1 Metric spaces

We assume the reader has had ample experience with calculus on the real line, including in-depth coverage of convergence, of open and closed sets, and of continuous functions. In this chapter, we begin the process of lifting these concepts to arbitrary metric spaces.

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1.1 Definitions and examples

Many of the concepts of analysis on the real line depend only on the notion of closeness (convergence, for example). When we are working on the real line, we usually take the distance between $x, y \in \mathbb{R}$ to be $|x - y|$. However, many of the results of calculus hold for any set X and any distance function $d(x, y)$ that satisfies four axioms.

Definition 1.1.1. A *metric space* (X, d) consists of a set X and a *metric* $d : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$,

(i) $d(x, y) \geq 0$;

(ii) $d(x, y) = 0$ if and only if $x = y$;

(iii) $d(x, y) = d(y, x)$; and

(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

Items (i) and (ii) together are sometimes expressed by saying that d is *positive definite*.

Item (iii) says that d is *symmetric*.

Item (iv) is the *triangle inequality*, and is usually the only metric space axiom that requires much effort to verify.

A set X can have a multitude of metrics placed on it, which is why we must include the metric when we specify the metric space we are considering. As a first example, consider $X = \mathbb{R}$. We know that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = |x - y|$$

is a metric on $\mathbb{R}$, and it is called the *standard metric on* $\mathbb{R}$. However, the function $d(x, y) = 2|x - y|$ would work just as well as a metric on $\mathbb{R}$. A more exotic metric is given below.

Example 1.1.2 (Discrete metric)**.** Given any set X, the *discrete metric* on X is the function $d_{\text{disc}} : X \times X \to \mathbb{R}$ defined by

Exercise 1.1.2 asks the reader to verify that the discrete metric is in fact a metric.

$$d_{\text{disc}}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

A metric space equipped with the discrete metric is called a *discrete metric space*.

Because we have the discrete metric, we can turn *every* set into a metric space in at least one way.

We could invent quite a few more metrics on $\mathbb{R}$. For example, the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

The reader is asked to verify that these are metrics in Exercises 1.1.4 and 1.1.5.

$$d(x, y) = \min\{1, |x - y|\},$$

or the function

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Thus, even on our familiar set $\mathbb{R}$, there are a number of choices for the metric to use, although if a metric on $\mathbb{R}$ is not specified, the standard metric is to be assumed.

Before continuing with more examples, we need to make two definitions.

Definition 1.1.3 (Subspace of a metric space)**.** Let (X, d) be a metric space and $Y \subseteq X$. The pair $(Y, d|_{Y \times Y})$ is itself a metric space (Exercise 1.1.6), and is called a *subspace* of (X, d).

With Definition 1.1.3, we can restrict any metric on $\mathbb{R}$ to obtain a metric on $[0, 1]$, $\mathbb{Q}$, $\mathbb{Z}$, etc. Frequently we denote the resulting metric simply by d, instead of by $d|_{Y \times Y}$, if no confusion could result by doing so.

Next we define neighborhoods. Except for the change in metric, these are defined exactly how they were on the real line.

Definition 1.1.4 (Neighborhood)**.** Let (X, d) be a metric space. Given a real number $\epsilon > 0$ and a point $x_0 \in X$, the *ϵ-neighborhood of x_0* is the set

$$N_\epsilon(x) = \{x \in X : d(x, x_0) < \epsilon\}.$$

In Definition 1.1.4, x_0 is the *center* of the neighborhood and ϵ is the *radius* of the neighborhood. Neighborhoods always contain their centers, because we insist on radii being positive.

Neighborhoods are often called *open balls*, because of what they look like in $\mathbb{R}^2$ with the *Euclidean distance* or *Euclidean metric*; given points

$x = (x_1, x_2)$ and $y = (y_1, y_2)$ in the plane, the Euclidean distance between them is

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Thus in this metric, the open neighborhood of the origin with radius 1 is the open unit disc,

$$N_1((0,0)) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

It is not entirely straight-forward to prove that the Euclidean metric satisfies the triangle inequality, which is why we delay that proof until the end of the next section.

This is very special metric on $\mathbb{R}^2$, and we will have a lot more to say about it in the next section. Before that, we present two other metrics, which are quite useful themselves, below.

As we define further metrics, you should ask yourself what the neighborhoods of points look like.

Definition 1.1.5 (The ℓ^1 metric on $\mathbb{R}^2$)**.** Given points $x, y \in \mathbb{R}^2$, the ℓ^1 *metric* is the function d_1 defined by

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|.$$

Exercises 1.1.7 and 1.1.8 ask the reader to verify that the ℓ^1 and ℓ^∞ metrics are metrics on $\mathbb{R}^2$.

Definition 1.1.6 (The ℓ^∞ metric on $\mathbb{R}^2$)**.** Given points $x, y \in \mathbb{R}^2$, the ℓ^∞ *metric* is the function d_∞ defined by

$$d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

We consider metrics on one more set, the vector space $C([0,1])$ of continuous functions from $[0,1]$ to $\mathbb{R}$.

We have defined the ℓ^1 metric (and the ℓ^∞ metric) on both $\mathbb{R}^2$ and $C([0,1])$. These are different functions (their domains are different), but it should always be clear which metric we are discussing.

Definition 1.1.7 (The ℓ^1 metric on $C([0,1])$)**.** Given functions $f, g \in C([0,1])$, the ℓ^1 *metric* is the function d_1 defined by

$$d_1(f, g) = \int_0^1 |f(t) - g(t)|\, dt.$$

Exercises 1.1.9 and 1.1.10 ask the reader to verify that the ℓ^1 and ℓ^∞ metrics are metrics on $C([0,1])$.

Definition 1.1.8 (The ℓ^∞ metric on $C([0,1])$)**.** Given functions $f, g \in C([0,1])$, the ℓ^∞ *metric* is the function d_∞ defined by

$$d_\infty(f, g) = \max\{|f(t) - g(t)| : t \in [0,1]\}.$$

We conclude the section by proving a result.

Proposition 1.1.9. *Let (X, d) be a metric space and $x, y \in X$. If $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.*

Proof. Suppose that $x \neq y$. We know from part (i) of the definition of a metric space that $d(x, y) \geq 0$ always, and because $x \neq y$, part (ii) shows that $d(x, y) \neq 0$. Therefore $d(x, y) > 0$. Letting $\epsilon = d(x, y)$, we see that we cannot have $d(x, y) < \epsilon$, proving the proposition. □

Exercises

Exercise 1.1.1. Let (X, d) be a metric space. Prove that if $x_1, \ldots, x_n \in X$, then

$$d(x_1, x_n) \leq \sum_{k=1}^{n-1} d(x_k, x_{k+1}).$$

Exercise 1.1.2. Prove that if X is any set, then (X, d_{disc}) is indeed a metric space.

Exercise 1.1.3. Describe the neighborhoods in a discrete metric space (X, d_{disc}).

Exercise 1.1.4. Prove that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = \min\{1, |x - y|\},$$

is a metric on $\mathbb{R}$.

