2 Compactness

2.1 Open covers

WE NOW COME to the important notion of compactness.

Definition 2.1.1 (Compact set)**.** Let (X, d) be a metric space and $S \subseteq X$ a set. An *open cover* of S is a family $\mathcal{U}$ of open sets such that

$$S \subseteq \bigcup_{U \in \mathcal{U}} U.$$

The set K is *compact* if for every open cover $\mathcal{U}$ of K, there is a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that

$$K \subseteq \bigcup_{U \in \mathcal{V}} U.$$

In this case, $\mathcal{V}$ is called a *finite subcover of* K.

The short way to describe compactness is that every open cover admits a *finite subcover*.

Sometimes we talk about a space itself being compact.

Exercise 2.1.1 asks you to prove that finite sets are always compact.

Definition 2.1.2 (Compact space)**.** The metric space (X, d) is called *compact* if the set X is compact in (X, d).

Example 2.1.3. The set $K = \{1, 1/2, 1/3, \ldots, 1/n, \ldots, 0\}$ is compact in the metric space $(\mathbb{R}, d)$.

Example 2.1.3 is generalized by Exercise 2.1.3. Exercise 2.1.4 implies that the set K in this example is *not* compact in $(\mathbb{R}, d_{\text{disc}})$.

Proof. Let $\mathcal{U}$ be any open cover of K. Then there must be a set $U_0 \in \mathcal{U}$ with $0 \in U_0$. Since U_0 is open, there is an $\epsilon > 0$ such that $N_\epsilon(0) \subseteq U_0$. Choose $N \in \mathbb{N}$ so that $1/N < \epsilon$. Thus $1/n \in N_\epsilon(0) \subseteq U_0$ for all $n \geq N$. For each $1 \leq n \leq N-1$, there is some set $U_n \in \mathcal{U}$ containing $1/n$. It follows that $\mathcal{V} = \{U_0, U_1, U_2, \ldots, U_{N-1}\} \subseteq \mathcal{U}$ is a finite subcover of K. Therefore K is compact. □

Often it is convenient to view covers as an indexed family of sets. In this case an open cover of the set S consists of an index set I and a collection of open sets $\mathcal{U} = \{U_i : i \in I\}$ whose union contains S. A subcover is then a collection $\mathcal{V} = \{U_j : j \in J\}$, for some subset $J \subseteq I$. A set K is compact if, for each collection $\{U_i : i \in I\}$ such that

$$K \subseteq \bigcup_{i \in I} U_i,$$

there is a finite subset $J \subseteq I$ such that

$$K \subseteq \bigcup_{j \in J} U_j.$$

To show that a set is *not* compact, we must exhibit an open cover that does not admit a finite subcover.

Example 2.1.4. Show that the set $(0,1]$ in the metric space $(\mathbb{R}, d)$ is not compact.

For another non-example of compactness, do Exercise 2.1.4, which states that a subset K of a discrete metric space X is compact if and only if K is finite.

Proof. Consider the family of sets defined by

$$U_n = \left(2^{-n}, 2\right)$$

for all $n \in \mathbb{N}$. It is readily checked that

$$(0,1] \subseteq \bigcup_{n=0}^{\infty} U_n,$$

and of course each U_n is open. Thus $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is an open cover of $(0,1]$.

Why doesn't our proof that $(0,1]$ is not compact apply to the closed interval $[0,1]$? (We'll see in the next section that $[0,1]$ *is* compact.)

Now suppose that $J \subseteq \mathbb{N}$ is a finite set. This means that there is an integer N such that $J \subseteq \{0, 1, 2, \dots, N\}$, and therefore

$$\bigcup_{j \in J} U_j \subseteq \bigcup_{j=0}^{N} U_j = \left(2^{-N}, 2\right).$$

Thus there is no finite subset J of the index set $\mathbb{N}$ such that

$$(0,1] \subseteq \bigcup \{U_j : j \in J\},$$

and we conclude that $(0,1]$ is not compact. □

Our next result is frequently useful for establishing that a given set is compact.

Proposition 2.1.5. *In every metric space, every closed subset of a compact set is compact.*

Proof. Let (X, d) be a metric space, let $K \subseteq X$ be compact in (X, d), and suppose that C is a closed set in (X, d) satisfying $C \subseteq K$.

