5 Power series

We consider here a particular type of convergence of functions: that of power series where they converge. We limit our consideration to *real functions*—those whose domain and codomain are subsets of $\mathbb{R}$, and we consider only the usual absolute value metric on $\mathbb{R}$.

5.1 Taylor polynomials

Taylor polynomials approximate a function by matching its derivatives at a given point. The derivatives of e^x at the point $x = 0$ are all equal to 1, and so as we've all seen in calculus class, its nth Taylor polynomial centered at the point 0 is

$$t_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + \frac{x^n}{n!}.$$

It is then easy to check that for all $k \leq n$, the kth derivative of e^x evaluated at $x = 0$ and the kth derivative of $t_n(x)$ evaluated at $x = 0$ are both equal (to 1).

Before presenting the general construction of Taylor polynomials, we need some preliminaries. First, as Taylor polynomials are defined by matching derivatives, our functions must *have* derivatives, so we make the following inductive definition.

Remember that $n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1$ denotes n *factorial*. This function grows *very fast*. *Stirling's approximation* states that

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

With elementary methods, it is not hard to get the usually-sufficient lower bound

$$n! \geq \left(\frac{n}{e}\right)^n.$$

A crude simple bound is that $(2n)!$ has n terms that are all at least n, and so

$$(2n)! \geq n^n.$$

Definition 5.1.1 (k-times differentiability)**.** Let $A \subseteq \mathbb{R}$ be open. The function $f : A \to \mathbb{R}$ is *once differentiable at the point* $a \in A$ if it is differentiable at a. For $k \geq 2$, the function f is k *times differentiable at the point* $a \in A$ if and only if it is differentiable at a and f' is $(k-1)$ times differentiable at a.

As usual, we say that the function f is simply *k times differentiable on A* if it is k times differentiable at every point $a \in A$. If the domain A is clear, we shorten this and only say that f is *k times differentiable*.

Example 5.1.2. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|^3$ is 2 times differentiable (or, twice differentiable) on all of $\mathbb{R}$, but it is not 3 times differentiable at $x = 0$.

Proof. This function is differentiable on all of $\mathbb{R}$, and $f'(x) = 3x|x|$. Since f' is differentiable, f is 2 times differentiable. However, $f''(x) = 6|x|$ is not differentiable at 0, so f is not 3 times differentiable at 0. Moreover, f is not k times differentiable for any $k \geq 3$. □

We began the section by loosely defining the nth Taylor polynomial of a function as a polynomial that matches its value and first n derivatives at a given point. This information is enough to uniquely determine a polynomial of degree at most n (Exercise 5.1.2), and this is our definition of Taylor polynomials.

Definition 5.1.3 (Taylor polynomial)**.** Suppose that the function f is n times differentiable at the point a. The *nth Taylor polynomial for f centered at a* is the unique polynomial $t_n(x)$ of degree at most n for which

$$t_n^{(k)}(a) = f^{(k)}(a)$$

for all $0 \leq k \leq n$.

Of course, since $t_n(x)$ is a polynomial of degree at most n, we can express it in the form

$$t_n(x) = \sum_{k=0}^{n} a_k (x-a)^k$$

Just as all polynomials of degree at most n can be expressed in the basis

$$\{x^k : 0 \leq k \leq n\},$$

they can equally well all be expressed in the basis

$$\{(x-a)^k : 0 \leq k \leq n\}.$$

By taking derivatives and evaluating them at a, we see that, for all $0 \leq k \leq n$,

$$t_n^{(k)}(a) = k!\, a_k.$$

Setting this quantity equal to $f^{(k)}(a)$ and solving for a_k gives us a formula for the coefficients of $t_n(x)$.

Recall that $0! = 0^0 = 1$.

Proposition 5.1.4 (Taylor's formula)**.** *Suppose that the function f is n times differentiable at the point a. Then the nth Taylor polynomial for f centered at a is*

$$t_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

In light of the previous chapter, it is natural to ask:

Question 5.1.5. Given a function f and a point $a \in \mathbb{R}$, where/when does the sequence $\{t_n(x)\}_{n=0}^{\infty}$ of Taylor polynomials for f centered at a converge to f? Where we do have convergence, is it uniform, or just pointwise?

Of course, in order for the sequence $\{t_n(x)\}$ of Taylor polynomials to even be *defined*, f must have derivatives of all orders at the point a.

