A Sharp for the Chang Model

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Inner Model Theory & Large Cardinals
A 50 Year Celebration

Outline

These are the slides which I used for my talk at the MAMLS meeting at Harvard University, “Inner Model Theory & Large Cardinals—A 50 Year Celebration” on February 19, 2011. I have corrected (but not, I believe, all) of the typos and have added some notes.

The forcing used in this paper is based on that of Gitik in his proof of the independence of the SCH. The argument that a generic set for the forcing can be constructed using the indiscernibles coming from the iteration is based on work of Carmi Merimovich.

Definition

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The Chang model $\mathbb{C}$ is the smallest model of ZF containing the ordinals and containing all of its countable subsets.

It can be obtained by modifying the definition of L at ordinals α of cofinality ω:

$$\mathbb{C}_{\alpha+1} = \mathrm{Def}^{\mathbb{C}_\alpha \cup [\alpha]^{<\omega_1}}(\mathbb{C}_\alpha \cup [\alpha]^{<\omega_1}),$$

Or (following Chang) use the infinitary logic $L_{\omega_1,\omega}$:

$$\mathbb{C}_{\alpha+1} = \mathrm{Def}^{\mathbb{C}_\alpha}_{L_{\omega_1,\omega}}(\mathbb{C}_\alpha).$$

Simple Observations

- $\mathcal{R} \subseteq \mathbb{C}$, and hence $L(\mathcal{R}) \subseteq \mathbb{C}$.
- $\mathbb{C}$ satisfies DC (Dependent choice), and any integer game decided in V is decided in $\mathbb{C}$.
- Any member of $\mathbb{C}$ is definable in $\mathbb{C}$ by using a countable sequence of ordinals as a parameter.
- $[\mathbb{C}]^{<\omega_1} \subset \mathbb{C}$
- If $[M]^{<\omega_1} \subseteq M$ and the ordinals are contained in M, then $\mathbb{C}^M = \mathbb{C}$.

Examples

- $\mathbb{C}^L = L$.
- $\neg 0^\sharp \implies \mathbb{C} = L([\omega_2]^\omega)$.
- In $L[U]$, let $i^U_\omega \colon L[U] \to \mathrm{Ult}_\omega(L, U) = L[U_\omega]$.
  Then

$$\begin{aligned}\mathbb{C}^{L[U]} &= L[U_\omega][\langle\, i^U_n(\kappa) \mid n < \omega \,\rangle] \\ &= L[\langle\, i^U_n(\kappa) \mid n < \omega \,\rangle]\end{aligned}$$

  and

$$\mathrm{HOD}^{\mathbb{C}^{L[U]}} = L[U_\omega].$$

The point is that U_ω can be reconstructed from its Prikry sequence $\langle\, i^U_n(\kappa) \mid n \in \omega \,\rangle$, which is countable.

Examples, continued

With more measures, we have the theorem of Kunen:

Theorem (Kunen)

If there are uncountably many measurable cardinals, then the Axiom of Choice is false in $\mathbb{C}$.

Examples, continued

Nevertheless, if $V = L[\mathcal{U}]$ for a sequence $\mathcal{U}$ of measures then

- $\mathbb{C} = L[\text{an iterate of } \mathcal{U}](\{\text{all countable sequences of critical points}\})$
  $= L[(\{\text{all countable sequences of critical points}\})$
- $\mathrm{HOD}^{\mathbb{C}}$ is an iterate of $L[\mathcal{U}]$.

and if $K = L[\mathcal{U}]$ then, by the covering lemma,

- $\mathbb{C} = L([\omega_2]^\omega)(\{\text{countable sequences of critical points}\})$.

Where does this fail?

Woodin gave an upper bound:

Theorem (Woodin)

If there is a measurable Woodin cardinal, then there is a sharp for the Chang model.

What this means

There is a closed, unbounded class I of ordinals such that the following holds: Say $B \subseteq I$ is *suitable* if

- B is closed and countable, and
- every successor member of B either is a successor point of I or has uncountable cofinality.

Then

- If B and B' are suitable and have the same length, then $\mathbb{C} \models (\phi(B) \iff \phi(B'))$ for all formulas ϕ.
- The theory of $\mathbb{C}$ (with parameters for suitable sets) is fixed.
- There are Skolem functions (not in $\mathbb{C}$) such that $\mathbb{C}$ is the hull of $\mathcal{R}$ and $[I]^{<\omega_1}$.

Question

Question (Woodin)

- With only measurable cardinals, $K^{\mathrm{HOD}^{\mathbb{C}}}$ is an iterate of K.
- With a measurable Woodin cardinal, we have a sharp for $\mathbb{C}$.

