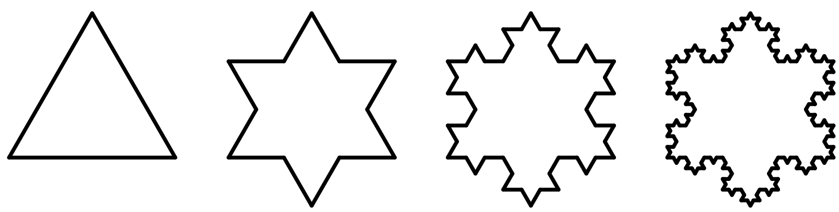
Homework 6

Introduced in 1904 by the Swedish mathematician Helge von Koch (1870–1924), the *Koch snowflake* was one of the earliest fractals to have been described. We start with an equilateral triangle. Then we divide each of the three sides into three equal line segments, and replace the middle portion with a smaller equilateral triangle. We then iterate this construction, dividing each of the line segments of the new figure into thirds and replacing the middle with an equilateral triangle, and then iterate this again and again, forever. The first four iterations are shown below.



1 Find a series representing the area of the Koch snowflake and find its value. Then find a series representing the perimeter of the Koch snowflake and find its value.

2 Suppose that there were only finitely many primes, say $\{p_1, p_2, \ldots, p_m\}$. Under this assumption, prove, using the fact that every positive integer n has a unique prime factorization[1], that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \left(\sum_{n=0}^{\infty} \frac{1}{p_1^n} \right) \left(\sum_{n=0}^{\infty} \frac{1}{p_2^n} \right) \cdots \left(\sum_{n=0}^{\infty} \frac{1}{p_m^n} \right),$$

where the right-hand side is the product of all series of the form $\sum 1/p_i^n$ for the primes in our list $\{p_1, p_2, \ldots, p_m\}$.

3 Show that the conclusion to the previous problem contradicts facts we know about series, thus proving that there are infinitely many primes. *(Yes, there are easier proofs, but those will not get any points for this assignment.)*

[1] This refers to the fact that every positive integer n can be written as a product $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for some choice of primes $p_1, p_2, \ldots, p_k$ and nonnegative integers $a_1, a_2, \ldots, a_k$.