Definition 5.1.6 (Infinite differentiability)**.** Let $A \subseteq \mathbb{R}$ be open. The function $f : A \to \mathbb{R}$ is *infinitely differentiable at the point* $a \in A$ if it is k times differentiable at a for every $k \geq 1$.

As usual, we say that the function f is *infinitely differentiable on* A if it is infinitely differentiable at every point $a \in A$. If the domain A is clear, we shorten this and say only that f is *infinitely differentiable* (or, *smooth*).

The reader should be able to think of a large number of infinitely differentiable functions: polynomials, exponentials, sine, cosine, etc. It is these functions for which we can construct an infinite sequence of Taylor polynomials, and thus it is these for which we can ask Question 5.1.5. The following tool will help provide answers.

There are several versions of the remainder theorem; this is the *Lagrange form*.

Theorem 5.1.7 (Remainder theorem)**.** *Suppose that* $a \in \mathbb{R}$, $r > 0$, *and that* $f : (a-r, a+r) \to \mathbb{R}$ *is* $n+1$ *times differentiable on* $(a-r, a+r)$. *Let* $t_n(x)$ *be the nth Taylor polynomial for* f *centered at* a. *Then for every* $x \in (a-r, a+r)$,

$$f(x) - t_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

for some point ξ *(depending on* x*) that lies between* a *and* x.

Proof. Viewing $x \in (a-r, a+r)$ as fixed, choose $K \in \mathbb{R}$ so that

$$f(x) - t_n(x) = \frac{C}{(n+1)!}(x-a)^{n+1}.$$

We are allowed to choose such a C because we can simply solve the equation for C. (To repeat, we are thinking of x as fixed.)

Our goal is to prove that $C = f^{(n+1)}(\xi)$ for some ξ between a and x.

Define the function $\varphi : (a-r, a+r) \to \mathbb{R}$ by

$$\varphi(y) = \frac{C}{(n+1)!}(x-y)^{n+1} - \left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k \right).$$

We then have

$$\varphi(a) = \underbrace{\frac{C}{(n+1)!}(x-a)^{n+1}}_{f(x)-t_n(x)} - \underbrace{\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right)}_{f(x)-t_n(x)} = 0$$

by our choice of K, and also

$$\varphi(x) = \underbrace{\frac{C}{(n+1)!}(x-x)^{n+1}}_{0} - \underbrace{\left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-x)^k \right)}_{f(x)-f(x)} = 0.$$

Since the function φ is continuous (as a function of y) on the closed interval between a and x and it is differentiable on the open interval between a and x, we can apply Rolle's theorem to conclude that there is some point ξ between a and x for which $\varphi'(\xi) = 0$.

Now we simply compute the derivative of $\varphi(y)$ to reach our desired conclusion. By the product rule, we have that for $k \geq 1$,

$$\frac{d}{dy}\left(\frac{f^{(k)}(y)}{k!}(x-y)^k\right) = \frac{f^{(k+1)}(y)}{k!}(x-y)^k - \frac{f^{(k)}}{(k-1)!}(x-y)^{k-1}.$$

Thus the derivative of the sum in the definition of $\varphi(y)$ telescopes:

$$\frac{d}{dy}\left(\sum_{k=0}^{n}\frac{f^{(k)}(y)}{k!}(x-y)^k\right) = \frac{f^{(n+1)}(y)}{n!}(x-y)^n.$$

Therefore,

$$\varphi'(\xi) = -\frac{C}{n!}(x-\xi)^n + \frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n = 0,$$

showing that $C = f^{(n+1)}(\xi)$ for some point ξ between a and x, as desired. □

We conclude by using the remainder theorem to show that on any *bounded* domain, the sequence of Taylor polynomials for e^x centered at 0 converges to the function e^x, and moreover, that this convergence is uniform.

Example 5.1.8. Let $t_n(x)$ denote the nth Taylor polynomial for e^x centered at 0. Prove that for every $r > 0$, the sequence $\{t_n(x)\}$ converges uniformly to e^x on the interval $(-r, r)$.