Exercise 1.1.5. Prove that if (X, d) is a metric space, then the function $d_* : X \times X \to \mathbb{R}$ defined by

$$d_*(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is also metric on X.

Hint. To verify the triangle inequality in Exercise 1.1.5, you might want to first establish that if a and b are nonnegative real numbers satisfying $a \leq b$, then $a/(1 + a) \leq b/(1 + b)$.

Exercise 1.1.6. Let (X, d) be a metric space and $Y \subseteq X$. Prove that $(Y, d|_{Y \times Y})$ is also a metric space.

Exercise 1.1.7. Prove that the ℓ^1 metric from Definition 1.1.5 is indeed a metric on the set $\mathbb{R}^2$.

Exercise 1.1.8. Prove that the ℓ^∞ metric from Definition 1.1.6 is indeed a metric on the set $\mathbb{R}^2$.

Exercise 1.1.9. Prove that the ℓ^1 metric from Definition 1.1.7 is indeed a metric on the set $C([0, 1])$.

Exercise 1.1.10. Prove that the ℓ^∞ metric from Definition 1.1.8 is indeed a metric on the set $C([0, 1])$.

Exercise 1.1.11. Let (X, d) be a metric space. Prove that if $x, y, z \in X$, then

$$|d(x, z) - d(y, z)| \leq d(x, y).$$

Exercise 1.1.12. Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Define $d : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Prove that d is a metric on $X \times Y$.

Exercise 1.1.13. Let (X, d_X) be a metric space, let Y be any set, and let $f : Y \to X$ be an injection. Define a function $d_Y : Y \times Y \to \mathbb{R}$ by

$$d_Y(a, b) = d_X(f(a), f(b)).$$

Prove that d_Y is a metric on Y.

1.2 Normed vector spaces

An important class of metric spaces arises from vector spaces. Of course, $\mathbb{R}^n$ is the quintessential finite dimensional vector space, but the results we establish here apply to any real vector space, even infinite dimensional spaces such as the vector space $C([0,1])$ of continuous functions from $[0,1]$ to $\mathbb{R}$.

Recall that a *real* vector space is one in which the scalars are real numbers.

Definition 1.2.1 (Vector space norm)**.** A *norm* on the real vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ such that, for all vectors $x, y \in V$ and all scalars $c \in \mathbb{R}$,

(i) $\|x\| \geq 0$,
(ii) $\|x\| = 0$ if and only if $x = 0$,
(iii) $\|cx\| = |c|\,\|x\|$, and
(iv) $\|x+y\| \leq \|x\| + \|y\|$.

If V is a real vector space and $\|\cdot\| : V \to \mathbb{R}$ is a norm on V, then the pair $(V, \|\cdot\|)$ is a *normed vector space*.

The last of the norm axioms is also known as the *triangle inequality*.

Example 1.2.2. The functions $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ mapping $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\|x\|_1 = \sum_{k=1}^{n} |x_k|,$$

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} x_k^2}, \text{ and}$$

$$\|x\|_\infty = \max\{|x_k| : 1 \leq k \leq n\}$$

are norms on $\mathbb{R}^n$. These are known as the **1**-*norm*, **2**-*norm*, and **∞**-*norm*, respectively. It is not difficult to verify that the triangle inequality holds for the 1-norm and the ∞-norm. For the 2-norm it takes a bit more work, and we leave this verification until the end of the section.

Note that $\|x\|_\infty \leq \|x\|_1$ for all $x \in \mathbb{R}^n$.

Our first result shows that we can turn every norm into a metric.

Proposition 1.2.3 (Norms induce metrics)**.** *If $(V, \|\cdot\|)$ is a normed vector space, then the function $d : V \times V \to \mathbb{R}$ defined by*

$$d(x,y) = \|x-y\|$$

is a metric on V.

Proof. With the exception of the triangle inequality, it is evident that d satisfies the axioms of a metric. To prove that d satisfies the triangle inequality, let $x, y, z \in V$ be given. Using the triangle inequality for the norm, we have

$$\begin{aligned} d(x,z) &= \|x - z\| \\ &= \|(x-y) + (y-z)\| \\ &\leq \|x-y\| + \|y-z\| \\ &= d(x,y) + d(y,z), \end{aligned}$$

as desired. □

When we view a normed vector space $(V, \|\cdot\|)$ as a metric space, our default is to use the metric *induced by* the norm, as in the previous result.

We have just seen that norms induce metrics. Next we look at a useful way to induce a norm.

Definition 1.2.4. Let V be a real vector space. The function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

is an *inner product on* V if, for all vectors $x, y, z \in V$ and all scalars $c \in \mathbb{R}$,

(i) $\langle x, x \rangle \geq 0$,
(ii) $\langle x, x \rangle = 0$ if and only if $x = 0$,
(iii) $\langle x, y \rangle = \langle y, x \rangle$, and
(iv) $\langle cx + y, z \rangle = c\langle x, z \rangle + \langle y, z \rangle$.

Another term for an inner product is a *scalar product*.

The fourth inner product axiom says that the inner product is linear in its first component, but by the symmetry axiom we see that inner products must actually be linear in both components.

We present two examples of inner products next; the reader is asked to verify that they satisfy the inner product axioms in Exercises 1.2.1 and 1.2.2.

Example 1.2.5. On $\mathbb{R}^n$, the function $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$$

is an inner product. When we consider $\mathbb{R}^2$ or $\mathbb{R}^3$, this is often called the *dot product*.

Example 1.2.6. On the vector space $C([0,1])$, the function $\langle \cdot, \cdot \rangle$ defined by

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\,dt$$

is an inner product.

Our next result is incredibly useful.

Theorem 1.2.7 (Cauchy–Schwarz inequality)**.** *Suppose that $\langle \cdot, \cdot \rangle$ is an inner product on a real vector space V. Then for all $x, y \in V$, we have*

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \, \langle y, y \rangle.$$

Proof. Take any two vectors $x, y \in V$ and let $t \in \mathbb{R}$ an arbitrary real number. By the nonnegativity property of inner products, we see that

$$\langle tx + y, tx + y \rangle \geq 0.$$

Expanding the left-hand side of the above inequality using the symmetry and linearity properties of inner products shows that

$$t^2 \langle x, x \rangle + 2t \langle x, y \rangle + \langle y, y \rangle \geq 0.$$

Viewing x and y as fixed, the above inequality states that a certain quadratic in the variable t is always nonnegative. Therefore the discriminant of the quadratic (that is, the quantity $b^2 - 4ac$ in the quadratic formula) must be non-positive. Thus we see that

$$4\langle x, y \rangle^2 - 4\langle x, x \rangle \langle y, y \rangle \leq 0.$$

Upon simplification, this is precisely the inequality we sought to prove. □

With the Cauchy–Schwarz inequality in hand, our final result of the section shows how inner products induce norms (which then induce metrics). This finally verifies that the 2-norm $\| \cdot \|_2$ is a norm, and therefore that Euclidean distance d_2 is a metric on $\mathbb{R}^n$.