Let $\mathcal{U}$ be an arbitrary open cover of C, so we need to find a finite subcover. Since C is closed, its complement $X \setminus C$ is open. Therefore,

$$\mathcal{U} \cup \{X \setminus C\}$$

is an open cover of X, and thus also of K. Because K is compact, there is a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \cup \{X \setminus C\}$ covers K, and thus also C. We certainly don't need $X \setminus C$ to cover C, so it follows that $\mathcal{V} \subseteq \mathcal{U}$ is a finite subcover of C, proving that C is indeed compact. □

Our final result of the section shows that compactness is *intrinsic* and thus, unlike with open and closed sets, we can speak of compact sets without worrying about the ambient space (although we must still worry about the *metric*, because changing metrics can change which sets are compact).

Theorem 2.1.6. *Let (X, d) be a metric space. The set $K \subseteq Y \subseteq X$ is compact in $(Y, d|_{Y \times Y})$ if and only if it is compact in (X, d).*

Proof. First suppose that K is compact in the smaller space $(Y, d|_{Y \times Y})$. To prove that K is compact in (X, d), let $\mathcal{U}$ be an open cover of K in (X, d). Define

$$\mathcal{W} = \{U \cap Y : U \in \mathcal{U}\}.$$

Each of the sets $U \cap Y$ in $\mathcal{W}$ is relatively open with respect to Y because it is the intersection of Y with an open set of (X, d), so each set $U \cap Y$ in $\mathcal{W}$ is open in $(Y, d|_{Y \times Y})$. Moreover, since $K \subseteq Y$, it follows that $\mathcal{W}$ is an open cover of K in $(Y, d|_{Y \times Y})$. By the compactness of K in $(Y, d|_{Y \times Y})$, it follows that there is a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that

$$K \subseteq \bigcup_{U \in \mathcal{V}} (U \cap Y).$$

Thus $\mathcal{V}$ is a finite subcover for K in the space (X, d).

Conversely, suppose that K is compact in the larger space (X, d). Let $\mathcal{U}$ be an open cover of K in $(Y, d|_{Y \times Y})$. For each set $U \in \mathcal{U}$, since it is open in $(Y, d|_{Y \times Y})$, there is an open set U' in (X, d) such that $U = U' \cap Y$. The family $\mathcal{U}' = \{U' : U \in \mathcal{U}\}$ is thus an open cover of K in (X, d). Because K is compact in (X, d), there must be a finite subcover of $\mathcal{U}'$. Thus there is a finite subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that

$$K \subseteq \bigcup_{U \in \mathcal{V}} U'.$$

Since $K \subseteq Y$, $K \cap Y = K$, so intersecting both sides of the above inclusion with Y, we obtain

$$K \subseteq \bigcup_{U \in \mathcal{V}} (U' \cap Y) = \bigcup_{U \in \mathcal{V}} U.$$

This shows that $\mathcal{V}$ is a finite subcover of K in the space $(Y, d|_{Y\times Y})$, completing the proof. □

Exercises

Exercise 2.1.1. Prove that in every metric space, finite sets are always compact.

Exercise 2.1.2. Prove that in every metric space, finite unions of compact sets are always compact.

Exercise 2.1.3. Prove that if (X, d) is a metric space and $(a_n)_{n=1}^{\infty}$ is a sequence in X that converges to $a \in X$, then the set $\{a_1, a_2, \dots, a\}$ is compact.

Exercise 2.1.4. Show that a set in a discrete metric space (X, d_{disc}) is compact if and only if it is finite.

Exercise 2.1.5 (The finite intersection property)**.** Suppose that (X, d) is a compact metric space and that $\mathcal{C}$ is a family of closed sets of X. Prove that if

$$\bigcap_{C\in\mathcal{F}} C \neq \emptyset$$

for each finite subfamily $\mathcal{F} \subseteq \mathcal{C}$, then in fact

$$\bigcap_{C\in\mathcal{C}} C \neq \emptyset.$$

Consider taking complements in Exercise 2.1.5.

Exercise 2.1.6 (The nested intersection property)**.** Use the finite intersection property to deduce the nested intersection property: if $C_1 \supseteq C_2 \supseteq$ is a nested decreasing sequence of non-empty compact sets in a metric space (X, d), then $\bigcap C_n$ is non-empty as well.

Exercise 2.1.7. Give an example to show that the finite intersection property of Exercise 2.1.5 does not necessarily hold if (X, d) is not compact.

Exercise 2.1.8. Suppose that the metric space (X, d) contains a (countable) sequence K_1, K_2, ... of compact subsets such that $X = \cup K_n$. Prove that every open cover of X has a finite or countable subcover.

