.**

Exercise 16.1. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x,y) = (xy, x - y).$$

Compute $Df(x,y)$. Verify that the inverse function theorem applies at the point $(1,1)$. What is the conclusion? Find an open set U containing $(1,1)$ on which f is one-one and find the inverse of f restricted to U (viewed as a map onto its range). [Hint: Note if $xy = ab$ and $x - y = a - b$, then $x^2 + y^2 = a^2 + b^2$ and hence both (x,y) and (a,b) are points of intersection of the same line and circle.]

Find the image of the set $\{(x,y) : \frac{1}{2} < x < 2, \frac{1}{2} < y < 2\}$ under f.

Exercise 16.2. What does the inverse function theorem say about the mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$F(\rho, \theta, \phi) = \rho(\sin(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)).$$

Fixing $\rho = 1$ gives a mapping $G : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$G(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\phi)).$$

What is the image of G?

Exercise 16.3. Consider the mapping $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = 1 - (x^2 + y^2 + z^2).$$

Show that the set $S = \{(x, y, z) \in \mathbb{R}^3 : \quad f(x, y, z) = 0\}$ is a surface. Find, for each point $s \in S$, an open set $U \subseteq \mathbb{R}^2$ and an embedding $g : U \to S$ onto an open set in S containing s. The collection (g, U) is a set of *local parameterizations*.

16.6. **Problems.**

Problem 16.1. In real coordinates, the complex function $z \mapsto z^2$ takes the form $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (x^2 - y^2, 2xy)$. Prove that if $(a, b) \neq (0, 0)$, then there is an open set U containing (a, b) such that $F(U)$ is open and $F|_U : U \to F(U)$ is one-one.

In the case that $(a, b) = (1, 0)$ find such a U and compute the inverse of $F : U \to F(U)$.

As above, with $(a, b) = (-1, 0)$.

Problem 16.2. Suppose $f = f(x, y, z) : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable. Show, if $f(a, b, c) = 0$ and $\frac{\partial f}{\partial z}(a, b, c) \neq 0$, then the relation $f(x, y, z) = 0$ defines $z = g(x, y)$ near the point (a, b, c). Show further that, at the point (a, b, c),

$$\frac{\partial g}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}.$$

This is implicit differentiation. Note that the necessary assumption that the denominator above is not 0 (at (a, b, c)) is sufficient to establish that, at least locally, z is indeed a function of (x, y) (an issue which is not directly addressed in most calculus texts).

Show $yz = \log(x + z) - \log(3)$ defines z as a function of (x, y) near $(2, 0, 1)$ and find $\frac{\partial z}{\partial x}$ at this point.

Problem 16.3. Fix $R > r > 0$ and define $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x, y, z) = (R^2 - r^2 + x^2 + y^2 + z^2)^2 - 4R^2(x^2 + y^2).$$

Show the set $F(x, y, z) = 0$ is a surface - called a *torus*.

Use

$$\begin{aligned} x(u, v) =&(R + r\cos v)\cos u \\ y(u, v) =&(R + r\cos v)\sin u \\ z(u, v) =&r\sin v, \end{aligned}$$

to informally identify a set of local parametrizations (see Exercise 16.3).

Problem 16.4. Note that the point $(x, y, u, v, w) = (1, 1, 1, 1, -1)$ satisfies the system of equations

$$\begin{aligned} u^5 - xv^2 + y + w =&0 \\ v^5 - yu^2 + x + w =&0 \\ w^4 + y^5 - x^4 - 1 =&0 \end{aligned}$$

Explain why there exists an open set containing $(x, y) = (1, 1)$ and continuously differentiable functions $u(x, y)$, $v(x, y)$ and $w(x, y)$ such that $u(1, 1) = 1 = v(1, 1)$ and $w(1, 1) = -1$ and such that (x, y, u, v, w) satisfy the system of equations.

Problem 16.5. Consider the folium of Descartes, described implicitly as

$$f(x, y) = x^3 + y^3 - 3axy = 0,$$

(for a fixed $a > 0$). Show that the implicit function theorem says that f is locally the graph of a function for any point $(x_0, y_0) \neq (0, 0)$ with $f(x_0, y_0) = 0$. Hence it is in principle possible to solve for x as a function of y or y as a function of x near any point, except possibly $(0, 0)$, on the curve. A couple of parametric representations can be found at http://en.wikipedia.org/wiki/Folium_of_Descartes from which it is evident that near $(0, 0)$ the curve $f(x, y) = 0$ is not the graph of function.

Problem 16.6. The parabolic folium is described implicitly as

$$x^3 = a(x^2 - y^2) + bxy$$

(for $a, b > 0$). What does the implicit function theorem say?

Problem 16.7. What can be said about solving the system

$$x^2 - y^2 + 2u^3 + v^2 = 3$$
$$2xy + y^2 - u^2 + 3v^4 = 5$$

for (u, v) in terms of (x, y) near the point $(x, y, u, v) = (1, 1, 1, 1)$?