Proposition 1.2.8 (Inner products induce norms)**.** *If $\langle \cdot, \cdot \rangle$ is an inner product on the real vector space V, then the function $\| \cdot \| : V \to \mathbb{R}$ defined by*

$$\|x\| = \sqrt{\langle x, x \rangle}$$

is a norm on V.

With the norm defined in the Proposition 1.2.8, the Cauchy–Schwarz inequality says that

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|.$$

Proof. We verify the triangle inequality, leaving the other properties of a norm to the gentle reader. Let $x, y \in V$ be arbitrary. Using the linearity of the inner product in both variables, we have that

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle. \end{aligned}$$

Now we use the Cauchy–Schwarz inequality to bound the term $\langle x, y \rangle$, giving us

$$\begin{aligned} \|x + y\|^2 &\leq \|x\|^2 + 2\|x\| \|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2, \end{aligned}$$

from which the triangle inequality follows by taking square roots. □

This norm is our default when we are working in a vector space that has an inner product (an *inner product space*). In the case of $\mathbb{R}^n$ with the usual inner product (defined in Example 1.2.5), the resulting norm is the *Euclidean norm* $\|\cdot\|_2$ that we have already introduced,

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} x_k^2},$$

and the resulting metric is the *Euclidean distance* or ℓ^2 *metric*,

$$d_2(x, y) = \|x - y\|_2 = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

The term *Euclidean space* refers to the metric space $(\mathbb{R}^n, d_2)$ for some value of n. Note that $(\mathbb{R}^n, d_2)$ is, as a metric space, distinct from both $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_\infty)$. As mentioned above, d_2 is the default metric on $\mathbb{R}^n$, because it is the metric that is induced by the inner product.

Exercises

Exercise 1.2.1. Verify the claim made in Example 1.2.5, that on $\mathbb{R}^n$, the function $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k$$

is an inner product.

Exercise 1.2.2. Verify the claim made in Example 1.2.6, that in the vector space $C([0, 1])$, the function

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$$

is an inner product.

Exercise 1.2.3. Prove that if V is a vector space with an inner product, then the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$ satisfies the *parallelogram law*,

$$\|v + w\|^2 + \|v - w\|^2 = 2(\|v\|^2 + \|w\|^2).$$

Exercise 1.2.4. The parallelogram law in plane geometry states that the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals. Explain why the equality in Exercise 1.2.3 is called the parallelogram law.

Exercise 1.2.5. Is the 1-norm $\|\cdot\|_1$ on $\mathbb{R}^n$ for $n \geq 2$ induced by an inner product? In other words, is there any inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\|x\|_1 = \sqrt{\langle x, x \rangle}$$

for all vectors $x \in \mathbb{R}^n$? Why or why not?

Exercise 1.2.6. Use the Cauchy–Schwarz inequality to prove that for all vectors $x \in \mathbb{R}^n$, we have $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

Exercise 1.2.7. By Proposition 1.2.3,

$$d(f,g) = \left(\int_0^1 |f-g|^2 dt\right)^{1/2}$$

defines a metric on the space of polynomials $\mathcal{P}$. For $n \in \mathbb{N}$, define

$$p_n(t) = \sqrt{2n+1}\, t^n.$$

Find $d(p_n, p_m)$.

1.3 Convergence and limits

SEQUENCES OF REAL NUMBERS should be familiar to the reader now. Sequences over arbitrary sets are a straight-forward generalization.

Definition 1.3.1. A *sequence from* $\boldsymbol{X}$ is a function with codomain X and domain of the form $\{n \in \mathbb{Z} : n \geq n_0\}$ for some starting index $n_0 \in \mathbb{Z}$.

If a is a sequence, it is customary to denote $a(n)$ by a_n, and to denote the entire sequence by

$$(a_n), (a_n)_n, \text{ or } (a_n)_{n=n_0}^{\infty},$$

depending on the amount of clarity required. When X comes equipped with a metric, we define convergence in the obvious way.

Definition 1.3.2 (Convergence in metric spaces)**.** The sequence (a_n) *converges to* $\boldsymbol{a} \in \boldsymbol{X}$ in the metric space (X, d) if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $d(a_n, a) < \epsilon$ for all $n \geq N$.

The following result follows quickly from Proposition 1.1.9; we leave its formal proof as Exercise 1.3.1.

Proposition 1.3.3 (Uniqueness of limits)**.** *Let (X, d) be a metric space. If the sequence (a_n) from X converges to both a and a' in (X, d), then $a = a'$.*

Proposition 1.3.3 allows us to talk about *the* limit of a sequence in the following definition.

Definition 1.3.4 (Convergence, further definitions)**.** The sequence (a_n) from X *converges* in the metric space (X, d) if there is some $a \in X$ such that (a_n) converges to a in (X, d). In this case, a is called the *limit* of (a_n), and we write

$$\lim a_n = a, \lim_{n\to\infty} a_n = a, \text{ or } a_n \to a.$$

If the sequence (a_n) does not converge, then it is said to *diverge*.

We begin with an easy limit.

Example 1.3.5. In the metric space $(\mathbb{R}, d)$, the sequence $(1/n)_{n=1}^{\infty}$ converges to 0.

Proof. Let $\epsilon > 0$ be given. By the Archimedean property, there is an integer $N \in \mathbb{N}$ such that $N > 1/\epsilon$, and thus for all $n \geq N$, we have $d(a_n, 0) = |1/n| < \epsilon$. □

However, this sequence can be made to diverge by changing the metric on $\mathbb{R}$. We leave the proof of the following to the reader as Exercise 1.3.2.

Example 1.3.6. In the metric space $(\mathbb{R}, d_{\text{disc}})$, the sequence $(1/n)_{n=1}^{\infty}$ diverges.

It is frequently useful to recast convergence in arbitrary metric spaces as convergence in $(\mathbb{R}, d)$, as the following result allows us to do.

Proposition 1.3.7. *Let (X, d) be a metric space. The sequence (a_n) from X converges to a in (X, d) if and only if the sequence $(d(a_n, a))$ converges to 0 in $\mathbb{R}$ (with the standard metric).*

Proof. Suppose first that $a_n \to a$ in (X, d). Then for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $d(a_n, a) = |d(a_n, a)| < \epsilon$ for all $n \geq N$, but this is precisely the same as $d(a_n, a) \to 0$ in $\mathbb{R}$. Conversely, if $d(a_n, a) \to 0$ in $\mathbb{R}$, then for every ϵ there is an $N \in \mathbb{N}$ such that $d(a_n, a) = |d(a_n, a)| < \epsilon$ for all $n \geq N$, and this is precisely what it means to have $a_n \to a$ in (X, d). □

Example 1.3.8. The sequence (a_n) defined by $a_n = (1/n, 1/n)$ converges to $(0, 0)$ in the three metric spaces $(\mathbb{R}^2, d_1)$, $(\mathbb{R}^2, d_2)$, and $(\mathbb{R}^2, d_\infty)$.