Proof. Let $r > 0$ be given and $x \in (-r, r)$ be arbitrary. Because all derivatives of e^x are itself, the remainder theorem tells us that

$$|t_n(x) - e^x| = \frac{e^\xi}{(n+1)!}\xi^{n+1}$$

for some ξ between 0 and x. Because $x \in (-r, r)$, we have $e^\xi < e^r$ and $|\xi^{n+1}| < r^{n+1}$, so we may conclude that

$$|t_n(x) - e^x| < e^r\frac{r^{n+1}}{(n+1)!}.$$

As $n \to \infty$, the righthand side above tends to 0: the e^r is simply a constant, while the factorial $(n+1)!$ dominates the exponential r^{n+1}. Thus for any given $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|t_n(x) - e^x| < \epsilon$ for all $x \in (-r, r)$, which is the definition of uniform convergence. □

Exercises

Exercise 5.1.1. For every $k \geq 1$, construct a function $f : \mathbb{R} \to \mathbb{R}$ that is k times differentiable but not $k+1$ times differentiable.

Exercise 5.1.2. Prove that a polynomial $p(x)$ of degree at most n is uniquely determined by its value and first n derivatives at a given point $a \in \mathbb{R}$. That is, prove that if $p(x)$ and $q(x)$ are both polynomials of degree at most n and for some point $a \in \mathbb{R}$ we have $p^{(k)}(a) = q^{(k)}(a)$ for and all $0 \leq k \leq n$, then $p(x)$ and $q(x)$ have identical coefficients.

Exercise 5.1.3. Let $p(x)$ be a polynomial and let $t_n(x)$ denote the nth Taylor polynomial for $p(x)$ centered at 0. Prove that the sequence $\{t_n(x)\}$ converges uniformly to $p(x)$ on all of $\mathbb{R}$.

Exercise 5.1.4. Let $t_n(x)$ denote the nth Taylor polynomial for $\sin x$ centered at 0. Prove that for any $r > 0$, the sequence $\{t_n(x)\}$ converges uniformly to $\sin x$ on the interval $(-r, r)$.

Exercise 5.1.5. Let $t_n(x)$ denote the nth Taylor polynomial for e^x centered at 0. Show that, contrary to Example 5.1.8, the sequence $\{t_n(x)\}$ does *not* converge uniformly to e^x on all of $\mathbb{R}$.

5.2 Numerical series and the root test

WE CONCLUDED THE PREVIOUS SECTION by using the remainder theorem to prove that the sequence of Taylor polynomials for e^x centered at 0 converges uniformly to e^x. In order to obtain more general answers about where and when this happens, is is necessary to consider series.

Given a sequence $(a_n)_{n=m}^{\infty}$ of real numbers, we form the *series*

We call these numerical series, because they consist of numbers, to differentiate them from power series.

$$\sum_{n=m}^{\infty} a_n = a_m + a_{m+1} + \cdots ,$$

which we abbreviate to $\sum a_n$ if the lower bound of the summation is clear. We analyze a series via its sequence of *partial sums*, which is the sequence $(s_k)_{k=m}^{\infty}$ defined by

$$s_k = \sum_{n=m}^{\infty} a_n.$$

Definition 5.2.1 (Convergence of series)**.** The series $\sum a_n$ is said to *converge* if and only if the sequence (s_k) of partial sums converges, and in this case we write

$$\sum_{n=m}^{\infty} a_n = \lim_{n\to\infty} s_n.$$

Otherwise the series *diverges*.

If (s_k) diverges to ∞, then we sometimes say that series $\sum a_n$ *diverges to infinity* as well.

We begin with a fundamental example, *geometric series*, on which much of the theory is based. For any number $r \in \mathbb{R}$, one can check that the identity

$$(1-r)(1+r+r^2+\cdots+r^k) = 1 - r^{k+1}$$

holds by expanding the left-hand side. Defining the sequence (a_n) by $a_n = r^n$, we see that whenever $r \neq 1$, we have

$$s_k = \sum_{n=0}^{k} a_n = \sum_{n=0}^{k} r^n = \frac{1-r^{k+1}}{1-r}.$$

From this equation we can see the behavior of the partial sums (s_k).

Theorem 5.2.2 (Geometric series)**.** *The geometric series $\sum r^n$ converges if and only if $|r| < 1$.*

The geometric series $\sum r_k$ diverges to infinity if $r \geq 1$ (although in the case $r = 1$, one cannot use our equation for the partial sums to see this). It simply does not converge if $r \leq -1$.

By definition, the series $\sum a_n$ converges if and only if its sequence (s_k) of partial sums converges. Since a sequence of real numbers converges if and only if it is Cauchy, we see that $\sum a_n$ converges if and only if the sequence (s_k) is Cauchy. The following result simply translates this Cauchy criterion on partial sums to a condition on the terms of the sequence itself.