Inspired by Exercise 2.1.8, can you give an example of a metric space (X, d) that is *not* a countable union of compact sets?

2.2 Sequential compactness

We now introduce the notion of sequential compactness, which we will prove is equivalent to compactness in this section. We begin with the definition, which should remind readers of the Bolzano–Weierstrass theorem (a connection we make explicit in the next section).

Definition 2.2.1 (Sequentially compact sets)**.** Let (X, d) be a metric space. The set $K \subseteq X$ is *sequentially compact* if every sequence in K has a subsequence that converges to a limit in K.

Expanding on the definition a bit, the set K is sequentially compact if for every sequence (a_n) from K, there is some $a \in K$ for which there is a subsequence (a_{n_k}) that converges to a.

Note that definition of sequential compactness is *intrinsic* in that it depends only on the set K and the metric, not on the ambient space X. We prove—one direction at a time—that compactness and sequential compactness are equivalent for metric spaces.

Proposition 2.2.2. *Every compact subset of a metric space is sequentially compact.*

Proof. Let (X, d) be a metric space and suppose that the set $K \subseteq X$ is compact. Further let $(a_n)_{n=0}^{\infty}$ be a sequence from K. For every $x \in X$ and $\epsilon > 0$, we consider the set of indices

$$I_\epsilon(x) = \{n \in \mathbb{N} : d(a_n, x) < \epsilon\}$$

for which a_n is within ϵ of x.

First (in order to rule this case out), we assume that for every point $x \in K$, there is an $\epsilon_x > 0$ for which the set $I_{\epsilon_x}(x)$ is finite. In this case, the family

$$\{N_{\epsilon_x}(x) : x \in K\}$$

of neighborhoods is an open cover of K. Since K is compact, this cover must admit a finite subcover. Therefore there must be a finite subset $F \subseteq K$ such that $\{N_{\epsilon_x}(x) : x \in F\}$ covers K, and thus

$$K \subseteq \bigcup_{x \in F} N_{\epsilon_x}(x).$$

Now since every term of the sequence (a_n) lies in K, every term of this sequence must lie in at least one of these finitely many neighborhoods. In other words, for every $n \in \mathbb{N}$, there must be at least one point $x \in F$ such that $n \in I_{\epsilon_x}(x)$. However, since there are only finitely many points in F, this implies that at least one of the sets $I_{\epsilon_x}(x)$ must be infinite, a contradiction.

Therefore we may now assume that there is at least one point $x \in K$ such that the set $I_\epsilon(x)$ is infinite for every $\epsilon > 0$. This condition guarantees the existence of a subsequence that converges to x, as we show now.

Choose an index n_1 so that $d(a_{n_1}, x) < 1$. This can be done because there are infinitely many indices in the set $I_1(x)$. Now, because there are infinitely many indices in the set $I_{1/2}(x)$, we can choose an index $n_2 > n_1$ such that $d(a_{n_2}, x) < 1/2$. We can then find an index $n_3 > n_2$ such that $d(a_{n_3}, x) < 1/3$. Continuing in this manner, we construct a subsequence of (a_n) that converges to $x \in K$, establishing that K is indeed sequentially complete. □

The remainder of this section is devoted to proving the converse of Proposition 2.2.2. We begin with a lemma.

Lemma 2.2.3 (Lebesgue number lemma)**.** *Let (X, d) be a metric space. If the set $K \subseteq X$ is sequentially compact, then for every open cover $\mathcal{U}$ of K, there is a number $\delta > 0$ (depending only on $\mathcal{U}$) such that for every point $x \in K$ there is some set $U \in \mathcal{U}$ with $N_\delta(x) \subseteq U$.*

Since $\mathcal{U}$ is an open cover of K, we know that for every $x \in K$ there is some $\delta_x > 0$ and $U \in \mathcal{U}$ such that $N_{\delta_x}(x) \subseteq U$. However, the Lebesgue number lemma shows that we can choose *the same* δ for every point. This should be somewhat surprising.

Proof. Suppose to the contrary that there is no such number $\delta > 0$. Thus for every integer $n \geq 1$ there is a point $x_n \in K$ such that $N_{1/n}(x_n)$ is not a subset of any of the sets in $\mathcal{U}$. Because K is sequentially compact, the sequence (x_n) has a subsequence that converges in K; suppose that the subsequence $(x_{n_j})_{j=1}^{\infty}$ converges to $x \in K$.