Proof. We have

$$\begin{aligned} d_1((1/n, 1/n), (0,0)) &= 2/n, \\ d_2((1/n, 1/n), (0,0)) &= \sqrt{2}/n, \text{ and} \\ d_\infty((1/n, 1/n), (0,0)) &= 1/n. \end{aligned}$$

The result follows immediately from Proposition 1.3.7. □

This is an example of a more general fact, that the ℓ^1, ℓ^2, and ℓ^∞ metrics are in some sense equivalent on $\mathbb{R}^d$. We begin with the following consequence of the Cauchy–Schwarz inequality.

Proposition 1.3.9. *For every vector $x \in \mathbb{R}^d$, we have*

$$\|x\|_1 \leq \sqrt{d}\|x\|_2 \leq d\|x\|_\infty \leq d\|x\|_1.$$

A reader who has done Exercise 1.2.6 has already seen the hard part of the proof of Proposition 1.3.9.

Proof. Let x be an arbitrary vector in $\mathbb{R}^d$. By taking $e \in \mathbb{R}^d$ to be the all-1 vector and letting $|x|$ denote the vector obtained by taking the entry-wise absolute value of x, we see that

$$\|x\|_1 = \sum_{k=1}^{d} |x_k| = \langle |x|, e \rangle.$$

By the Cauchy–Schwarz inequality, $|\langle |x|, e \rangle| \leq \|x\|_2 \|e\|_2$, and we know that $\|e\|_2 = \sqrt{d}$, so

$$\|x\|_1 = \langle |x|, e \rangle = |\langle |x|, e \rangle| \leq \|x\|_2 \|e\|_2 = \sqrt{d}\|x\|_2.$$

Similarly, we have

$$\|x\|_2 = \sqrt{\sum x_k^2} \le \sqrt{\sum \|x\|_\infty^2} = \sqrt{d}\|x\|_\infty$$

and

$$\|x\|_\infty \le \sum |x_k| = \|x\|_1.$$

Putting these inequalities together yields the result. □

We can now establish that convergence in $\mathbb{R}^d$ is the same under all three of these metrics.

Theorem 1.3.10 (Equivalence of ℓ^1, ℓ^2, and ℓ^∞)**.** *Let (a_n) denote a sequence from $\mathbb{R}^d$ and $a \in \mathbb{R}^d$. The following are equivalent.*

(i) The sequence (a_n) converges to a in $(\mathbb{R}^d, d_1)$.

(ii) The sequence (a_n) converges to a in $(\mathbb{R}^d, d_2)$.

(iii) The sequence (a_n) converges to a in $(\mathbb{R}^d, d_\infty)$.

Proof. By Proposition 1.3.7, it suffices to show that if any one of $\|a_n - a\|_1$, $\|a_n - a\|_2$, or $\|a_n - a\|_\infty$ converges to 0 in $(\mathbb{R}, d)$, then the other two do as well. This follows immediately from Proposition 1.3.9. □

We conclude by relating convergence in $\mathbb{R}^d$ (under one of these three metrics) to component-wise convergence.

Proposition 1.3.11. *Let (a_n) denote a sequence from $\mathbb{R}^d$ where for each n, $a_n = (a_{n,1}, \ldots, a_{n,d})$. Then (a_n) converges in $(\mathbb{R}^d, d_1)$, $(\mathbb{R}^d, d_2)$, or $(\mathbb{R}^d, d_\infty)$ if and only if, for every $1 \le k \le d$, the sequence $(a_{n,k})_n$ converges in $(\mathbb{R}, d)$. Moreover, if (a_n) does converge in any of these three metric spaces, then*

$$\lim_{n\to\infty} a_n = \left(\lim_{n\to\infty} a_{n,1}, \ \ldots, \ \lim_{n\to\infty} a_{n,d} \right).$$

Proof. By our previous result, it suffices to consider a sequence (a_n) that converges to converges to $x = (x_1, \ldots, x_d)$ in the metric space $(\mathbb{R}^d, d_\infty)$. For every $1 \le k \le d$, we have

$$|a_{n,k} - x_k| \le \|a_n - x\|_\infty.$$

Since we have assumed that $\|a_n - x\|_\infty \to 0$, it follows that $|a_{n,k} - x_k| \to 0$ for every $1 \le k \le d$.

Conversely, suppose that for every $1 \le k \le d$, the sequence $(a_{n,k})$ converges in $(\mathbb{R}, d)$ to x_k and let $x = (x_1, \ldots, x_d)$. Then we have

$$\|a_n - x\|_\infty = \max\{|a_{n,k} - x_k| : 1 \le k \le d\}.$$

Since we know that $|a_{n,k} - x_k| \to 0$ for every $1 \le k \le d$, it follows that $\|a_n - x\|_\infty \to 0$, proving that $(a_n) \to x$ in $(\mathbb{R}^d, d_\infty)$. □

Exercises

Exercise 1.3.1. Prove Proposition 1.3.3.

Exercise 1.3.2. Prove the result stated in Example 1.3.6: the sequence $(1/n)$ diverges in the metric space $(\mathbb{R}, d_{\text{disc}})$.

Exercise 1.3.3. Let (X, d) be a metric space and (a_n) a sequence from X. Prove that (a_n) converges to the point $a \in X$ if and only the sequence $(a_1, a, a_2, a, \dots)$ converges to a.

Exercise 1.3.4. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \pi & \text{if } x = 0, \\ 0 & \text{if } x = \pi, \\ x & \text{otherwise,} \end{cases}$$

and define $d_\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d_\pi(a, b) = |f(a) - f(b)|$. The function d_π is a metric on $\mathbb{R}$ by Exercise 1.1.13. Prove that in the metric space $(\mathbb{R}, d_\pi)$, the sequence $(1/n)$ converges to π.

1.4 Open and closed sets

OPEN SETS are of fundamental importance in all of what follows. We begin with the definition.

Definition 1.4.1. A set $U \subseteq X$ in a metric space (X, d) is *open* if and only if for every $x \in U$ there is an $\epsilon > 0$ such that

$$N_\epsilon(x) \subseteq U.$$

A set is open if every point in the set has a neighborhood that is also in the set.

Closed sets are defined as the complements of open sets, so much of what we prove for open sets can be translated to closed sets.

Definition 1.4.2. A set $K \subseteq X$ in a metric space (X, d) is *closed* if and only if its complement $X \setminus K$ is open.

It is very important to keep in mind that *open and closed are not opposites*. Some sets are both open and closed, while "most" sets are neither. A few examples are in order.

Example 1.4.3. In every metric space (X, d), the sets $\emptyset$ and X are both open and closed.

Proof. The set $\emptyset$ is vacuously open; there is no point $x \in \emptyset$ that could fail the definition. Now consider the set X. For every $x_0 \in X$ and every $\epsilon > 0$, we have

$$N_\epsilon(x_0) = \{x \in X : d(x, x_0) < \epsilon\},$$

so $N_\epsilon(x_0) \subseteq X$ by its very definition. Finally, since $X \setminus \emptyset = X$ is open, $\emptyset$ is closed, and since $X \setminus X = \emptyset$ is open, X is closed. □

Example 1.4.4. In $\mathbb{R}^2$ with the Euclidean metric, the set

$$E = \{(x_1, x_2) : x_1, x_2 > 0\}$$

is open.