Proposition 5.2.3 (The Cauchy criterion for series)**.** *The series $\sum a_n$ converges if and only if for every $\epsilon > 0$, there is some $K \in \mathbb{N}$ such that*

$$\left| \sum_{n=k}^{\ell} a_n \right| < \epsilon$$

for all $\ell \geq k \geq K$.

The quantity appearing in the Cauchy criterion for series is $|s_\ell - s_{k-1}|$.

Note that if we take $\ell = k$ here, we see that if $\sum a_n$ converges, then we must have $a_n \to 0$. In other words, if a series is to have a chance to converge, its terms must go to 0.

Our next task is to establish the root test. To state this test in its strongest possible form, we must first recall the notion of a *limit supremum*.

Definition 5.2.4 (Limit supremum)**.** Let (a_n) be a sequence of nonnegative real numbers. If (a_n) is unbounded, then we define

$$\limsup a_n = \infty.$$

Otherwise, if (a_n) is bounded, then we define the sequence (α_k) by

$$\alpha_k = \sup\{a_n : n \geq k\}.$$

The sequence (α_k) is nonincreasing and bounded, so it has a limit, and we define

$$\limsup a_n = \lim_{k \to \infty} \alpha_k.$$

The sequence (α_k) is nonincreasing simply because α_{k+1} is the supremum of a subset of the numbers of which α_k is the supremum.

With the notion of limit supremums established, we can now state and prove the root test.

Theorem 5.2.5 (Root test)**.** *Given a sequence $(a_n)_{n=m}^{\infty}$, define*

$$L = \limsup |a_n|^{1/n}.$$

If $L < 1$, then the series $\sum a_n$ converges. If $L > 1$, then the series $\sum a_n$ diverges.

If $L = 1$ in the root test, then it does not say whether the series $\sum a_n$ converges; the test is inconclusive. Two series with $L = 1$ are $\sum 1/n$ and $\sum 1/n^2$. You may remember from calculus that one of these series converges while the other diverges to infinity.

Proof. Suppose first that $L < 1$. Choose a number ρ such that $L < \rho < 1$. There must be some $N \in \mathbb{N}$ such that

$$|a_n|^{1/n} < \rho$$

for all $n \geq N$. To show that $\sum a_n$ converges, we now verify that it satisfies the Cauchy criterion. Let $\epsilon > 0$ be given. Because the series $\sum \rho^n$ converges, there is some $K \in \mathbb{N}$ such that

$$\left| \sum_{n=k}^{\ell} \rho^n \right| = \sum_{n=k}^{\ell} \rho^n < \epsilon$$

for all $\ell \geq k \geq K$. Now we see that for all $\ell \geq k \geq \max\{N, K\}$, we have

$$\epsilon > \sum_{n=k}^{\ell} \rho^n > \sum_{n=k}^{\ell} |a_n| \geq \left| \sum_{n=k}^{\ell} a_n \right|.$$

To see why such an N must exist, suppose to the contrary that it does not. Then we can find an infinite subsequence (a_{n_j}) such that

$$|a_{n_j}|^{1/n_j} \geq \rho$$

for all j. However, this implies that $\limsup |a_n|^{1/n} \geq \rho > L$.

This verifies that $\sum a_n$ satisfies the Cauchy criterion and therefore converges.

To finish the proof, suppose that $L > 1$ and choose a number ρ such that $L > \rho > 1$. For each $N \in \mathbb{N}$, there must be some $n \geq N$ such that $|a_n|^{1/n} > \rho$, and thus $|a_n| > \rho^n$. If follows that there must be an infinite subsequence (a_{n_k}) for which $|a_{n_k}| > \rho^{n_k}$ for all k. However, since $\rho > 1$, this means that the sequence (a_n) does not converge to 0, and thus $\sum a_n$ must diverge. □

We conclude this section by considering two classical examples of numerical series where the root test does not apply. One of them diverges while the other converges.

Example 5.2.6. The harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges.