There is some open set $U_x \in \mathcal{U}$ with $x \in U_x$, so there is some $\epsilon > 0$ such that $N_\epsilon(x) \subseteq U_x$. Choose k such that both $1/n_k < \epsilon/2$ and $d(x_{n_k}, x) < \epsilon/2$. Then

$$N_{1/n_k}(x_{n_k}) \subseteq N_{\epsilon/2}(x_{n_k}) \subseteq N_\epsilon(x) \subseteq U_x,$$

but this is a contradiction to our assumption that $N_{1/n}(x_n)$ is not a subset of any of the sets in $\mathcal{U}$, and this contradiction completes our proof. □

We can now complete the proof that compactness and sequential compactness are equivalent (for metric spaces).

Proposition 2.2.4. *Every sequentially compact subset of a metric space is compact.*

Proof. Let (X, d) be a metric space and suppose that the set $K \subseteq X$ is sequentially compact. Further let $\mathcal{U}$ be an arbitrary open cover of K. From the Lebesgue number lemma, we know that there is some number $\delta > 0$ such that for every $x \in K$ there is some set $U \in \mathcal{U}$ such that $N_\delta(x) \subseteq U$.

If there were some finite subset $F \subseteq K$ such that

$$K \subseteq \bigcup_{x \in F} N_\delta(x),$$

then we would be done, because each of the finite number of sets $N_\delta(x)$ for $x \in F$ is contained in one of the sets in $\mathcal{U}$, so we could find a finite subcover of K.

Now suppose to the contrary that the above does not hold. Choose $x_0 \in K$ arbitrarily, and then for each $n \in \mathbb{N}$, choose

$$x_{n+1} \in K \setminus \left(\bigcup_{j=0}^{n} N_\delta(x_j) \right).$$

We can always choose such a point x_{n+1} because we have assumed that

$$K \not\subseteq \bigcup_{j=0}^{n} N_\delta(x_j)$$

for every $n \in \mathbb{N}$. Therefore we have found a sequence $(x_n)_{n=1}^{\infty}$ such that $d(x_j, x_k) \geq \delta$ for all indices $j \neq k$. However this means that (x_n) cannot have a convergent subsequence, which contradictions our hypothesis that K is sequentially compact. This contradiction shows that there is a finite subset $F \subseteq K$ such that

$$K \subseteq \bigcup_{x \in F} N_\delta(x),$$

and we have already shown how that implies that K is compact. □

More generally, a set E is said to be *totally bounded* if for every number $\epsilon > 0$, there is a finite subset $F \subseteq E$ such that

$$E \subseteq \bigcup_{x \in F} N_\epsilon(x).$$

Our proof shows that sequentially compact sets are always totally bounded.

Exercises

Exercise 2.2.1. Let (X, d) be a metric space and suppose that there is a number $r > 0$ and a sequence (a_n) from X such that $d(a_n, a_m) \geq r$ for $n \neq m$. Prove that X is not compact.

Exercise 2.2.2. Prove that every compact subset of a metric space is complete.

For Exercise 2.2.2, recall that if a Cauchy sequence has a subsequence that converges to the point x, then the entire sequence converges to x.

Exercise 2.2.3. The metric space (X, d) is called *separable* if there is a finite or countable set $S \subseteq X$ such that $\overline{S} = X$, and in this case the set S is called *dense*. For example, the real line under the usual metric is separable because $\mathbb{Q}$ is countable and $\overline{\mathbb{Q}} = \mathbb{R}$. Prove that every compact metric space is separable.

For Exercise 2.2.3, use the fact established in our proof of Proposition 2.2.4 that (sequentially) compact sets are totally bounded. Thus for every integer $n \geq 1$, there is a finite set $F_n \subseteq X$ such that

$$X \subseteq \bigcup_{x \in F_n} N_{1/n}(x).$$

Then set $F = \bigcup F_n$ and argue that $\overline{F} = X$.

2.3 Heine–Borel and completeness

NOW THAT WE KNOW that compactness and sequential compactness are equivalent (for metric spaces), we relate sets that satisfy these properties to other concepts we have seen. First we show that all compact sets are closed and bounded. Then we show that the converse is true in $\mathbb{R}^d$ for our usual metrics. First, we must define what we mean by a bounded set in a general metric space.

Definition 2.3.1. The set E in the metric space (X, d) is *bounded* if there is a point $x_0 \in X$ and a real number $r > 0$ such that $E \subseteq N_r(x_0)$. A sequence (a_n) from X is *bounded* if the set $\{a_n : n \in \mathbb{N}\}$ is bounded.