Proof. Let $(x_1, x_2) \in E$ be arbitrary and define $\epsilon = \min\{x_1, x_2\}$. By the definition of E, we have $\epsilon > 0$. For any $(y_1, y_2) \in N_\epsilon((x_1, x_2))$, we have

$$\begin{aligned} \epsilon &> d_2((y_1, y_2), (x_1, x_2)) \\ &= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} \\ &\geq \max\{|y_1 - x_1|, |y_2 - x_2|\}. \end{aligned}$$

It follows that $y_1 > x_1 - \epsilon \geq 0$ and $y_2 > x_2 - \epsilon \geq 0$, so $(y_1, y_2) \in E$, completing the proof. □

Example 1.4.5. In $\mathbb{R}$ with the standard metric, the set $[0,1)$ is neither open nor closed.

Proof. This set is not open because for every $\epsilon > 0$, the set $N_\epsilon(0) = (-\epsilon, \epsilon)$ contains negative numbers and thus is not contained in $[0,1)$. This set is not closed because $\mathbb{R} \setminus [0,1) = (-\infty, 0) \cup [1, \infty)$ is not open; for every $\epsilon > 0$, the set $N_\epsilon(1) = (1-\epsilon, 1+\epsilon)$ contains numbers less than 1. □

Next we show that neighborhoods are open in every metric space (though what neighborhoods look like depends on the metric).

Proposition 1.4.6 (Neighborhoods are open)**.** *If (X,d) is a metric space, $x_0 \in X$, and $r > 0$, then the set*

Open balls are indeed open sets.

$$N_r(x_0) = \{x \in X : d(x, x_0) < r\}$$

is an open set.

Proof. We must show, for every $x \in N_r(x_0)$, that there is an $\epsilon > 0$ (depending on x) such that

$$N_\epsilon(x) \subseteq N_r(x_0).$$

Accordingly, let $x \in N_r(x_0)$ be given, so $d(x, x_0) < r$. Set

$$\epsilon = r - d(x, x_0) > 0.$$

Suppose now that $y \in N_\epsilon(x)$, so $d(x,y) < \epsilon$. Using the triangle inequality, we have

$$d(x_0, y) \leq d(x_0, x) + d(x, y) < d(x_0, x) + \epsilon = r.$$

Therefore $y \in N_r(x_0)$, and we have shown that $N_\epsilon(x) \subseteq N_r(x_0)$, completing the proof. □

Proposition 1.4.7 (Finite sets are closed)**.** *If (X,d) is a metric space and A is a finite set, then A is closed.*

Proof. Let (X,d) be a metric space and suppose that $A = \{a_1, a_2, \ldots, a_n\} \subseteq X$. We need to prove that the complement $X \setminus A$ is open. Let $x \in X \setminus A$ be arbitrary. Because $x \notin A$, $d(x, a_k) > 0$ for all $1 \leq k \leq n$ (by part (ii) of the definition of metric spaces). Therefore, the quantity

$$\epsilon = \min\{d(x, a_k) : 1 \leq k \leq n\}$$

is greater than 0. Because $N_\epsilon(x) \subseteq X \setminus A$, it follows that $X \setminus A$ is open and therefore A is closed. □

We conclude by considering unions and intersections of open sets.

Proposition 1.4.8 (Unions of open sets)**.** *In any metric space, the union of any number of open sets is open.*

Proof. Let (X, d) be a metric space containing the family $\mathcal{U}$ of open sets. Consider an arbitrary point x in the union of these sets,

$$x \in \bigcup_{U \in \mathcal{U}} U.$$

By "family of open sets", we mean that the elements of $\mathcal{U}$ are sets themselves, and each of them is open in (X, d).

Because x lies in this union, there is some $U \in \mathcal{U}$ such that $x \in U$. Then because U is open, there is some ϵ such that

$$N_\epsilon(x) \subseteq U \subseteq \bigcup_{U \in \mathcal{U}} U,$$

and this proves that the union is open. □

A corollary of Propositions 1.4.6 and 1.4.8 is that a set is open if and only if it can be expressed as a union of neighborhoods.

Proposition 1.4.9 (Intersections of open sets)**.** *In any metric space, the intersection of any finite number of open sets is open.*

Proof. Let (X, d) be a metric space containing the open sets $U_1, U_2, \ldots,$ U_n. Consider an arbitrary point x in the intersection of these sets,

$$x \in \bigcap_{k=1}^{n} U_k.$$

Because x lies in this intersection, it lies in every U_k. Thus for every k we can find a number $\epsilon_k > 0$ such that $N_{\epsilon_k}(x) \subseteq U_k$. Setting

$$\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_k\} > 0,$$

it follows that

$$N_\epsilon(x) \subseteq \bigcap_{k=1}^{n} U_k,$$

completing the proof. □

Note the difference between Propositions 1.4.8 and 1.4.9. We are allowed arbitrary unions but only finite intersections. For closed sets this is reversed; see Exercises 1.4.4 and 1.4.5.

Exercises

Exercise 1.4.1. Show that the set $E = \{(x, y) \in \mathbb{R}^2 : x \neq y\}$ is open in $(\mathbb{R}^2, d_2)$.

Exercise 1.4.2. Show that the set $E = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is closed in $(\mathbb{R}^2, d_2)$.

Exercise 1.4.3. Let (X, d) be a metric space, $x_0 \in X$, and $r > 0$. Prove that the *closed neighborhood*

$$\overline{N}_r(x_0) = \{x \in X : d(x, x_0) \leq r\}$$

is indeed closed in (X, d). Deduce—without using Proposition 1.4.7—that singleton sets are always closed.

Pay close attention to the overline in $\overline{N}_r(x_0)$; we will see in the next section that $\overline{N_r(x_0)}$ is a potentially different set.

Exercise 1.4.4. Prove that in any metric space, the union of any finite number of closed sets is closed.

Exercises 1.4.3 and 1.4.4 provide another proof that finite sets are closed.

Exercise 1.4.5. Prove that in any metric space, the intersection of any number of closed sets is closed.

Exercise 1.4.6. Determine (with justification) the open subsets of a discrete metric space (X, d_{disc}).

Exercise 1.4.7. Determine (with justification) whether the set

$$\{(x, y) : xy = 1\}$$

is open and/or closed in the metric space $\mathbb{R}^2$ with the Euclidean metric.

Exercise 1.4.8. The metrics d and d' on the set X are said to be *strongly equivalent* if there are real numbers $c, C > 0$ such that

$$cd(x, y) \leq d'(x, y) \leq Cd(x, y)$$

for all $x, y \in X$. Prove that if d and d' are strongly equivalent on the set X, then the metric spaces (X, d) and (X, d') have the same open sets.

Exercise 1.4.9. Using Proposition 1.3.9, prove that the metrics d_1, d_2, and d_∞ are all strongly equivalent on $\mathbb{R}^d$.

