Growth Rates of Permutation Classes

From Countable to Uncountable

Vince Vatter

University of Florida

Gainesville, FL

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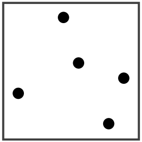
The Containment Order



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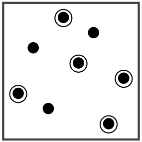

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- A downset in this order is a *permutation class*.
- Every permutation class $\mathcal{C}$ can be defined by the minimal set of permutations B it *avoids* (its *basis*):

$$\mathcal{C} = \mathrm{Av}(B) = \{\pi \;:\; \beta \not\leqslant \pi \text{ for all } \beta \in B\}.$$

- $\mathcal{C}_n$ is the set of permutations in $\mathcal{C}$ of length n.
- The generating function of $\mathcal{C}$ is

$$\sum_{n \in \mathbb{N}} |\mathcal{C}_n| x^n = \sum_{\pi \in \mathcal{C}} x^{|\pi|}.$$

Growth Rates

- **If the limit exists**, the *growth rate* of $\mathcal{C}$ is

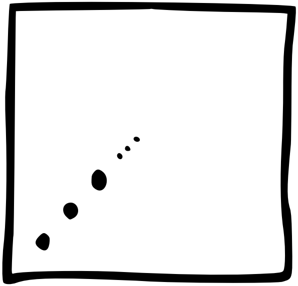
$$\mathrm{gr}(\mathcal{C}) = \lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}.$$

- Growth rates are known to exist for classes that avoid a *single* basis element.
- Otherwise we settle for the *upper growth rate*,

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}.$$

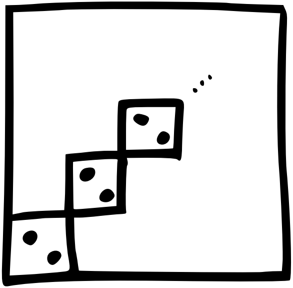
The Marcus–Tardos Theorem. *Every proper permutation class has a finite upper growth rate.*

Av(21)



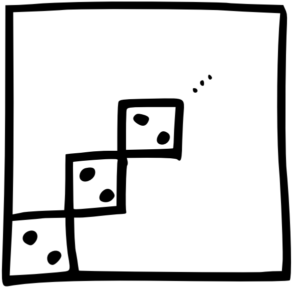
- Enumeration: 1, 1, 1, 1, 1, ...
- Generating function: $1 + x + x^2 + \cdots = \dfrac{1}{1-x}$.
- Growth rate: 1.

$\mathrm{Av}(231, 312, 321)$



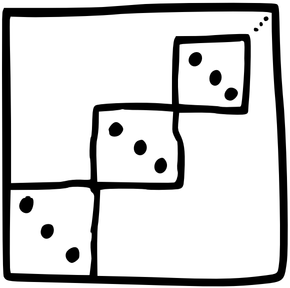
- Enumeration: 1, 1, 2, 3, 5, 8, …

$\mathrm{Av}(231, 312, 321)$



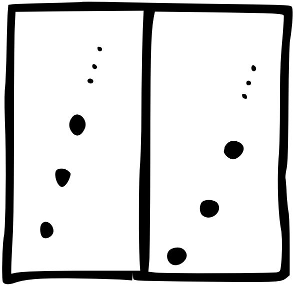
- Enumeration: $1, 1, 2, 3, 5, 8, \ldots$ (Fibonacci numbers).
- Generating function: $\dfrac{1}{1 - x - x^2}$.
- Growth rate: $\phi \approx 1.62$ (golden ratio).

Av(231, 312, 4321)



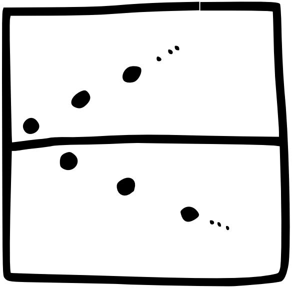
- Enumeration: 1, 1, 2, 4, 7, 13, … (3-generalized Fibonacci numbers).
- Generating function: $\dfrac{1}{1 - x - x^2 - x^3}$.
- Growth rate: ≈ 1.84.

