4 Sequences of functions

4.1 Types of convergence

As we've previously seen on the real line, there are two types of convergence of functions. We begin with the weaker of the two.

Definition 4.1.1 (Pointwise convergence)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. The sequence (f_n) of functions from X to Y *converges pointwise to the function* $f : X \to Y$ if for every $x \in X$, the sequence $(f_n(x))$ converges to $f(x)$ in (Y, d_Y).

Nothing in the definition of pointwise convergence actually requires that X be a metric space—it could be a set without any additional structure at all. However, for later concepts it will be necessary that X be a metric space.

Note that if the sequence $(f_n(x))$ is going to converge pointwise, then it *must* converge pointwise to the function $f : X \to Y$ defined by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

Given that we know that limits of sequences are unique, this immediately implies that limits of sequences of functions must be unique.

We now (re)visit two of the standard examples of pointwise convergence. Both of these examples illustrate how the definition of pointwise convergence is fairly weak.

Example 4.1.2. Define the sequence $f_n : [0, 1] \to \mathbb{R}$ of functions by

$$f_n(x) = x^n.$$

This sequence converges pointwise to the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } x \in [0, 1) \text{, and} \\ 1 & \text{if } x = 1. \end{cases}$$

Note that each of the functions f_n is continuous, but that the limit function f is not.

Example 4.1.3. Let the sequence $(f_n)_{n\geq 1}$ of functions from $[0,1]$ to $\mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 2n & \text{if } x \in [1/2n, 1/n], \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

This sequence converges pointwise to the identically zero function. However, for every $n \geq 1$ the function f_n is (Riemann) integrable, and

$$\int_0^1 f_n = \frac{2n}{2n} = 1,$$

but the integral of the limit function (the zero function) from 0 to 1 is 0. Thus this example shows that we can have

$$\lim_{n\to\infty} \int_a^b f_n \neq \int_a^b \lim_{n\to\infty} f_n.$$

The previous examples show that pointwise convergence is not powerful enough to prove theorems about; the limit function need not bear much resemblance to the functions in the sequence.

Definition 4.1.4 (Uniform convergence)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. The sequence (f_n) of functions from X to Y *converges uniformly to the function* $f : X \to Y$ if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \epsilon$ for every $n \geq N$ and every $x \in X$.

Again, we don't need X to be a metric space in the definition of uniform convergence, but we'll want it to be later.

We hid the ϵ when we defined pointwise convergence in Definition 4.1.1, but of course there is one inside the condition that the sequence $(f_n(x))$ converge to $f(x)$ at every point $x \in X$. To define pointwise convergence analogously to our definition of uniform convergence, we would say that for every $\epsilon > 0$ and every $x \in X$, there is an $N_x \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \epsilon$ for every $n \geq N_x$. Thus the difference between the two definitions is the placement of the quantifier attached to the point x. In pointwise convergence, the value $N_x \in \mathbb{N}$ is allowed to depend on the point $x \in X$, while in uniform convergence, we must use the same value $N \in \mathbb{N}$ for every point $x \in X$.

The following result follows immediately from the definitions, so we state it without proof.

Proposition 4.1.5. *Let (X, d_X) and (Y, d_Y) be metric spaces. If the sequence (f_n) of functions from X to Y converges uniformly to the function $f : X \to Y$, then it also converges pointwise to f.*

Proposition 4.1.5 is trivial, but useful for showing that sequences of functions do *not* converge uniformly. If the sequence (f_n) of functions does not converge pointwise, then by Proposition 4.1.5, it cannot converge uniformly.

Suppose that the sequence (f_n) converges pointwise to some function f. By Proposition 4.1.5 and the uniqueness of pointwise limits of sequences of functions, if this sequence *were to converge uniformly*, then it could only converge uniformly to the same limit, f.

Many important properties transfer from a sequence of functions to its uniform limit (if it exists). We conclude this section by considering one in particular.

Definition 4.1.6. Let (X, d_X) and (Y, d_Y) be metric spaces. The function $f : X \to Y$ is *bounded* if $f(X)$ is a bounded set; that is, if there exists a neighborhood $N_r(y) \subseteq Y$ such that $f(X) \subseteq N_r(y)$.