1.5 Interior, exterior, boundary, and closure

Here we study further aspects of point-set topology. In addition to being useful on their own, these concepts offer alternative ways to look at open and closed sets.

Definition 1.5.1 (Interior)**.** Let (X, d) be a metric space and $E \subseteq X$. The *interior* of E is the set of points that have a neighborhood contained in E, and is denoted by E° or $\operatorname{int} E$. Thus,

$$E^\circ = \operatorname{int} E = \{x \in X : \text{there is an } \epsilon > 0 \text{ for which } N_\epsilon(x) \subseteq E\}.$$

Proposition 1.5.2 (Properties of interiors)**.** *Let (X, d) be a metric space and $E \subseteq X$. The following all hold.*

(a) We have $E^\circ \subseteq E$.
(b) The interior E° is an open set.
(c) We have $E^\circ = E$ if and only if E is itself open.

Proof. Part (a) is straight-forward: if $x \in E^\circ$ then there is some $\epsilon > 0$ so that $x \in N_\epsilon(x) \subseteq E$.

To prove part (b), consider an arbitrary point $x \in E^\circ$. By definition, there is some $\epsilon > 0$ so that $N_\epsilon(x) \subseteq E$. We claim that this implies that $N_{\epsilon/2}(x) \subseteq E^\circ$. To prove this, let $y \in N_{\epsilon/2}(x)$ be arbitrary. Since $N_{\epsilon/2}(y) \subseteq N_\epsilon(x) \subseteq E$, we see that $y \in E^\circ$, as claimed.

Part (c) now follows easily from the other two parts. If $E = E^\circ$, then since E° is open by part (b), E is open. Conversely, suppose that E is open and let $x \in E$ be arbitrary. Since E is open, there is some $\epsilon > 0$ so that $N_\epsilon(x) \subseteq E$. This shows that $x \in E^\circ$, and since x was an arbitrary element of E, we see that $E \subseteq E^\circ$. Since part (a) gives us the reverse inclusion ($E^\circ \subseteq E$), we conclude that $E^\circ = E$. □

By parts (a) and (b) of Proposition 1.5.2, E° is an open set contained in E. Our next result shows that it can be characterized as the *largest* open set contained in E.

Proposition 1.5.3. *Let (X, d) be a metric space and $E \subseteq X$. If the set $U \subseteq E$ is open, then $U \subseteq E^\circ$.*

Proof. Suppose that the set $U \subseteq E$ is open and let $x \in U$ be arbitrary. Because U is open, there is some $\epsilon > 0$ so that $N_\epsilon(x) \subseteq U \subseteq E$, which means that $x \in E^\circ$, proving the result. □

The exterior of a set can be defined as the *interior of its complement*.

Definition 1.5.4 (Exterior)**.** Let (X, d) be a metric space and $E \subseteq X$. The *exterior* of E is the set of points that have a neighborhood contained in $X \setminus E$, and is denoted by $\operatorname{ext} E$. Thus,

$$\operatorname{ext} E = \{x \in X : \text{there is an } \epsilon > 0 \text{ for which } N_\epsilon(x) \subseteq X \setminus E\}.$$

Because $\operatorname{ext} E = \operatorname{int}(X \setminus E)$, the following properties follow immediately from Proposition 1.5.2 applied to $X \setminus E$.

Proposition 1.5.5 (Properties of exteriors)**.** *Let (X, d) be a metric space and $E \subseteq X$. The following all hold.*

(a) We have $\operatorname{ext} E \subseteq X \setminus E$.

(b) The exterior $\operatorname{ext} E$ is an open set.

(c) We have $\operatorname{ext} E = X \setminus E$ if and only if E is itself closed.

Note that because $E^\circ \subseteq E$ and $\operatorname{ext} E \subseteq X \setminus E$, no point can lie in both the interior and the exterior of a set. Points may lie in neither the interior nor the exterior, however.

Definition 1.5.6 (Boundary)**.** Let (X, d) be a metric space and $E \subseteq X$. The *boundary* of E is the set of points that lie neither in the interior nor the exterior of E, and is denoted by ∂E.

It follows from the definition that

$$\partial E = \partial(X \setminus E).$$

We know that the interior of E is contained in E and the exterior of E is contained in its complement $X \setminus E$. The boundary points may or may not be contained in E, but in either case we have

$$E \subseteq (E^\circ \cup \partial E) \quad \text{and} \quad X \setminus E \subseteq (\operatorname{ext} E \cup \partial E)\,.$$

By negating the definitions of interior and exterior, we arrive at the following result. We leave its proof to the reader as Exercise 1.5.6.

Proposition 1.5.7 (Characterization of boundary points)**.** *Let (X, d) be a metric space and $E \subseteq X$. We have $x \in \partial E$ if and only if for every $\epsilon > 0$, the neighborhood $N_\epsilon(x)$ contains both a point of E and a point of $X \setminus E$.*

The notion of boundary allows us to give an alternative characterization of open and closed sets.

Proposition 1.5.8 (Open and closed in terms of boundary)**.** *Let (X, d) be a metric space and $E \subseteq X$. Then we have the following.*

(a) The set E is open if and only if it contains none of its boundary: $E \cap \partial E = \emptyset$.

(b) The set E is closed if and only if it contains all of its boundary: $\partial E \subseteq E$.

Proof. By Proposition 1.5.2 (c), the set E is open if and only if $E = E^\circ$. Since every point of E lies either in the interior of E or on its boundary, we have $E = E^\circ$ if and only if $E \cap \partial E = \emptyset$, proving part (a).

By Proposition 1.5.5 (c), the set E is closed if and only if $\operatorname{ext} E = X \setminus E$. Since every point of $X \setminus E$ lies either in the exterior of E or on its boundary, we have $\operatorname{ext} E = X \setminus E$ if and only if $\partial E \subseteq E$, proving part (b). □

Finally we come to the closure of a set. Just as the interior of a set can be defined as its largest open subset, the closure can be defined as its smallest closed superset.

Definition 1.5.9 (Closure). Let (X,d) be a metric space and $E \subseteq X$. The *closure* of E, denoted by $\overline{E}$, is defined as

$$\overline{E} = E \cup \partial E = X \setminus \operatorname{ext} E.$$

In some other mathematical contexts, $\overline{E}$ is used to denote the complement of E, but in analysis we generally reserve $\overline{E}$ to denote its closure. We use $X \setminus E$ to denote the complement of the set E.

There are several equivalent definitions of the closure. The reader is asked to prove the following analogue of Proposition 1.5.7 in Exercise 1.5.7.

Proposition 1.5.10 (Characterization of closure). *Let (X,d) be a metric space and $E \subseteq X$. We have $x \in \overline{E}$ if and only if for every $\epsilon > 0$, the neighborhood $N_\epsilon(x)$ contains a point of E.*

We also have an analogue of Propositions 1.5.2 and 1.5.5 for closures.