- What happens inside this large gap? Does $\mathrm{HOD}^{\mathbb{C}}$ continue to be an iterate of $K^{\mathbb{C}}$?

When Woodin asked this in a conversation at the Mittag-Leffler Institute, James Cummings and Ralf Schindler stated that arguments of Gitik suggested that this would break down at $o(\kappa) = \kappa^{+\omega}$.
I demurred, thinking that this situation was different.
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Note: Woodin observes that with some large cardinal strength we have $\mathrm{HOD}^{\mathbb{C}} = K^{\mathbb{C}}$, so it is not necessary to specify $K^{\mathrm{HOD}^{\mathbb{C}}}$ in the next slide. I do not think that this is true at the level of measurable cardinals. In any case, the main point is that $\mathbb{C}$ has all the large cardinal strength of V.

First, Pushing the Lower Bounds

- An argument due to Gitik allows extenders of length less than $\kappa^{+\omega}$ to be reconstructed from their indiscernibles.
- Hence $K^{\mathsf{HOD}^{\mathbb{C}}}$ is an iterate of K if there is no extender of length $\kappa^{+\omega}$.
- This can easily be extended to “no extender of length $\kappa^{+\omega+1}$.”
- This breaks down at an extender of length $\kappa^{+(\omega+1)}$ (calculated in a mouse before the extender is added)..

A Conjecture...

Conjecture

Suppose that $M = L_\alpha(\mathcal{R})[\mathcal{E}]$ is the least mouse over the reals such that $\mathcal{E}$ has a last extender E of length $\kappa^{+(\omega+1)}$.

Then M is a sharp for $\mathbb{C}$.

- Note that M projects onto $\mathcal{R}$ by a Σ_1 function — In particular, if M satisfies the CH then $|M| = \omega_1$.
- Hence all the cardinal calculations will be made in M (0r iterates of M) below the final extender.
- It may well be that the extender E should survive for a few levels of construction after E is added.

A Bit of Explaination . . .

$0^\sharp$, the sharp for L, can be viewed in (at least) three ways:

- A closed proper class $I = \{ \iota_n \mid \nu \in \Omega \}$ of indiscernibles for L, such that every member of L is definable with parameters from $[I]^{<\omega}$.
- The theory of $(L, \iota_n)_{n\in\omega}$.
- The least model $M = L_{\kappa+1}[U]$ satisfying that U is a measure on κ. Then M is countable, and $\iota_\nu = \{ i^U_\nu \mid \nu \in \Omega \}$.

A Bit of Explaination . . .

- That E is an extender of length $(\kappa^{+(\omega+1)})^M$ means that $\mathcal{P}^{\omega+1}(\kappa) \subseteq M$ and $\mathrm{Ult}(M, E) = \{\, i^E(f)(\nu) \mid f \in M \;\&\; \nu \in [\kappa, \kappa^{+(\omega+1)} \,\}$.

- Thus, if $M_\Omega = \mathrm{Ult}_\Omega(M, E)$ then every member of M can be written $i_\Omega(f)(a)$ for some $f \in M$ and

$$a \in \bigcup \left\{ \left[\, i_\nu(\kappa), i_\nu(\kappa^{+(\omega+1)}) \right) \mid \nu \in \Omega \right\}.$$

- Thus, since ${}^\omega M \subseteq M$, the Chang model $\mathbb{C}$ can be obtained by adding every countable subset of $\bigcup \left\{ \left[\, i_\nu(\kappa), i_\nu(\kappa^{+(\omega+1)}) \right) \mid \nu \in \Omega \right\}$.

An Upper bound

Theorem

Suppose that $M = L_\alpha(\mathcal{R})[\mathcal{E}]$ is a mouse over the the reals with a last extender E of length $\kappa^{+\omega_1}$.

Then there is a sharp for the Chang model $\mathbb{C}$.

I will try to outline some ideas of the proof.

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Woodin pointed out that in one respect it may be incorrect to call this a "sharp" for the Chang model: the theory of the Chang model is only fixed relative to the set of reals, and forcing to add reals may change this theory. He also observed that with a class of Woodin cardinals the theory actually would be fixed under small generic extensions.