Bounded sets were defined in Definition 2.3.1; the set S in the metric space (Y, d_Y) is *bounded* if there exists a point $y \in Y$ and a real number $r > 0$ such that $S \subseteq N_r(y)$.

Theorem 4.1.7. *Let (X, d_X) and (Y, d_Y) be metric spaces. If every function $f_n : X \to Y$ is bounded and the sequence (f_n) converges uniformly to the function $f : X \to Y$, then f is also bounded.*

Proof. Because the sequence (f_n) converges uniformly to f, there is some $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in X$,

Here we essentially set $\epsilon = 1$ in the definition of uniform convergence.

$$d_Y(f_n(x), f(x)) < 1.$$

The function f_N is bounded, so there is some point $y \in Y$ and some radius $r > 0$ such that

$$f_N(X) \subseteq N_r(y).$$

Therefore by the triangle inequality, for every point $x \in X$ we have

$$d_Y(f(x), y) \leq d_Y(f(x), f_N(x)) + d_Y(f_N(x), y) < 1 + r,$$

and thus $f(X) \subseteq N_{r+1}(y)$, proving the theorem. □

Exercises

Exercise 4.1.1. Prove that the pointwise convergence in Example 4.1.2 is not uniform.

Exercise 4.1.2. Prove that the pointwise convergence in Example 4.1.3 is not uniform.

Exercise 4.1.3. Give an example to show that the conclusion of Theorem 4.1.7 does not hold if "converges uniformly" is replaced with "converges pointwise".

Exercise 4.1.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and for $a \in \mathbb{R}$, define the shifted function $f_a : \mathbb{R} \to \mathbb{R}$ by $f_a(x) = f(x - a)$. Prove that f is continuous if and only if for every sequence (a_n) in $\mathbb{R}$ that converges to 0, the sequence (f_{a_n}) converges pointwise to f.

Exercise 4.1.5. As in the previous problem, let $f : \mathbb{R} \to \mathbb{R}$ be a function and for $a \in \mathbb{R}$, define the shifted function $f_a : \mathbb{R} \to \mathbb{R}$ by $f_a(x) = f(x - a)$. Prove that f is *uniformly* continuous if and only if for every sequence (a_n) in $\mathbb{R}$ that converges to 0, the sequence (f_{a_n}) converges *uniformly* to f.

Exercise 4.1.6. Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that if every function $f_n : X \to Y$ is bounded and the sequence (f_n) converges uniformly to the function $f : X \to Y$, then the sequence (f_n) is *uniformly bounded* in the sense that there is some neighborhood $N_r(y) \subseteq Y$ such that $f_n(X) \subseteq N_r(y)$ for all $n \in \mathbb{N}$.

4.2 The uniform metric

We have accumulated several notions of limits: one for sequences, one for functions, and two for sequences of functions. Here we show how the notion of uniform convergence of sequences of functions can be viewed as a special case of convergence of sequences, in a metric space consisting of the functions themselves.

Pointwise convergence can also be viewed as a special case of convergence of sequences, but to do so, one must work in a topological space.

We need to restrict ourselves to bounded functions. Thus given metric spaces (X, d_X) and (Y, d_Y), we define

$$\mathcal{B}(X, Y) = \{f : f \text{ is a bounded function from } X \text{ to } Y\}.$$

Recall from Definition 4.1.6 that the function $f : X \to Y$ is bounded if and only if its range $f(X)$ is contained in some neighborhood in Y.

If $f, g \in \mathcal{B}(X, Y)$, then one can show (Exercise 4.2.1) that the set

$$\{d_Y(f(x), g(x)) : x \in X\} \subseteq \mathbb{R}$$

is bounded above. Therefore (so long as $X \neq \emptyset$), this set has a supremum. This allows us to make the following definition.

Note that for any domain (X, d_X), the set $\mathcal{B}(X, \mathbb{R})$ is a vector space (in fact this is true whenever the codomain is a field).