Proposition 1.5.11 (Properties of closures). *Let (X,d) be a metric space and $E \subseteq X$. The following all hold.*

(a) We have $E \subseteq \overline{E}$.
(b) The closure $\overline{E}$ is a closed set.
(c) We have $\overline{E} = E$ if and only if E is itself closed.

Proof. Part (a) is immediately from the definition of $\overline{E}$. Part (b) follows because $X \setminus \overline{E} = \operatorname{ext} E$ is open by Proposition 1.5.5 (b). Part (c) follows immediately from the other two parts. □

Exercises

Exercise 1.5.1. Determine the interior, closure and boundary of an interval $(a,b]$ in $\mathbb{R}$. What is the boundary of $(0,1]$ as a subset of the metric space $(0,\infty)$ with the standard metric?

Exercise 1.5.2. Let E be a set in a discrete metric space (X, d_{disc}). Determine the interior, boundary and closure of E.

Exercise 1.5.3. Let (X,d) be a metric space and $E \subseteq X$. Prove that the point x lies in $\overline{E}$ if and only if there is a sequence (a_n) from E that converges to x.

Exercise 1.5.4. Let (X,d) be a metric space and $E \subseteq X$. Prove that

$$E^\circ = \bigcup \{U \subseteq E : U \text{ is open}\}.$$

Exercise 1.5.5. Let (X,d) be a metric space and $E \subseteq X$. Prove that

$$\overline{E} = \bigcap \{K \supseteq E : K \text{ is closed}\}.$$

Exercise 1.5.6. Prove Proposition 1.5.7.

Exercise 1.5.7. Prove Proposition 1.5.10.

Exercise 1.5.8. Let X be a nonempty set and d the discrete metric. Fix a point $x_0 \in X$. Is it true that

$$\overline{N_1(x_0)} = \{x \in X : d(x, x_0) \leq 1\}?$$

In Exercise 1.4.3 we defined

$$\overline{N}_1(x_0) = \{x \in X : d(x, x_0) \leq 1\}.$$

The notation $\overline{N_1(x_0)}$ in Exercise 1.5.8 denotes instead the closure of the open neighborhood $N_1(x_0)$.

Exercise 1.5.9. Determine $\partial\partial(0, 1]$ in $(\mathbb{R}, d)$.

Exercise 1.5.10. Let

$$S = \{(x, 0) \in \mathbb{R}^2 : 0 < x < 1\}.$$

Find ∂S and $\partial\partial S$ in $(\mathbb{R}^2, d_2)$.

1.6 Relative topology

No matter what metric space we are working with, finite sets are always closed (Proposition 1.4.7). Beyond this, though, the choice of metric space affects which sets are open and closed, and therefore also affects the definitions of interior, exterior, boundary, and closure. When we change the metric on a space, all bets are off. The discrete metric on a set behaves very differently from other metrics. However, we can say something about what happens when we keep the metric the same, but change the *space*.

First we consider a few examples. The first is trivial, but worth keeping in mind.

Example 1.6.1. Let (X, d) be a metric space and $E \subseteq X$. By restricting the metric d to E, we obtain the metric space $(E, d|_{E\times E})$. No matter what topological properties E had in the metric space (X, d), in the metric space $(E, d|_{E\times E})$, the set E is both open and closed.

For a more interesting example, we consider the difference between the real line on its own and the real line embedded in the plane.

Example 1.6.2. Let $Y = \{(0, y) : y \in \mathbb{R}\}$ denote the y-axis in the plane with the Euclidean metric d_2. By restricting this metric to Y, we obtain the subspace $(Y, d_2|_{Y\times Y})$. This space is, essentially, the real line with the standard metric, $(\mathbb{R}, d)$. Now consider the open line segment from $(0, 0)$ to $(0, 1)$,

$$E = \{(0, y) : 0 < y < 1\}.$$

When we view E as set in the metric space $(Y, d_2|_{Y\times Y})$, we see that it is open (it is essentially the same as the open interval $(0, 1)$ in $\mathbb{R}$ with the standard metric). However, in the metric space $(\mathbb{R}^2, d_2)$, every point of E is a boundary point, so E is *not* open. (It is also not closed; why?)

$(Y, d_2|_{Y\times Y})$ and $(\mathbb{R}, d)$ are not literally the *same* metric space, but they are *isometric*.

We now introduce some terminology for this.

Definition 1.6.3 (Relative topology)**.** Let (X, d) be a metric space, and $E \subseteq Y \subseteq X$. We say that E is *relatively open with respect to* $\boldsymbol{Y}$ if E is open in the metric space $(Y, d|_{Y\times Y})$. We analogously say that E is *relatively closed with respect to* $\boldsymbol{Y}$ if E is closed in the metric space $(Y, d|_{Y\times Y})$.

The main result of this section gives us an alternative characterization of relatively open and closed sets.

Proposition 1.6.4 (Characterization of relatively open/closed sets)**.** *Let (X, d) be a metric space and $E \subseteq Y \subseteq X$.*

(a) The set E is relatively open with respect to Y if and only if there is a set $U \subseteq X$ that is open in (X, d) such that $E = U \cap Y$.

For example, in Example 1.6.2, one could take $U = N_{1/2}((0, 1/2))$. This is an open set in $(\mathbb{R}^2, d_2)$ and $U \cap Y = E$.

(b) *The set E is relatively closed with respect to Y if and only if there is a set $K \subseteq X$ that is closed in (X, d) such that $E = K \cap Y$.*

Proof. We prove part (a), leaving part (b) for Exercise 1.6.1. In our proof we need to consider neighborhoods in both (X, d) and $(Y, d|_{Y\times Y})$, so for any point z and any $\epsilon > 0$, we define

$$N_\epsilon^X(z) = \{x \in X : d(x,z) < \epsilon\} \text{ and}$$
$$N_\epsilon^Y(z) = \{y \in Y : d|_{Y\times Y}(y,z) < \epsilon\} = N_\epsilon^X(z) \cap Y.$$

Suppose first that $E = U \cap Y$ for some set $U \subseteq X$ that is open in (X, d). This implies that $E \subseteq U$, and since U is open in (X, d), for every point $z \in E$ there must be some radius $\epsilon_z > 0$ for which $N_{\epsilon_z}^X(z) \subseteq U$. This implies that

$$N_{\epsilon_z}^Y(z) = N_{\epsilon_z}^X(z) \cap Y \subseteq U \cap Y = E,$$

and this implies that E is open in $(Y, d|_{Y\times Y})$.

Conversely, suppose that E is open in $(Y, d|_{Y\times Y})$. This means that for each point $z \in E$, there is some radius $\epsilon_z > 0$ such that $N_{\epsilon_z}^Y(z) \subseteq E$, and from this we see that

$$E = \bigcup_{z\in E} N_{\epsilon_z}^Y(z).$$

Now define

$$U = \bigcup_{z\in E} N_{\epsilon_z}^X(z).$$

Since U is a union of neighborhoods, it is open in (X, d). Moreover, we have

$$U \cap Y = \left(\bigcup_{z\in E} N_{\epsilon_z}^X(z)\right) \cap Y = \bigcup_{z\in E} N_{\epsilon_z}^Y(z) = E,$$

completing the proof of this direction. □

A union of neighborhoods is always open because Proposition 1.4.6 states that neighborhoods are open, and Proposition 1.4.8 states that the union of any number of open sets is open.