Av(321, 2143, 3142)



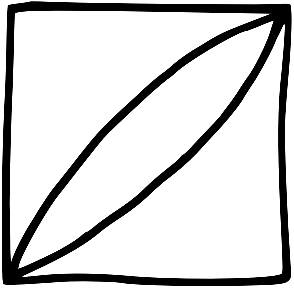
- Enumeration: 1, 1, 2, 5, 12, 27, . . . $(2^n - n)$.
- Generating function: $\frac{1 - x - x^2}{(1-2x)(1-x)^2}$.
- Growth rate: 2.

Av(132, 312)



- Enumeration: 1, 1, 2, 4, 8, 16, … (2^{n-1} for $n \geqslant 1$).
- Generating function: $\dfrac{1-x}{1-2x}$.
- Growth rate: 2.

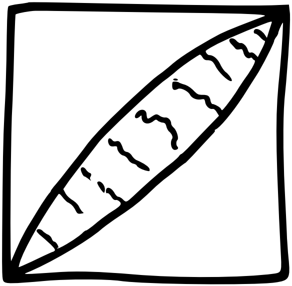
Av(321)



- Enumeration: 1, 1, 2, 5, 14, 42, … (Catalan numbers).
- Generating function: $\dfrac{1-\sqrt{1-4x}}{2x}$.
- Growth rate: 4.

Note: Larger than any class we'll consider in this talk.

Av(4231)



- Enumeration: 1, 1, 2, 6, 23, 103, . . . (no one knows).
- Generating function: ?
- Growth rate: between 9.81 and 13.74, maybe around 11.60? (Bevan, Bóna, and Guttmann. Improvements promised in 2017 by Bevan, Brignall, Pantone, and Price.)

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Common question: *Compute the growth rate of this or that class.*

Our question: *Describe the set of all growth rates.*

Very Small Classes

"Theorem". *There are no growth rates between* 0 *and* 1.

Proof. Either the class is finite (growth rate 0) or it is infinite (growth rate $\geqslant 1$). □

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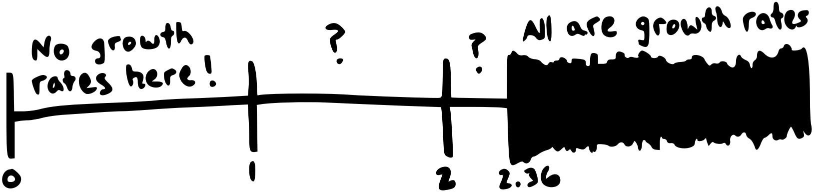

The Fibonacci Dichotomy

Theorem. (Kaiser and Klazar 2003) *If* $|\mathcal{C}_n| < F_n$ *(the* n*th combinatorial Fibonacci number) for any integer* n *then* $|\mathcal{C}_n|$ *is eventually polynomial.*

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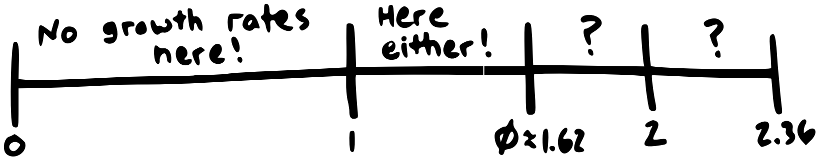
Corollary. *There are no growth rates between* 1 *and* ϕ.



The Full Kaiser and Klazar Result

Theorem. (Kaiser and Klazar 2003) *If* $\mathrm{gr}(\mathcal{C}) < 2$ *then, up to a polynomial factor,* $|\mathcal{C}_n|$ *grows like the* k*-generalized Fibonacci numbers* $F_n^{(k)}$ *for some* k*.*

- Recurrence: $F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}$.
- The regular Fibonacci numbers are $F_n^{(2)}$.
- Generating function: $1/(1 - x - x^2 - \cdots - x^k)$.
- Growth rates $\to 2$ as $k \to \infty$.