Definition 4.2.1 (Uniform metric)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. For $f, g \in \mathcal{B}(X, Y)$, we define the *uniform metric* by

$$d_\infty(f, g) = \sup\{d_Y(f(x), g(x)) : x \in X\}.$$

We have gotten ahead of ourselves a bit by calling this the uniform metric, and should check that it actually is a metric.

Proposition 4.2.2. *Let (X, d_X) and (Y, d_Y) be metric spaces. The function*

$$d_\infty : \mathcal{B}(X, Y) \times \mathcal{B}(X, Y) \to \mathbb{R}$$

is a metric on $\mathcal{B}(X, Y)$.

Proof. We prove that d_∞ satisfies the triangle inequality. The other properties of a metric are, as usual, easy to see by inspection. Suppose that $f, g, h \in \mathcal{B}(X, Y)$. For any point $x \in X$, the triangle inequality that d_Y satisfies implies that

$$\begin{aligned} d_Y(f(x), h(x)) &\leq d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \\ &\leq d_\infty(f, g) + d_\infty(g, h). \end{aligned}$$

As this holds for every $x \in X$, it implies that $d_\infty(f, g) + d_\infty(g, h)$ is an upper bound for the set

$$\{d_Y(f(x), h(x)) : x \in X\}.$$

Therefore $d_\infty(f, h) \leq d_\infty(f, g) + d_\infty(g, h)$, as desired. □

Our next result shows that, indeed, convergence in $(\mathcal{B}(X,Y), d_\infty)$ is the same as uniform convergence of functions.

Proposition 4.2.3. *Let (X, d_X) and (Y, d_Y) be metric spaces. The sequence (f_n) from $\mathcal{B}(X,Y)$ converges to $f \in \mathcal{B}(X,Y)$ in the metric space $(\mathcal{B}(X,Y), d_\infty)$ if and only if it converges to f uniformly.*

Proof. First suppose that the sequence (f_n) from $\mathcal{B}(X,Y)$ converges to $f \in \mathcal{B}(X,Y)$ in the metric space $(\mathcal{B}(X,Y), d_\infty)$. Given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n \geq N$, then $d_\infty(f_n, f) < \epsilon$. In particular, for every $n \geq N$ and every $x \in X$,

$$d_Y(f_n(x), f(x)) \leq d_\infty(f_n, f) < \epsilon.$$

Thus (f_n) converges to f uniformly.

Conversely suppose that (f_n) converges to f uniformly. Then given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for every $n \geq N$ and every $x \in X$,

$$d_Y(f_n(x), f(x)) < \epsilon.$$

This implies that $d_\infty(f_n, f) \leq \epsilon$ for all $n \geq N$, and thus (f_n) converges to f in the metric space $(\mathcal{B}(X,Y), d_\infty)$. □

We normally use a strict inequality for convergence, but it doesn't matter; having $d_\infty(f_n, f) \leq \epsilon$ for every $\epsilon > 0$ is enough to guarantee convergence.

Our previous result justifies considering the uniform metric. Our next result shows that completeness of the codomain carries over to the space $(\mathcal{B}(X,Y), d_\infty)$.

Theorem 4.2.4. *Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is complete, then $(\mathcal{B}(X,Y), d_\infty)$ is also complete.*

Proof. Let (f_n) be a Cauchy sequence in $(\mathcal{B}(X,Y), d_\infty)$. We first establish that (f_n) has a pointwise limit. For every point $x \in X$, and every $m, n \in \mathbb{N}$, we have

$$d_Y(f_m(x), f_n(x)) \leq d_\infty(f_m, f_n),$$

so the sequence $(f_n(x))$ is a Cauchy sequence in (Y, d_Y). Since (Y, d_Y) is complete, this sequence converges to some point which we denote by $f(x)$. Thus there is a function $f : X \to Y$ such that (f_n) converges pointwise to f.

We have not proved that f is itself bounded, but this will follow at the end of the proof.