- For simplicity, we will assume that $M \models \mathrm{CH}$. If not, then we could begin with a generic collapse of $\mathcal{R}$ onto ω_1.
- Let $M_\Omega = \mathrm{Ult}_\Omega(M, E)$ be the class iteration of M (with the top extender stripped off). Let I be the set of critical points of $i^E_\Omega \colon M \to M_\Omega$.
- If B is suitable, then let $M_B \prec M_\Omega$ be the Skolem hull in M_Ω of members of (and indiscernibles belonging to members of) B.
- If $\delta + 1 = \mathrm{otp}(B)$ and B' is the set containing the first $\delta + 1$ members of I, then $M_{B'} = \mathrm{Ult}_\delta(M, E) \cong M_B$ (with the top extender $i_\delta(E)$ cut off).
- Let $\mathbb{C}_B$ be the Chang model in the model obtained by closing M_B under countable sequences.

Some Observations

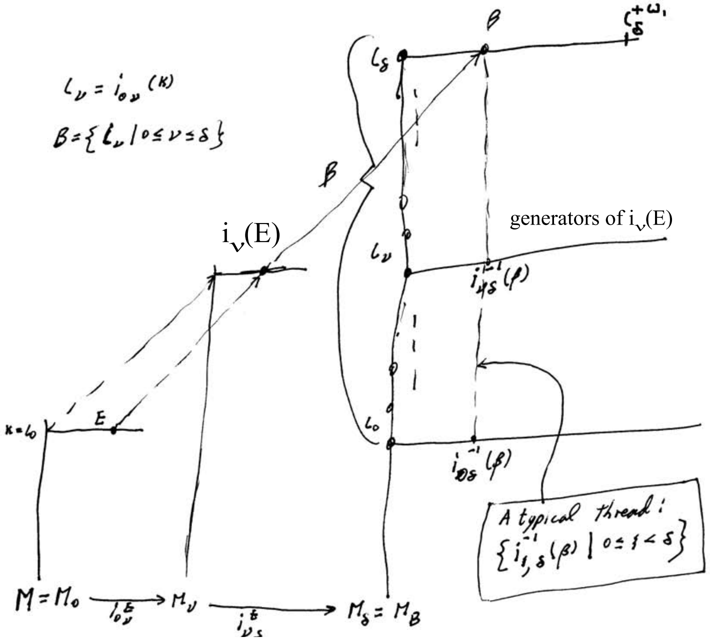
If $B = \{ \iota_\xi : \xi \leq \delta \}$ is the first $\delta + 1$ members of I then

- M_B is transitive.
- $\mathbb{C}_B = \mathbb{C}_\alpha$ where $\alpha = \mathrm{Ord}(M_B)$.

and $\mathbb{C}_B$ can be obtained from M_B by

- Start with M_B.
- Add all further countable threads, i.e., the sets of the form $\left\{ i^{-1}_{\xi,\delta}(\nu) \mid \nu < \delta \ \& \ \beta \in a \right\}$ for $a \in \left[[\iota, \iota^{+\omega_1}) \right]^{<\omega_1}$.

  The result of this step will contain all of its countable subsets, since $M = M_0$ contains all of *its* countable subsets.
- Then $\mathbb{C}$ as defined inside this model is $\mathbb{C}_B$.



$\iota_\nu = i_{0\nu}(\kappa)$
$B = \{\iota_\nu \mid 0 \le \nu \le \delta\}$
β
$\iota_\delta^{+\omega_1}$
ι_δ
B
$i_\nu(E)$
generators of $i_\nu(E)$
ι_ν
$i_{\nu\delta}^{-1}(\beta)$
$\kappa = \iota_0$
E
ι_0
$i_{0\delta}^{-1}(\beta)$
A typical thread:
$\{i_{\xi,\delta}^{-1}(\beta) \mid 0 \le \xi < \delta\}$
$M = M_0 \xrightarrow{i_{0\nu}^E} M_\nu \xrightarrow{i_{\nu\delta}^E} M_\delta = M_B$

More Observations

Proposition

- $[\Omega]^\omega \subseteq \bigcup_B \mathbb{C}_B$.
- $\mathbb{C}_B \models ZF + V = \mathbb{C}$.
- *If B and B' have the same order type, then $\mathbb{C}_B \cong \mathbb{C}_{B'}$.*

Lemma (Main Lemma)

If $B' \subset B$ then $\mathbb{C}_{B'} \prec \mathbb{C}_B$.

Corollary

$\mathbb{C} = \bigcup_B \mathbb{C}_B$ — *and all the rest.*

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The crucial point is that this consideration allows us to deal directly only with countable iterations. This is important because the forcing I describe only works for countably many indiscernibles.