The name of this series is due to Pythagoras's first experiments with music. Pythagoras noticed that striking a glass half-full of water produced a note one octave higher than striking a glass full of water. A glass one-third-full of water similarly produced a note at a "perfect fifth" of a whole glass, while a glass one-quarter-full produced a note two octaves higher, and a glass one-fifth-full produced a "major third." These higher frequencies are referred to as *harmonics*, and all musical instruments produce harmonics in addition to the *fundamental frequency* which they are playing (the instrument's "timbre" describes the amounts in which these different harmonics occur). This is what led Pythagoras to call the series $1 + 1/2 + 1/3 + \cdots$ the harmonic series.

To establish this, we can simply group the terms together so that every group sums to at least $1/2$:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq 2\cdot\frac{1}{4}=\frac{1}{2}} \\
&\quad + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq 4\cdot\frac{1}{8}=\frac{1}{2}} \\
&\quad + \underbrace{\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}}_{\geq 8\cdot\frac{1}{16}=\frac{1}{2}} \\
&\quad + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots .
\end{aligned}$$

In general, this grouping is always possible because

$$\underbrace{\frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}}_{2^k \text{ terms}} \geq 2^k \cdot \frac{1}{2^{k+1}} = \frac{1}{2}.$$

This proof of the divergence of the harmonic series is due to the French philosopher Nicolas Oresme (1323–1382), and stands as one of the pinnacles of medieval mathematical achievement. (There was not much medieval mathematical achievement.)

Example 5.2.7. The series $\sum 1/n^2$ converges.

This series is often shown to converge by the integral test, but here we give a proof using comparison. Letting (s_k) denote the partial sums of the series, we have

$$s_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \sum_{n=2}^{k} \frac{1}{n^2 - n} = 1 + \sum_{n=2}^{k} \left(\frac{1}{n-1} - \frac{1}{n} \right).$$

The sum on the right of this inequality telescopes, and thus we have

$$s_k < 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right) = 2 - \frac{1}{k}.$$

While we show that the series $\sum 1/n^2$ converges, we do not compute its value here. For series that aren't geometric, such questions are generally extremely difficult, and this series is no exception. Finding $\sum 1/n^2$ became known as the Basel problem after it was posed by Pietro Mengoli (1626–1686) in 1644. In 1735, at the age of twenty-eight, Leohnard Euler showed that

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.645,$$

one of the first results of his career.

From this inequality, we see that (s_k) is a nondecreasing sequence of real numbers bounded above by 2. Therefore (s_k) has a limit, and thus (by definition), $\sum 1/n^2$ converges.

Exercises

Exercise 5.2.1. Show if (s_n) is an increasing sequence of real numbers and if (s_n) has a bounded subsequence, then (s_n) converges. Interpret this result in terms of series.

Exercise 5.2.2. Prove that if (a_n) is a sequence of nonnegative numbers and the series $\sum a_n$ converges, then so does the series $\sum a_n^2$.

Exercise 5.2.3. Suppose that (a_n) and (b_n) are sequences of real numbers and that both of the series $\sum a_n^2$ and $\sum b_n^2$ converge. Prove that the series $\sum a_n b_n$ also converges.

For Exercise 5.2.3, you may want to use the inequality $2|ab| \leq a^2 + b^2$, or apply the Cauchy–Schwarz inequality to the partial sums.

Exercise 5.2.4. Show that if (a_n) is a decreasing sequence of positive real numbers and

$$\sum_{n=0}^{\infty} a_n$$

converges, then $\lim_{n\to\infty} na_n = 0$.

For Exercise 5.2.4, one may want to observe that $\lim a_n = 0$ and that for every $m \in \mathbb{N}$ and every $k \geq m$,

$$\sum_{n=m}^{k} a_n \geq (k - m + 1)a_k.$$

5.3 Power series and uniform convergence

We now define power series, which are the principal reason we are interested in series.

Definition 5.3.1 (Power series)**.** Let (a_n) be a sequence of real numbers and suppose that $a \in \mathbb{R}$ and $m \in \mathbb{N}$. The expression

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

is a *power series* with *center* a.

Changing the center of a power series merely shifts its behavior left or right, so for convenience we often specialize to the case of power series centered at the origin. We also suppress the summation limits when they are clear, so our generic power series is written as $\sum a_n x^n$.

Given a power series $\sum a_n x^n$, we define the set

$$D = \{x \in \mathbb{R} : \sum a_n x^n \text{ converges}\}.$$

Since the power series converges for $x \in D$, it defines a real-valued function with domain D. Our first result tells us about this domain.

Note that every power series converges at its center, so the set D is never empty.