Now let $\epsilon > 0$ be given. Because (f_n) is Cauchy, there is some $N \in \mathbb{N}$ such that

$$d_\infty(f_m, f_n) < \epsilon/2$$

for all $m, n \geq N$. For any particular point $x \in X$, we can use the pointwise convergence of (f_n) to f to find an index $m_x \geq N$ such that

$$d_Y(f_{m_x}(x), f(x)) < \epsilon/2.$$

Therefore, for every $n \geq N$ and every $x \in X$, we have

$$d_Y(f_n(x), f(x)) \leq d_Y(f_n(x), f_{m_x}(x)) + d_Y(f_{m_x}(x), f(x)) < \epsilon.$$

This proves that (f_n) converges uniformly to f. Theorem 4.1.7 then implies that $f \in \mathcal{B}(X, Y)$. This completes the proof that (f_n) converges to f in the metric space $(\mathcal{B}(X, Y), d_\infty)$. $\square$

Exercises

Exercise 4.2.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Prove that the set

$$\{d_Y(f(x), g(x)) : x \in X\} \subseteq \mathbb{R}$$

is bounded above for all $f, g \in \mathcal{B}(X, Y)$.

Exercise 4.2.1 allows us to actually define the function

$$d_\infty : \mathcal{B}(X, Y) \times \mathcal{B}(X, Y) \to \mathbb{R}.$$

Exercise 4.2.2. Prove that if (f_n) and (g_n) are both convergent sequences in $(\mathcal{B}(X, \mathbb{R}), d_\infty)$, then the sequence $(f_n + g_n)$ is also convergent in this metric space.

Exercise 4.2.3. Construct a sequence (f_n) from $\mathcal{B}([0, 1], \mathbb{R})$, each of which is discontinuous at every point of $[0, 1]$, that converges uniformly to a continuous function f.

4.3 The space of continuous functions

As we've already remarked, a sequence of continuous functions can converge pointwise to a noncontinuous function. However, this cannot happen if the convergence is uniform:

Theorem 4.3.1 (Uniform limit theorem). *Let (X, d_X) and (Y, d_Y) be metric spaces. If every function $f_n : X \to Y$ is continuous at the point $a \in X$ and the sequence (f_n) converges uniformly to the function $f : X \to Y$, then f is also continuous at the point a.*

Theorem 4.3.1 implies that if each function $f_n : X \to Y$ is continuous on all of X, and the sequence (f_n) converges uniformly to $f : X \to Y$, then f is also continuous on all of X.

Proof. Let $\epsilon > 0$ be arbitrary. Because the sequence (f_n) converges uniformly to f, there is some $N \in \mathbb{N}$ such that for all $n \geq N$ and $x \in X$,

$$d_Y(f_n(x), f(x)) < \epsilon/3.$$

Since the function f_N is continuous at the point a, there is some $\delta > 0$ such that if $d_X(x, a) < \delta$, then

$$d_Y(f_N(x), f_N(a)) < \epsilon/3.$$

Therefore, if $d_X(x, a) < \delta$, then by the triangle inequality, we see that $d_Y(f(x), f(a))$ is at most

$$d_Y(f(x), f_N(x)) + d_Y(f_N(x), f_N(a)) + d_Y(f_N(a), f(a)) < \epsilon.$$

This proves that f is indeed continuous at a. □

Theorem 4.3.1 can be used to show that a sequence of functions that converges pointwise does *not* converge uniformly, as demonstrated below.

Example 4.3.2. Prove that the sequence (f_n) of functions defined by $f_n(x) = x^n$ (and considered first in Example 4.1.2) does not converge uniformly.

Proof. As we saw in Example 4.1.2, the sequence (f_n) converges pointwise to the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1), \text{ and} \\ 1 & \text{if } x = 1. \end{cases}$$

However, each of the functions f_n is continuous at the point $a = 1$, while f is not continuous there. Therefore the convergence must not be uniform, by Theorem 4.3.1. □

In the previous section we considered the space $\mathcal{B}(X,Y)$ of bounded functions from X to Y, equipped with the uniform metric. We are frequently interested in the subspace of $\mathcal{B}(X,Y)$ consisting only of continuous functions, and so we define

$$\mathcal{C}(X,Y) = \{f : f \text{ is a bounded continuous function from } X \text{ to } Y\}.$$

In the case where $X = Y$, we often shorten this notation, defining

$$\mathcal{C}(X) = \{f : f \text{ is a bounded continuous function from } X \text{ to } X\}.$$

Note that if the domain (X, d_X) is compact, then all continuous functions with domain X are bounded, so in this cases we wouldn't need to insist on the boundedness condition in our definitions of $\mathcal{C}(X,Y)$ and $\mathcal{C}(X)$.