You should convince yourself that this definition of bounded sets agrees with our previous definition when we restrict it to $\mathbb{R}$ with the usual metric. What are the bounded subsets of a discrete metric space?

We use the definition of sequential compactness in our proof of the following result, but the reader may want to think about the definition of compactness could be used instead.

Theorem 2.3.2. *In every metric space, all compact sets are closed and bounded.*

Proof. Suppose that K is a compact subset of the metric space (X, d). First we show that K is bounded. Fix an arbitrary point $x_0 \in X$. We then have

$$K \subseteq X = \bigcup_{n=1}^{\infty} N_n(x_0).$$

Since each neighborhood $N_n(x_0)$ is open, the family $\{N_n(x_0) : n \geq 1\}$ is an open cover of K. By the compactness of K, this cover admits a finite subcover, so there must be an $N \in \mathbb{N}$ such that

$$K \subseteq \bigcup_{n=1}^{N} N_n(x_0) = N_N(x_0),$$

which shows that K is bounded.

Next we prove that K is closed. Let $a \in \overline{K}$ be arbitrary. We know (by Proposition 1.5.10) that for every $n \in \mathbb{N}$ there is a point $a_n \in K \cap N_{1/n}(a)$. The sequence (a_n) clearly converges to a, and thus every subsequence of it must also converge to a. Because K is sequentially compact, this proves that $a \in K$, so $\overline{K} \subseteq K$, and thus K is closed. □

The fact that K is closed in Theorem 2.3.2 also follows easily from Exercise 1.5.3.

Next we recall (and extend) the Bolzano–Weierstrass theorem.

Theorem 2.3.3 (The Bolzano–Weierstrass theorem for $\mathbb{R}$)**.** *In $\mathbb{R}$ with the usual metric, every bounded sequence contains a convergent subsequence.*

We begin by extending Bolzano–Weierstrass to $\mathbb{R}^d$ with any of the standard metrics. The biggest difficult in this proof is notational.

Theorem 2.3.4 (The Bolzano–Weierstrass theorem for $\mathbb{R}^d$). *In* $(\mathbb{R}^d, d_1)$, $(\mathbb{R}^d, d_2)$, *or* $(\mathbb{R}^d, d_\infty)$, *every bounded sequence contains a convergent subsequence.*

Proof. Let (a_n) denote a bounded sequence in $\mathbb{R}^d$ under any of the three metrics d_1, d_2, or d_∞. Express the terms as $a_n = (a_{n,1}, \dots, a_{n,d})$, where the $a_{n,i} \in \mathbb{R}$ are the components of a_n. Because (a_n) is bounded, the real-valued sequence $(a_{n,1})_n$ is bounded, and thus the Bolzano–Weierstrass theorem for $\mathbb{R}$ implies that it has a convergent subsequence. Denote this subsequence by $(a_{n_1(k),1})_k$. Next consider the bounded real-valued sequence $(a_{n_1(k),2})_k$. By the Bolzano–Weierstrass theorem, this sequence has a convergence sub-subsequence, say $(a_{n_2(k),2})_k$. Continuing in this manner, we find convergent (sub-$\cdots$-sub-)subsequences

$$(a_{n_3(k),3})_k, \cdots, \text{ and } (a_{n_d(k),d})_k.$$

It follows that each of the components of the subsequence

$$(a_{n_d(k)})_k = (a_{n_d(k),1}, a_{n_d(k),2}, \dots, a_{n_d(k),d})_k$$

converges, and thus (by Proposition 1.3.11), so does the subsequence itself, completing the proof. □

The Bolzano–Weierstrass theorem leads to a characterization of the compact subsets of $\mathbb{R}^d$, (under the usual metrics), a result that is typically called the Heine–Borel theorem. Note that the Heine–Borel theorem specializes to include the real line under the usual metric.

Theorem 2.3.5 (The Heine–Borel theorem for $\mathbb{R}^d$). *In* $(\mathbb{R}^d, d_1)$, $(\mathbb{R}^d, d_2)$, *or* $(\mathbb{R}^d, d_\infty)$, *a set is compact if and only if it is closed and bounded.*

Proof. We know (by Theorem 2.3.2) that all compact sets in every metric space are closed and bounded, so it suffices to prove the converse. We also know (by Propositions 2.2.2 and 2.2.4) that compactness and sequential compactness are equivalent (for metric spaces). Thus is suffices to prove that a closed and bounded set in any of these metric spaces is sequentially compact.