Exercises

Exercise 1.6.1. Prove part (b) of Proposition 1.6.4.

Exercise 1.6.2. State and prove a characterization of *relatively clopen* sets akin to Proposition 1.6.4.

1.7 Cauchy sequences and completeness

THE DEFINITION of Cauchy sequences in general metric spaces is a straightforward generalization of their definition on the real line.

Definition 1.7.1. A sequence (a_n) in a metric space (X, d) is *Cauchy* if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(a_m, a_n) < \epsilon$ for all $m, n \geq N$.

Exercise 1.7.1 asks you to characterize the Cauchy sequences of discrete metric spaces.

Proposition 1.7.2. *Convergent sequences are Cauchy; that is, if (a_n) is a convergent sequence in the metric space (X, d), then (a_n) is Cauchy.*

Proof. Suppose that (a_n) is a convergent sequence in the metric space (X, d) and set $a = \lim a_n$. Let $\epsilon > 0$ be arbitrary. There is an $N \in \mathbb{N}$ such that $d(a_n, a) < \epsilon/2$ for all $n \geq N$. Therefore for $m, n \geq N$ we have (by the triangle inequality)

$$d(a_m, a_n) \leq d(a_m, a) + d(a, a_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

proving the result. □

Importantly, *Cauchy sequences do not necessarily converge*.

Example 1.7.3. Consider the sequence of successively better decimal approximations of π,

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots.$$

Viewed as a sequence in $(\mathbb{R}, d)$, this sequence is Cauchy (why?) and converges to π. However, viewed as a sequence in $(\mathbb{Q}, d)$, this sequence is still Cauchy, but no longer converges (why?).

In some sense, the Cauchy sequences are the sequences that "should converge". The metric spaces where every sequence that "should converge" actually does are special.

Definition 1.7.4. The metric space (X, d) is *complete* if every Cauchy sequence in X converges (in X).

Exercise 1.7.2 asks you to prove that every discrete metric space is complete.

We just saw that the metric space $(\mathbb{Q}, d)$ is not complete. We assume that the reader has already seen that $(\mathbb{R}, d)$ is complete.

Theorem 1.7.5. *The metric space $(\mathbb{R}, d)$ is complete.*

Many proofs of of Theorem 1.7.5 use the *least upper bound property*: every nonempty bounded subset of $\mathbb{R}$ has a least upper bound. Proofs of completeness for other metric spaces will be very different, since most spaces don't have notions of "least" or "upper bound".

Next we consider subsequences, for which we give the formal definition below.

Definition 1.7.6. Suppose that (a_n) is a sequence of points in the metric space (X, d). For any choice of indices

$$1 \leq n_1 < n_2 < n_3 < \cdots,$$

the sequence $(a_{n_j})_{j=1}^{\infty}$ is a *subsequence* of (a_n).

Proposition 1.7.7. *Let (a_n) be a Cauchy sequence in the metric space (X, d). If some subsequence of (a_n) converges in (X, d) to the point a, then (a_n) converges in (X, d) to a.*

Proof. See Exercise 1.7.3 □

Complete metric spaces have nice properties, as we will see throughout these notes. For now, we note that these spaces are *intrinsically closed*—no matter what larger space you embed a complete metric space in, it will always be closed.

Proposition 1.7.8 (Complete sets are closed)**.** *Let (X, d) be a metric space, and $Y \subseteq X$. If $(Y, d|_{Y \times Y})$ is complete, then Y is closed in (X, d).*

Proof. Recall that a set is closed if and only if it contains its boundary (Proposition 1.5.8 (b)). Let $x \in \partial Y$ be an arbitrary boundary point of Y in (X, d). Because x is a boundary point, for every $\epsilon > 0$, the neighborhood $N_\epsilon(x)$ intersects both Y and $X \setminus Y$ (by Proposition 1.5.7). We can therefore construct a sequence (a_n) by choosing, for every integer $n \geq 1$,

$$a_n \in N_{1/n}(x) \cap Y.$$

This sequence converges in the space (X, d) to the point x. Therefore it is a Cauchy sequence, so since $(Y, d|_{Y \times Y})$ is complete, (a_n) must also converge in Y. Since limits of sequences are unique, (a_n) can only possibly converge to x in $(Y, d|_{Y \times Y})$, and thus we must have $x \in Y$. This proves that $\partial Y \subseteq Y$, and so Y is closed. □

Conversely, we have the following result.

Proposition 1.7.9. *Suppose that (X, d) is a complete metric space. If the set Y is closed in (X, d), then the subspace $(Y, d|_{Y \times Y})$ is complete.*

Proposition 1.7.9 says that every closed subset of a complete space is complete.

Proof. Let Y be a closed set in the complete metric space (X, d) and consider an arbitrary Cauchy sequence (a_n) in Y. Because (a_n) is Cauchy, it converges in (X, d), say to the point $a \in X$.

The definition of convergence says that for every $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $d(a_n, a) < \epsilon$ for all $n \geq N$. Therefore, for every radius $\epsilon > 0$, the neighborhood $N_\epsilon(a)$ contains at least one point of Y (it contains all the points a_n with $n \geq N$, but they might all be the same). This means that $a \in \overline{Y}$ (by Proposition 1.5.10). Since $\overline{Y} = Y$ (because Y is closed), we have that $a \in Y$, and so the sequence (a_n) also converges in $(Y, d|_{Y \times Y})$. This proves that $(Y, d|_{Y \times Y})$ is complete. □

Proposition 1.5.10 states that $x \in \overline{E}$ if and only if for every $\epsilon > 0$, the neighborhood $N_\epsilon(x)$ contains a point of E.

Exercises

Exercise 1.7.1. Prove that a sequence (a_n) in a discrete metric space (X, d_{disc}) is Cauchy if and only if it is eventually constant.

Exercise 1.7.2. Prove that every discrete metric space (X, d_{disc}) is complete.

Exercise 1.7.3. Prove Proposition 1.7.7.

Exercise 1.7.4. The *diameter* of a nonempty set E in a metric space (X, d_X) is defined as

$$\text{diam}(E) = \sup\{d_X(x, y) : x, y \in E\}.$$

(If the set of values $d_X(x, y)$ is not bounded, then we define $\text{diam}(E) = +\infty$.)

Prove that if (X, d_X) is a complete metric space, $C_1 \supseteq C_2 \supseteq \cdots$ is a nested decreasing sequence of nonempty closed sets in X, and the sequence $(\text{diam}(C_n))_{n=1}^{\infty}$ converges to 0 in $\mathbb{R}$ (with the usual metric), then

$$\bigcap_{n=1}^{\infty} C_n$$

contains precisely one point.

Exercise 1.7.5. Show that the result in Exercise 1.7.4 fails if any of the hypotheses—completeness, closedness of the sets C_n, or that the diameters tend to 0—are removed.