Since $\mathcal{C}(X,Y) \subseteq \mathcal{B}(X,Y)$, the uniform metric is also a metric on $\mathcal{C}(X,Y)$. Moreover, our previous result implies the following.

Corollary 4.3.3. *The set $\mathcal{C}(X,Y)$ is closed in the metric space $(\mathcal{B}(X,Y), d_\infty)$.*

Proof. Let (f_n) be a sequence from $\mathcal{C}(X,Y)$ that converges to f with respect to the uniform metric d_∞. Then f is bounded by Theorem 4.1.7, and f is continuous by Theorem 4.3.1, so $f \in \mathcal{C}(X,Y)$, proving that $\mathcal{C}(X,Y)$ is closed. □

Recall that Theorem 4.2.4 showed that if (Y, d_Y) is complete, then $(\mathcal{B}(X,Y), d_\infty)$ is also complete. By Theorem 4.3.1, the same holds for $(\mathcal{C}(X,Y), d_\infty)$.

Corollary 4.3.4. *Let (X, d_X) and (Y, d_Y) be metric spaces. If (Y, d_Y) is complete, then $(\mathcal{C}(X,Y), d_\infty)$ is also complete.*

Corollary 4.3.4 is just an instance of Proposition 1.7.9: a closed subset of a complete space is itself complete.

Proof. Suppose that (Y, d_Y) is complete and that (f_n) is a Cauchy sequence in $(\mathcal{C}(X,Y), d_\infty)$. By Theorem 4.2.4, (f_n) converges uniformly to some function $f \in \mathcal{B}(X,Y)$. By Theorem 4.3.1, f is continuous, and thus $f \in \mathcal{C}(X,Y)$. □

Example 4.3.5. Let $0 \in \mathcal{C}([0,1])$ denote the identically-0 function. The *unit ball in* $\boldsymbol{\mathcal{C}([0,1])}$ is the set

$$B = \{f \in \mathcal{C}([0,1]) : d_\infty(f, 0) \leq 1\}.$$

Prove that B is complete, closed, and bounded, but not compact.

Note that the word "bounded" in Example 4.3.5 refers to B being a bounded subset of $\mathcal{C}([0,1])$, not to the functions *in* B being themselves bounded. (Of course, the functions in B *are* bounded, because they map $[0,1]$ to $[0,1]$.)

Proof. The set B is, by definition, a closed ball in $\mathcal{C}([0,1])$, so it is both closed and bounded. Since $\mathcal{C}([0,1])$ is complete by Corollary 4.3.4 and B is closed, B is complete. To see that B is not compact, consider the sequence (f_n) where $f_n(x) = x^n$. This sequence lies in B, but in Example 4.3.2 we saw that it has no convergent subsequence with respect to the d_∞ metric. □

Exercises

Exercise 4.3.1. Prove that if (f_n) converges to f in $(\mathcal{C}(X,Y), d_\infty)$ and if (x_n) is a sequence from (X, d_X) that converges to x, then the sequence $(f_n(x_n))$ converges to $f(x)$ in (Y, d_Y).

Exercise 4.3.2. Prove that if the space (X, d_X) is compact and the sequence (f_n) from $\mathcal{C}(X,Y)$ converges to f, then (f_n) is *equicontinuous*, meaning that given $\epsilon > 0$, there is some $\delta > 0$ such that, for every $n \in \mathbb{N}$, if $d_X(x,y) < \delta$, then $d_Y(f_n(x), f_n(y)) < \epsilon$. (Thus the collection $\{f_n : n \in \mathbb{N}\}$ is *uniformly uniformly continuous*.)