Proof of Main Lemma

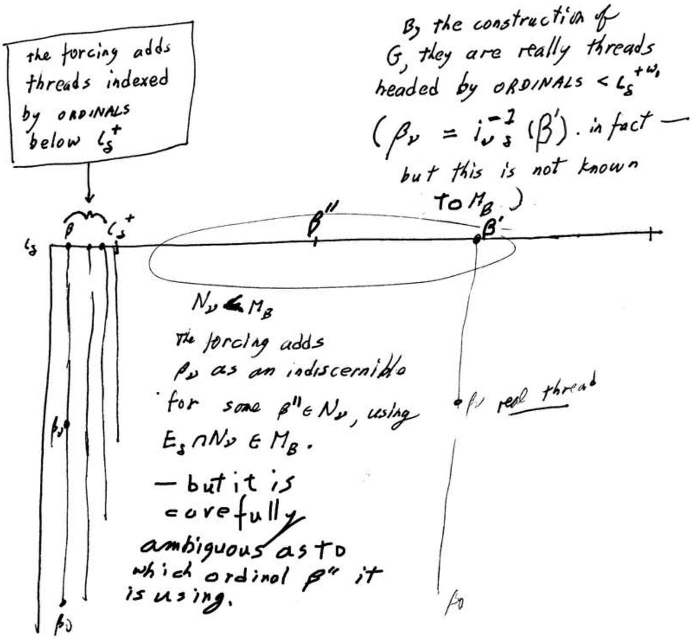
We can assume B is the first δ members of I for some countable δ:

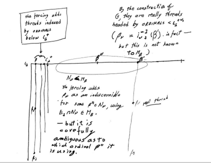
$$B = \{ \iota_\nu \mid \nu < \delta \}$$
$$B' = \{ \iota_{\sigma(\nu)} \mid \nu < \delta' \}.$$

Where $\sigma \colon \delta' \to \delta$ is continuous, strictly increasing, and maps successor ordinals to sucessor ordinals.

We want to define a forcing P in M_B, and a M_B-generic set $G \in V$, so that

$$\mathbb{C}^{M_B[G]} = \mathbb{C}_B.$$


the forcing adds threads indexed by ORDINALS below ι_s^+
By the construction of G, they are really threads headed by ORDINALS $< \iota_s^{+\omega_1}$
$(\beta_\nu = i_{\nu s}^{-1}(\beta'))$. in fact — but this is not known to M_B.)
β
ι_s^+
ι_s
β''
β'
$N_\nu \prec M_B$
The forcing adds β_ν as an indiscernible for some $\beta'' \in N_\nu$, using $E_s \cap N_\nu \in M_B$.
— but it is carefully ambiguous as to which ordinal β'' it is using.
β_ν
β_ν real thread
β_0
β_0



Perhaps some more explaination of this diagram should be given. The point is that we want to take threads from the actual iteration (labeled at β' in the picture and introduce them into the forcing. We can't do so directly—for one thing, E is not in the model M_B and so can't be used in the forcing. We simply assign the thread to an arbitrary ordinal β less than κ^+. However M_B does require some guidance for its forcing. We follow Gitik in having M_B regard the νth element of the thread as assigned to an ordinal β'' in a model $N_\nu \prec M_B$, using the extender $E_\nu = E \restriction N_\nu$ (which is a member of M_B). However (still following Gitik) it is necessary to introduce an ambiguity since β'' can't be fully like β'. This is done by using an equivalence relation $\leftrightarrow$ on the conditions.

Note that $\langle\, N_\nu \mid \nu < \delta \,\rangle$ is a member of M_B, but $\langle\, N_\nu \mid \nu < \omega_1 \,\rangle$ is not—indeed its union is probably all of M_B.

We start by defining a chain $\langle\, N_\nu \mid \nu \in \omega_1 \,\rangle$ in V with

- $N_{\nu'} \prec_? N_\nu \prec_? (M, E)$
- $\kappa^{+\nu} \subseteq N_\nu$
- $N_{\nu'} \subseteq N_\nu$
- $\vec{N} \restriction \gamma \in N_\gamma$ for each $\gamma < \omega_1$.
- $\vec{N} \restriction \gamma \in M$ for each $\gamma < \omega_1$.

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- $\vec{N} \restriction \gamma \in M$ for each $\gamma < \omega_1$.

Wooden asked about the ω_1-Chang model, the smallest model of ZF containing all ordinals and all of its subsets of size ω_1. At the time I answered that I though that Gitik's technique of recovering extenders from threads of length ω_1 might permit the Chang model to contain all large cardinal strength from the universe. Afterwards Ralf Schindler pointed out to me that GItik's technique runs into trouble with overlapping extenders, so that my suggestion is probably false.

Certainly the theory of the ω_1-Chang model can't be fixed under small generic extensions, since one can always force the CH to hold or for the reals not to be well orderable; or force the Souslin Hypothesis to be either true for false. Any of these would reflect to the ω_1-Chang model.