Theorem 5.3.2. *Given a sequence (a_n), define*

$$R = \frac{1}{\limsup |a_n|^{1/n}},$$

where R is interpreted as being 0 if the limit supremum is ∞ and as being ∞ if the limit supremum is 0. The series $\sum a_n x^n$ converges for $x \in (-R, R)$ and diverges for $x \notin [-R, R]$.

As in the root test, Theorem 5.3.2 does not specify the behavior of the power series when $|x| = R$.

Proof. For a fixed point $x \in \mathbb{R}$, we define

$$\begin{aligned} L &= \limsup |a_n x^n|^{1/n} \\ &= \lim_{k\to\infty} \left(\sup \left\{ |a_n x^n|^{1/n} : n \geq k \right\} \right) \\ &= |x| \lim_{k\to\infty} \left(\sup \left\{ |a_n|^{1/n} : n \geq k \right\} \right) \\ &= |x| \limsup |a_n|^{1/n}. \end{aligned}$$

By the root test, the power series converges if $L < 1$ and diverges if $L > 1$. If $\limsup |a_n|^{1/n} = 0$, then R is defined to be ∞, and we have $L = 0$ so the series converges for every $x \in (-\infty, \infty)$ and the theorem holds. If $\limsup |a_n|^{1/n} = \infty$, then R is defined to be 0, and we have $L = \infty$, so the series diverges everywhere but its center $x = 0$, so the theorem holds.

Otherwise if $\limsup |a_n|^{1/n} \neq 0, \infty$, then $L = |x|/R$, so the root test shows that the power series converges if $|x| < R$ and diverges if $|x| > R$, completing the proof of the theorem. □

Theorem 5.3.2 guarantees that the set of points at which a given power series converges is an interval, which we call the *interval of convergence*. The number R in Theorem 5.3.2 is called the *radius of convergence* of the power series.

More specifically, Theorem 5.3.2 shows that the interval of convergence can be any one of the four possibilities $(-R, R)$, $(-R, R]$, $[-R, R)$, or $[-R, R]$. Exercise 5.3.1 provides examples showing that all four of these possibilities can occur.

Given a power series $\sum a_n x^n$ with interval of convergence $I \subseteq \mathbb{R}$, we denote by $s(x)$ the function $s : I \to \mathbb{R}$ defined by

$$s(x) = \sum a_n x^n.$$

We also let $s_k(x)$ denote the kth partial sum of this series,

$$s_k(x) = \sum_{n=0}^{k} a_n x^n.$$

Note that these partial sums are polynomials, so they are defined on all of $\mathbb{R}$. On the interval I, the sequence $(s_k(x))$ of functions converges pointwise to $s(x)$, written

$$s(x) = \sum_{n=0}^{\infty} a_n x^n = \lim_{k\to\infty} s_k(x).$$

In fact, on any closed interval within the interval of convergence, this convergence is *uniform*, as our next result shows.

Theorem 5.3.3. *If the power series $s(x) = \sum a_n x^n$ has radius of convergence R, and $r \in (0, R)$, then the sequence $(s_k(x))$ of functions converges uniformly to $s(x)$ on the interval $[-r, r]$.*

Proof. Let $\epsilon > 0$ be given. To establish uniform convergence, we need to find a $K \in \mathbb{N}$ such that $|s_k(x) - s(x)| < \epsilon$ for every $k \geq K$ and every $x \in X$.

We begin by apply the root test to the sum $\sum |a_n r^n|$, which leads us to consider the limit supremum

$$\limsup |a_n r^n|^{1/n} = r \limsup |a_n|^{1/n}.$$

This quantity is less than 1 because $r < R$, so the series $\sum |a_n r^n|$ converges, and hence by definition the sequence (t_k) defined by

$$t_k = \sum_{n=0}^{k} |a_n r^n|$$

converges (because these are the partial sums of the series). Therefore the sequence (t_k) is Cauchy, so we can choose $K \in \mathbb{N}$ such that if $m \geq \ell \geq K$, then

$$\frac{\epsilon}{2} > |t_m - t_\ell| = \left| \sum_{n=\ell+1}^{\ell} |a_n r^n| \right| = \sum_{n=\ell+1}^{\ell} |a_n r^n|.$$