Exercise 4.3.3. Prove the following partial converse to Exercise 4.3.1. Suppose that (X, d_X) is compact, that (f_n) is a sequence from $\mathcal{C}(X,Y)$, and that $f \in \mathcal{C}(X,Y)$. Prove that if $(f_n(x_n))$ converges to $f(x)$ for every point $x \in X$ and every sequence (x_n) converging to x, then (f_n) converges to f uniformly. Give an example to show that the assumption that (X, d_X) is compact is needed.

4.4 Integration and differentiation

THEOREM 4.3.1 states that if (f_n) is a sequence of continuous functions that converges uniformly to the function f, then f is itself continuous.

Assuming that we are working with functions from $\mathbb{R}$ to $\mathbb{R}$ where we have defined integrals and derivatives, it is natural to ask if these properties carry over to uniform limits. We show here that—assuming sufficient hypotheses—they do. Both results can be viewed as interchanges of limits (the uniform limit with the limit from the derivative or integral).

We begin with the integral, which is a bit easier. We restrict our consideration to sequences in $\mathcal{B}([a,b],\mathbb{R})$ because (Riemann) integrable functions must be bounded.

Theorem 4.4.1 (Integrating uniform limits)**.** *Suppose the sequence (f_n) of integrable functions from $\mathcal{B}([a,b],\mathbb{R})$ converges uniformly to f. Then, f is integrable and*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

Proof. First note that f is itself bounded by Theorem 4.1.7. Let $\epsilon > 0$ be given and choose $N \in \mathbb{N}$ so that

$$d_\infty(f_n, f) < \frac{\epsilon}{3(b-a)}$$

for all $n \geq N$. This implies for every partition P of $[a,b]$, we have

$$L(f_N, P) - \frac{\epsilon}{3} \leq L(f,P) \leq U(f,P) \leq U(f_N,P) + \frac{\epsilon}{3}.$$

Because the function f_N is integrable, there is some partition P for which $U(f_N,P) - L(f_N,P) < \epsilon/3$. For this partition, the inequality above shows that

$$U(f,P) - L(f,P) \leq U(f_N,P) - L(f_N,P) + \frac{2\epsilon}{3} < \epsilon.$$

This proves that f is integrable on the interval $[a,b]$. To show that the integral of f is as claimed, note that for all $n \geq N$ (where N is as above), we have

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f| \leq \int_a^b \frac{\epsilon}{3(b-a)} = \frac{\epsilon}{3}.$$

As ϵ is arbitrary, the integral of f must be as claimed. □

In the proof of Theorem 4.4.1, we use $L(f,P)$ and $U(f,P)$ for the lower and upper (Riemann) sums of f with respect to the partition P. Recall that a bounded function f is (*Riemann*) *integrable* if and only if for every $\epsilon > 0$, there is a partition P so that $U(f,P) - L(f,P) < \epsilon$.

This leaves us to consider the derivative. We prove two theorems about the derivative, the second stronger than the first (because it has weaker—and thus more easily applied—hypotheses).

Theorem 4.4.2 (Differentiating uniform limits, first version)**.** *Let $I \subseteq \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions from I to $\mathbb{R}$. Suppose that (f_n) converges pointwise on I to a function f and that the sequence (f_n') of derivatives exists on I and converges uniformly on I to a function g. Then f is differentiable and $f' = g$.*

Proof. Let $c \in I$. We seek to show that $f'(c) = g(c)$. To this end, let $\epsilon > 0$ be given. Our goal is to find a $\delta > 0$ so that if $0 < |x - c| < \delta$, then

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon.$$

Consider an arbitrary point $x \in I$ distinct from c. With an eye to appealing to the Cauchyness of the sequence (f_n'), we apply the mean value theorem to the function $f_m - f_n$ to see that there exists a point z (which depends on both m and n) between c and x for which

$$f_m'(z) - f_n'(z) = \frac{(f_m(x) - f_n(x)) - (f_m(c) - f_n(c))}{x - c}.$$

In particular, this implies that

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \le d_\infty(f_m', f_n').$$

Because the sequence (f_n') converges uniformly, it is Cauchy with respect to the uniform metric, so there is some $N_1 \in \mathbb{N}$ so that

$$\left| \frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \le \frac{\epsilon}{3}$$

for all $m, n \ge N_1$. Now taking the limit as $m \to \infty$, we see that

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \le \frac{\epsilon}{3}$$

for all $n \ge N_1$.