Suppose that the set $K \subseteq \mathbb{R}^d$ is closed and bounded in $(\mathbb{R}^d, d_1)$, $(\mathbb{R}^d, d_2)$, or $(\mathbb{R}^d, d_\infty)$, and let (a_n) be a sequence from K. By the Bolzano–Weierstrass theorem for $\mathbb{R}^d$, (a_n) has a convergent subsequence, say $(a_{n_k})_k$. Let $a = \lim a_{n_k}$. We know that $a \in \overline{K}$ by Proposition 1.5.10, because every neighborhood of a contains some point $a_{n_k} \in K$. Furthermore, since K is closed we must have $K = \overline{K}$, so $a \in K$. This proves that K is sequentially compact. □

The Heine–Borel theorem is a special property of $\mathbb{R}^d$ (under the usual metrics) and does *not hold in general*. For example, in a discrete metric space (X, d_{disc}), every set is closed and bounded, but Exercise 2.1.4 shows that sets are compact if and only if they are finite. Other (non-)examples are described in Exercises 2.3.1 and 2.3.4.

We conclude our discussion of compactness by considering its relation to completeness. The proof of the following result was already requested in Exercise 2.2.2.

Proposition 2.3.6. *Every compact subset of a metric space is complete.*

Even though we know that $(\mathbb{R}, d)$ is complete, Proposition 2.3.6 does not immediately imply this, because $(\mathbb{R}, d)$ is not a compact space. However, this line of argument can still be made to work.

Corollary 2.3.7. *The metric space $(\mathbb{R}, d)$ is complete.*

Proof. Let (a_n) be a Cauchy sequence from $\mathbb{R}$. Cauchy sequences are necessarily bounded, so there is some number $r > 0$ such that all the terms of (a_n) are contained in the interval $[-r, r]$. Since this interval is closed and bounded, the Heine–Borel theorem tells us that it is compact. Thus by Proposition 2.3.6, the sequence (a_n) must converge in $[-r, r]$, and thus it must converge in $(\mathbb{R}, d)$ as well. □

To see that Cauchy sequences are always bounded, suppose that (a_n) is a Cauchy sequence in the metric space (X, d). Let $N \in \mathbb{N}$ be such that $d(a_m, a_n) < 1$ for all $m, n \geq N$. Set

$$r = \max\{1, d(a_1, a_N), \ldots, d(a_{N-1}, a_N)\}.$$

It follows that

$$\{a_n : n \in \mathbb{N}\} \subseteq N_r(a_N).$$

Exercise 2.3.2 asks the reader to extend this result to $\mathbb{R}^d$.

Corollary 2.3.8. *The metric spaces $(\mathbb{R}^d, d_1)$, $(\mathbb{R}^d, d_2)$, and $(\mathbb{R}^d, d_\infty)$ are all complete.*

Exercises

Exercise 2.3.1. Prove that in the metric space $(\mathbb{Q}, d)$, where d is the usual metric (on $\mathbb{R}$), the set $[0, \pi)$ is closed and bounded, but not compact.

Exercise 2.3.2. Prove Corollary 2.3.8, that the metric spaces $(\mathbb{R}^d, d_1)$, $(\mathbb{R}^d, d_2)$, and $(\mathbb{R}^d, d_\infty)$ are all complete.

Exercise 2.3.3. Suppose that the metric space (X, d) has the property that every closed and bounded subset of X is compact. Prove that (X, d) is complete.

Exercise 2.3.4. Define the metric d_* on $\mathbb{R}$ by

$$d_*(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

The metric d_* here is an instance of the construction from Exercise 1.1.5.

For every $x \in \mathbb{R}$, we have

$$d_*(x, 0) = \frac{|x|}{1 + |x|} < 1,$$

so

$$\mathbb{R} \subseteq N_1^{d_*}(0) = \{x \in \mathbb{R} : d_*(x, 0) < 1\},$$

and thus in the metric space $(\mathbb{R}, d_*)$, the set $\mathbb{R}$ is bounded. It is also closed. Prove, however, that $\mathbb{R}$ is not compact in this space.

It may be helpful to prove that the spaces $(\mathbb{R}, d)$ and $(\mathbb{R}, d_*)$ have the same open sets.

Exercise 2.3.5. Prove that any open cover of $(\mathbb{R}, d)$ has a finite or countably infinite subcover.

Exercise 2.3.6. Prove that any open cover of $(\mathbb{R}^d, d_2)$ has a finite or countably infinite subcover.

Exercises 2.3.5 and 2.3.6 are instances of Exercise 2.1.8.