Therefore for any $x \in [-r, r]$ and any indices $m \geq \ell \geq K$, we have

$$|s_m(x) - s_\ell(x)| = \left| \sum_{n=\ell+1}^{m} a_n x^n \right| \leq \sum_{n=\ell+1}^{m} |a_n x^n| \leq \sum_{n=\ell+1}^{m} |a_n r^n| < \epsilon/2.$$

Now let $x \in [-r, r]$ and $\ell \geq K$ be arbitrary. Because the sequence $(s_k(x))$ converges pointwise to $s(x)$, for this particular point x we can find an index $m_x \geq k$ such that $|s_{m_x}(x) - s(x)| < \epsilon/2$. Therefore we have

This part of the proof closely resembles our proof of Theorem 4.2.4.

$$|s_\ell(x) - s(x)| \leq |s_\ell(x) - s_{m_x}(x)| + |s_{m_x}(x) - s(x)| < \epsilon,$$

completing the proof. □

Because each partial sum $s_k(x)$ is a polynomial, polynomials are continuous, and uniform convergence preserves continuity, we immediately see that every power series is continuous within its radius of convergence:

It is natural to ask if the power series is also continuous at $x = \pm R$ if it happens to converge there. The answer is yes; this is known as *Abel's theorem* (and takes a bit more work to show).

Corollary 5.3.4. *If the power series $s(x) = \sum a_n x^n$ has radius of convergence R, then $s(x)$ is continuous on the open interval $(-R, R)$.*

We also immediately obtain term-by-term integration, because polynomials are integrable and uniform convergence preserves integrability (Theorem 4.4.1).

Corollary 5.3.5. *If the power series $s(x) = \sum a_n x^n$ has radius of convergence R, and $r \in (0, R)$, then $s(x)$ is integrable on the interval $[-r, r]$ and*

$$\int_0^r s(x)\,dx = \lim_{k\to\infty} \int_0^r s_k(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} r^{n+1}.$$

For the reader's convenience, we recall the following result.

Theorem 4.4.3 (Differentiating uniform limits, more general version). *Let $I \subseteq \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions from I to $\mathbb{R}$. Suppose that there exists a point $x_0 \in I$ such that $(f_n(x_0))$ converges, and that the sequence (f_n') of derivatives exists on I and converges uniformly on I to a function g. Then the sequence (f_n) converges uniformly on I to a function f that is differentiable at every point of I, and $f' = g$.*

Finally, we would like to justify term-by-term differentiation, but this requires a bit more work, because our theorem about the differentiation of uniform limits has rather complicated hypotheses. In particular, in order to establish that $s'(x)$ exists and equals $\lim s_k'(x)$ on the open interval $(-R, R)$, we need to show first that the sequence $(s_k(x))$ converges at some point in the interval, and second that the sequence of derivatives $(s_k'(x))$ converges uniformly to some function. The first of these tasks is easy; the sequence $(s_k(x))$ converges at $x = 0$ to $s(0) = a_0$. For the second, we note that

$$s_k'(x) = \sum_{n=0}^{k} n a_n x^{n-1} = \sum_{n=0}^{k-1} (n+1) a_{n+1} x^n,$$

and

$$\limsup |(n+1)a_{n+1}x^n|^{1/n} = \limsup |a_n x^n|^{1/n},$$

so the radius of convergence of the series $\sum (n+1) a_{n+1} x^n$ is the same as the radius of converges of the series $\sum a_n x^n$ by Theorem 5.3.2. Then by Theorem 5.3.3, we see that the sequence of derivatives $(s_k'(x))$ converges uniformly on the interval $[-r, r]$ for every $r \in (0, R)$. This allows us to derive our final consequence of the uniform convergence of power series.

Corollary 5.3.6. *If the power series $s(x) = \sum a_n x^n$ has radius of convergence R, then $s(x)$ is differentiable on the interval $(-R, R)$, and on this interval,*

$$s'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n.$$

Exercises

Exercise 5.3.1. Find the interval of convergence for the following power series.

$$\sum_{n=1}^{\infty} n^{-n} x^n, \qquad \sum_{n=1}^{\infty} \frac{x^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad \text{and} \qquad \sum_{n=1}^{\infty} n^n x^n.$$

Exercise 5.3.2. Let $\sum a_n (x-a)^n$ be a power series that converges uniformly on all of $\mathbb{R}$. Prove that the sequence (a_n) is eventually 0. (That is, there is an $N \in \mathbb{N}$ such that $a_n = 0$ for all $n \geq N$.)