Since $g(c) = \lim f_n'(c)$, there is some $N_2 \in \mathbb{N}$ so that if $n \ge N_2$, then

$$\left| f_n'(c) - g(c) \right| < \frac{\epsilon}{3}.$$

Choose $N \ge N_1, N_2$.

Since $f_N'(c)$ exists, there is some $\delta > 0$ so that if $0 < |x - c| < \delta$, then

$$\left| \frac{f_N(x) - f_N(c)}{x - c} - f_N'(c) \right| < \frac{\epsilon}{3}.$$

By the triangle inequality, we may conclude that if $0 < |x - c| < \delta$, then

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon,$$

proving that $f'(c) = g(c)$, as desired. □

Theorem 4.4.2 assumes that both sequences (f_n) and (f_n') converge uniformly. It is not necessary to assume that the sequence (f_n) converges uniformly, because the condition on the derivatives (f_n') almost implies this. We are required to assume *something* about the functions (f_n), however, because otherwise they could differ by a constant. We therefore require that the sequence (f_n) converge pointwise *somewhere*.

To see why we must require that $(f(x_0))$ converge for some point x_0, consider the sequence (f_n) defined by $f_n(x) = x + n$. We have $f_n'(x) = 1$ for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$, so the sequence (f_n') converges uniformly. The sequence (f_n), however, does not converge.

Theorem 4.4.3 (Differentiating uniform limits, more general version)**.** *Let $I \subseteq \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions from I to $\mathbb{R}$. Suppose that there exists a point $x_0 \in I$ such that $(f_n(x_0))$ converges, and that the sequence (f_n') of derivatives exists on I and converges uniformly on I to a function g. Then the sequence (f_n) converges uniformly on I to a function f that is differentiable at every point of I, and $f' = g$.*

Proof. It suffices to show that under the hypotheses of this theorem, the hypotheses of Theorem 4.4.2 hold. In particular, our goal is to prove that the sequence (f_n) converges uniformly on the interval I.

By applying the mean value theorem to the function $f_m - f_n$, we see that there exists a point z between x_0 and x for which

$$f_m'(z) - f_n'(z) = \frac{(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0))}{x - x_0}.$$

Solving for $f_m(x) - f_n(x)$ and taking absolute values, we see that

$$\begin{aligned} |f_m(x) - f_n(x)| &= |f_m(x_0) - f_n(x_0)| + |x - x_0| \cdot \left|f_m'(z) - f_n'(z)\right| \\ &\leq |f_m(x_0) - f_n(x_0)| + |x - x_0| \cdot d_\infty(f_m', f_n'). \end{aligned}$$

Let M denote the width of the interval I (we assumed that I is bounded). For every $x \in I$, we have that $|x - x_0| \leq M$, and thus for every $x \in I$, we have from the above inequality that

$$|f_m(x) - f_n(x)| \leq |f_m(x_0) - f_n(x_0)| + M \cdot d_\infty(f_m', f_n').$$

Since this holds for every $x \in I$, we have shown that

$$d_\infty(f_m, f_n) \leq |f_m(x_0) - f_n(x_0)| + M \cdot d_\infty(f_m', f_n').$$

Now let $\epsilon > 0$ be given. Because the sequence $(f_n(x_0))$ is Cauchy (with respect to the usual metric on $\mathbb{R}$) and the sequence (f_n') is Cauchy (with respect to the uniform metric), there is some $N \in \mathbb{N}$ so that if $m, n \geq N$, then both $|f_m(x_0) - f_n(x_0)| < \epsilon/2$ and $d_\infty(f_m', f_n') < \epsilon/2$. Therefore, for all $m, n \geq N$, we have

$$d_\infty(f_m, f_n) < \epsilon,$$

proving that the sequence (f_n) is Cauchy with respect to the d_∞ metric. It follows that the pointwise limit $f = \lim f_n$ exists, and that the sequence (f_n) converges uniformly to f. □