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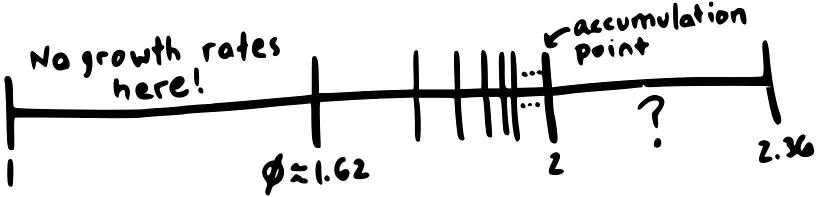

Up to Uncountability (The First Time)

Define κ = the unique positive root of $x^3 - 2x^2 + 1 \approx 2.20557$.

- (V 2011) κ is the first accumulation point of accumulation points of growth rates.
- (V 2011) There are only countably many classes of growth rate $< \kappa$, but uncountably many of growth rate κ.
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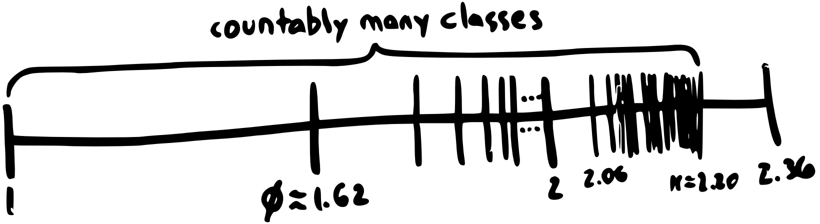
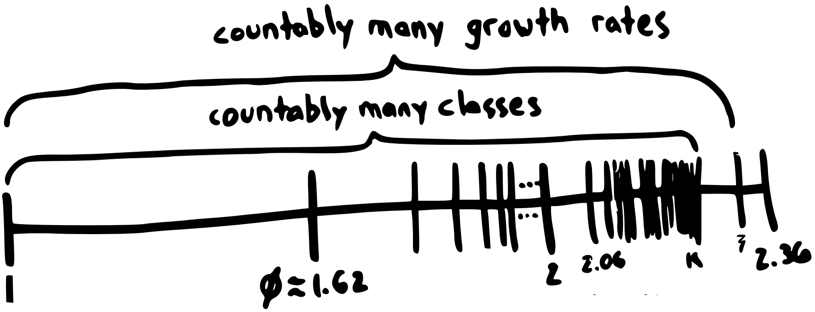


New Results

Define ξ = the positive root of $x^5 - 2x^4 - x^2 - x - 1 \approx 2.30522$.

Theorem. (V 2017+) *There are only countably many growth rates of permutation classes below ξ but uncountably many growth rates in every open neighborhood of it.*



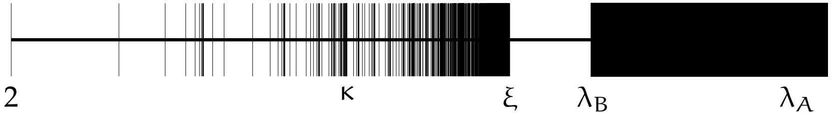
Theorem. (Pantone–V 2017+) *Complete list of growth rates up to ξ.*

A Very Broad Picture

For characterizing classes/growth rates at the low end of the spectrum, there are four basic ingredients:

1. Ramsey-ish results for the plane. *Conclusion: very rough pictures of the classes.*
2. Slicing results to refine the structure you get from above. *Conclusion: more detailed pictures of these classes.*
3. Order-theoretic arguments that allow you to strip the classes down to the parts responsible for their growth rates. *Conclusion: all such growth rates are achieved by "sum closed" classes.*
4. Results about sum closed classes. *Conclusion: the set of growth rates.*

Conclusion



- V. Growth rates of permutation classes: From countable to uncountable. arXiv:1605.04297 [math.CO]
- Pantone and V. Growth rates of permutation classes: Categorization up to the uncountability threshold. arXiv:1605.04289 [math.CO].